Practice Exam
Theoretical Physics 6a (QFT): SS 2019

08.07.2019

Exercise 1. $e^+e^- \to \phi^+\phi^-$ (50 points)

In the lecture, we derived the differential cross section ($d\sigma/d\Omega$) for the $e^+e^- \to \mu^+\mu^-$ in the collider frame, assuming the high energy limit ($\sqrt{s} \gg m_\mu$).
For this exercise, consider the case when the muons (spin 1/2) are replaced by spinless bosons ϕ (spin 0).

(a)(20 points) Determine the possible Feynman diagrams at leading order and calculate the corresponding S-matrix element (S_{fi}) for the $e^+e^- \to \phi^+\phi^-$ process using Feynman rules.
Hint: Notice that the scalar QED interaction needs to be considered in this case.

(b)(25 points) Calculate the unpolarized differential cross-section $d\sigma/d\Omega$ for this process in the the center of mass frame (collider frame).

(c)(05 points) Using the result obtained in the previous item, determine the total cross section.

Exercise 2. Conformal Symmetry (15 points)

Consider the Lagrangian for scalar ϕ^4 theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4 \tag{1}$$

a) *(5 p.)* Calculate the energy momentum tensor

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - g^{\mu\nu}\mathcal{L}, \tag{2}$$

and show explicitly by using the equation of motion for the field ϕ that

$$\partial_\mu T^{\mu\nu} = 0. \tag{3}$$

From which symmetry transformation follows the conservation law of the energy momentum tensor?

b) *(5 p.)* Calculate the trace T^μ_μ of the energy-momentum tensor and show that it can be written as a divergence in the case of $m = 0$.

c) *(5 p.)* Consider a infinitesimal scaling transformation

$$x' = (1-\epsilon)x. \tag{4}$$

This transformation induces a transformation law for the scalar field as:

$$\phi'(x') = (1+\epsilon d_\phi)\phi(x), \tag{5}$$

where d_ϕ is the scaling parameter of the field ϕ. Show that the variation of the field ϕ s given by:

$$\delta\phi = \phi'(x) - \phi(x) = \epsilon(d_\phi + x^\mu\partial_\mu)\phi \tag{6}$$

Exercise 3. 1-loop correction to the propagator in Yukawa theory (35 + 10 Bonus points)

Consider the interaction between a scalar field ϕ (with mass M) and a spin 1/2 field ψ (with mass m) described by the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}M^2\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \lambda\bar{\psi}\psi\phi, \tag{7}$$

where λ is a coupling constant.

a) *(5 p.)* Derive the Feynman rule corresponding to the interaction term.

b) *(10 p.)* The 1-loop correction to the scalar propagator induced by a fermion loop is presented in Fig. 3. Use the Feynman rules to derive the invariant amplitude $\mathcal{M}$ for this diagram, using the momentum labels as indicated on the figure.

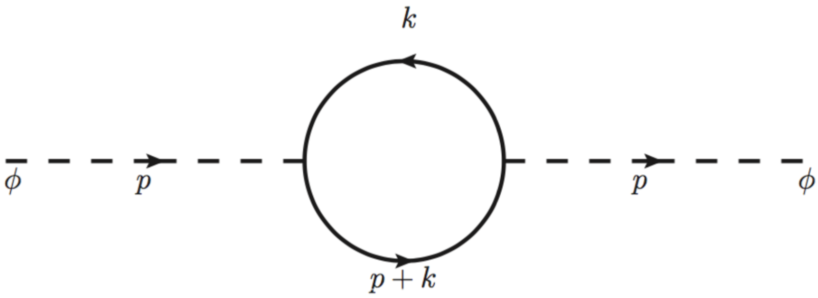

Figure 1: One-loop correction to the scalar propagator in Yukawa theory, given by the Lagrangian in Eq. (7).

c) *(10 p.)* Use the Feynman parameterization, and perform the one-loop integral using dimensional regularization (and using the formulas given at the end). Show that the result can be expressed as:

$$\mathcal{M}^{1-loop} = i\frac{4(d-1)\lambda^2\mu^{4-d}}{(4\pi)^{d/2}}\Gamma(1-d/2)\int_0^1 dx \frac{1}{[m^2 - p^2x(1-x)]^{1-d/2}}, \tag{8}$$

where d denotes the dimensionality of space-time, and μ is some arbitrary scale to keep the coupling λ dimensionless.

d) *(10 p.)* The scalar counterterms (CT) that have to be added to the diagram of Fig. 2 correspond with the Feynman rule:

$$\mathcal{M}^{CT} = i\left[p^2\delta_\phi - M^2(\delta_M + \delta_\phi)\right], \tag{9}$$

where δ_ϕ is the counterterm for the field ϕ, and δ_M is the counterterm for the scalar squared mass M^2.
Defining $\varepsilon \equiv 2 - d/2$, expand the above result for the invariant amplitude $\mathcal{M}$ in ε to extract the pole term in $1/\varepsilon$. Use the MS subtraction scheme, i.e. absorb only the divergent parts, and determine the MS expressions for the counterterms δ_ϕ and δ_M.

e) *(10 Bonus p.)* The renormalized propagator of the scalar field is given by

$$\frac{i}{p^2 - M^2 - \Sigma_R(p^2)}, \tag{10}$$

with the renormalized self-energy $\Sigma_R(p^2) = i(\mathcal{M}^{1-loop} + \mathcal{M}^{CT})$. Using the above result for the invariant amplitude, what is the expression for $\Sigma_R(p^2)$ in the $\overline{MS}$ scheme? You do not need to perform the Feynman parameter integral. What is the expression for the difference between the pole value (M_P^2) and the $\overline{MS}$ value ($M^2_{\overline{MS}}$) of the squared scalar mass ?
Note: This difference determines the shift in the Higgs mass (M) due to the heavy (mass m) top-quark loop.

Useful Formulas

The formula for the cross section for the process $a + b \to c + d$ is given by:

$$d\sigma = \frac{1}{(2E_a)(2E_b)v_{rel}} \frac{d^3\vec{p}_c}{(2\pi)^3(2E_c)} \frac{d^3\vec{p}_d}{(2\pi)^3(2E_d)} (2\pi)^4 \delta^4(p_a + p_b - p_c - p_d)|\mathcal{M}|^2, \tag{11}$$

where the four-momenta for each particle are given by $p_i = (E_i, \vec{p}_i)$, where v_{rel} stands for the incident flux, and $\mathcal{M}$ is the invariant amplitude.

$$\frac{1}{D_1 D_2} = \int_0^1 dx \frac{1}{[(1-x)D_1 + xD_2]^2} \tag{12}$$

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 - \Delta + i\epsilon)^n} = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)} \frac{(-1)^n}{\Delta^{n-d/2}} \tag{13}$$

$$\int \frac{d^d q}{(2\pi)^d} \frac{q^\mu q^\nu}{(q^2 - \Delta + i\epsilon)^n} = g^{\mu\nu} \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(n - 1 - d/2)}{2\Gamma(n)} \frac{(-1)^{n-1}}{\Delta^{n-1-d/2}} \tag{14}$$

$$\Gamma(z+1) = z\Gamma(z) \tag{15}$$

$$\Gamma(-1+\varepsilon) = -1 \left\{ \frac{1}{\varepsilon} + 1 - \gamma_E + \mathcal{O}(\varepsilon) \right\}, \tag{16}$$

with Euler constant $\gamma_E \approx 0.577$.