Problem Sheet 10

for the course
"Introduction to Lattice Gauge Theory"
Summer 2019

Lecturer: PD Dr. G. von Hippel

1. *Reading List*

    1. Gattringer/Lang, chapter 6 (p. 123–156) and section 11.1 (p. 267–279)
    2. Les Houches Lecture Notes, section2 1.6.2–1.6.3 (p. 83–85)

2. *Mesonic Correlators and Meson Masses*

    (a) In Problem (3)(d) on Problem Sheet 1, we derived the Wick theorem for free scalar field. Derive the corresponding theorem for free fermions. What is the crucial difference between the fermionic and bosonic cases?

    (b) Consider now a quark field q and the operator $O(x) = \overline{q}_x \gamma_5 q_x$. Derive an expression for the correlator $C_O(t) = \sum_{\mathbf{x}} \langle O(t, \mathbf{x}) O^\dagger(0) \rangle$ in terms of a path integral over the gauge fields only.

    (c) Consider now the case of two quark fields u, d of identical mass, and the operator $O(x) = \overline{u}_x \gamma_5 d_x$. Find an expression for $C_O(t)$ in terms of a path integral over the gauge fields only. How does this differ from the result of the preceding question?

    (d) Consider moreover the operators $O_\pm(x) = \overline{u}_x \gamma_5 u_x \pm \overline{d}_x \gamma_5 d_x$, and derive expressions for $C_{O_\pm}(t)$ in terms of a path integral over the gauge fields only. How do these expressions differ from each other?

    (e) Use the existence of a positive-definite transfer matrix for lattice QCD with Wilson fermions to show that the effective mass

    $$m_{\text{eff}}(t) = a^{-1} \log \frac{C_O(t)}{C_O(t+a)}$$

    tends to the mass E_1 of the lowest-lying energy eigenstate $|1\rangle$ with the quantum numbers of the operator O in the limit $t \to \infty$.

    (f) Taking u and d to be the up- and down-quark fields in the isospin limit, what are the lowest-lying energy eigenstates with the quantum numbers of $O_\pm$?

3. *Mesonic Decay Constants*

(a) Write down the Feynman diagram contributing to the decay $\pi^+ \to \mu^+ \nu_\mu$ to lowest order in the electroweak theory.

(b) Explain why the matrix element for this decay can be parameterized in terms of the pion decay constant f_π defined through $\langle 0|A_\mu(x)|\pi^+(p)\rangle = i f_\pi p_\mu \mathrm{e}^{-ip\cdot x}$, where the axial current is given by $A_\mu(x) = Z_A \overline{d}_x \gamma_\mu \gamma_5 u_x$ with Z_A an appropriately defined (finite) renormalization constant.

(c) Explain why this implies that f_π can be extracted from the correlator $C_{A_0}(t)$ of the axial current, and give an explicit expression allowing its extraction.