Problem Sheet 8

for the course

"Introduction to Lattice Gauge Theory"

Summer 2019

Lecturer: PD Dr. G. von Hippel

1. *Reading List*

   1. Smit, chapter 6 (p. 149-165)
   2. Gattringer/Lang, chapter 5 (p. 103-122)
   3. Les Houches Lecture Notes, sections 1.4 (p. 36-50) and 1.6.1 (p. 78-82)

2. *Naive Fermions and the Nielsen-Ninomiya Theorem*

   (a) Explain why the free Dirac action (in Euclidean metric)

   $$\mathcal{L} = \overline{\psi}\,(\not\partial + m)\,\psi$$

   must be discretized using a symmetric difference instead of a forward or backward difference. (Hint: consider the hermiticity properties of the Dirac operator!)

   (b) Compute the Fourier transform of the resulting lattice action on an infinite lattice. Considering the massless case for simplicity, show that the corresponding fermion propagator has 16 poles within the first Brillouin zone.

   (c) Explain why any discretization of the Dirac action that satisfies the demands of translation invariance, hermiticity, (ultra-)locality and chiral symmetry $\{D, \gamma_5\} = 0$ can be put into a one-to-one correspondence with a smooth vector field on the D-dimensional torus $\mathbb{T}^D$. (Hint: consider the Fourier transform!)

   (d) The Poincaré-Hopf theorem states that for a smooth vector field f on a manifold M the sum of the indices of the zeros of the vector field is equal to the Euler characteristic of M,

   $$\sum_{x \in f^{-1}(0)} \operatorname{sgn}\det\left(\frac{\partial f_\mu}{\partial x_\nu}\right) = \chi(M).$$

   The Euler characteristic of $\mathbb{T}^D$ vanishes, $\chi(\mathbb{T}^D) = 0$. Conclude the Nielsen-Ninomiya theorem: There is no discretization of the Dirac action that satisfies translation invariance, hermiticity, (ultra-)locality and chiral symmetry $\{D, \gamma_5\} = 0$ and has a propagator with only one pole.

3. *Wilson Fermions*

The Wilson action for fermions is defined by adding the Wilson term

$$-\frac{r}{2}a^5 \sum_x \sum_\mu \overline{\psi}_x \Delta_\mu^- \Delta_\mu^+ \psi_x$$

to the naive fermion action.

(a) Compute the Fourier transform of the resulting lattice action and convince yourself that adding the Wilson term has reduced the number of poles from 16 to one.

(b) Show that adding the Wilson term worsens the discretization error of the fermion action from $\mathrm{O}(a^2)$ to $\mathrm{O}(a)$.

(c) Show that by a suitable rescaling of the fermion fields the Wilson-Dirac operator can be written as $D = \mathbb{1} - \kappa H$, and give explicit expressions for κ and H. Discuss applications of this formulation in the heavy-quark limit.

(d) Carefully read section 1.4.3.1 in the Les Houches Lecture Notes and section 6.5 in the book by Smit. Which similarities and differences with the derivation of the transfer matrix for the gluonic action are notable?

4. *Gaussian Integrals and Pseudofermions*

(a) Show that for symmetric positive definite $D = A^\dagger A$ the Grassmann integral

$$\det(D) = \int \prod_i \mathrm{d}\eta_i \mathrm{d}\bar{\eta}_i \, \mathrm{e}^{-\bar{\eta} D \eta}$$

can be replaced by an integral

$$\det(D) = \int \prod_i \frac{\mathrm{d}\chi_i \mathrm{d}\chi_i^*}{\pi} \, \mathrm{e}^{-(A^{-1}\chi)^\dagger (A^{-1}\chi)}$$

with complex variables χ (so-called pseudofermions).

(b) Explain why for independent random variables ξ_i following a standard normal distribution the variables $\chi_i = (A\xi)_i$ follow a Gaussian distribution $P(\chi) \propto \mathrm{e}^{-(A^{-1}\chi)^\dagger (A^{-1}\chi)}$.

(c) Which Monte Carlo method to estimate Grassmann integrals does this suggest?