Hadronic Light-by-Light scattering in the anomalous magnetic moment of the muon

Harvey Meyer
J. Gutenberg University Mainz

Tau Lepton Conference, Amsterdam, 27 September 2018

Cluster of Excellence

THE LOW-ENERGY FRONTIER OF THE STANDARD MODEL

European Research Council
Established by the European Commission

Status of $(g-2)_\mu$ as a test of the Standard Model

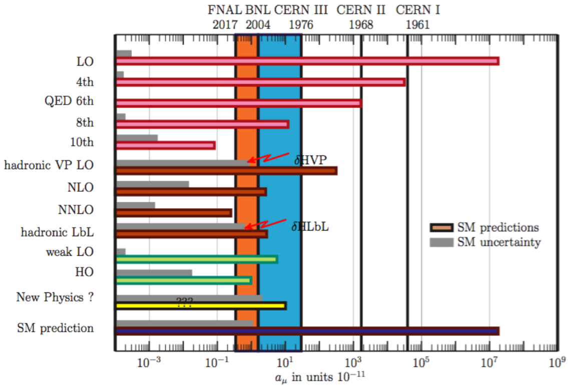
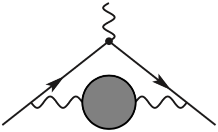
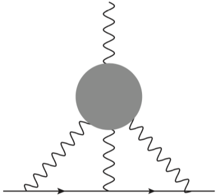


Fig. from Jegerlehner 1705.00263



Hadronic vacuum polarisation (HVP)



Hadronic light-by-light scattering (HLbL)

New experiments: $\times 4$ improvement in accuracy $\Longrightarrow$ theory effort needed:

- $a_\mu^{\rm exp} - a_\mu^{\rm SM} \approx 300 \cdot 10^{-11}$; $\delta a_\mu^{\rm exp,future} \approx 16 \cdot 10^{-11}$.
- HVP (=O(α^2)) target accuracy: $\lesssim 0.5\%$
- HLbL (=O(α^3)) target accuracy: $\lesssim 15\%$.

Approaches to $a_\mu^{\rm HLbL}$

1. **Model calculations:** (the only approach until 2014)
   - based on pole- and loop-contributions of hadron resonances

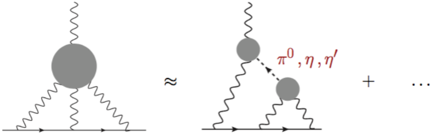
2. **Dispersive representation:** Bern approach; Mainz approach; Schwinger sum rule.
   - identify and compute individual contributions
   - determine/constrain the required input (transition form factors, $\gamma^*\gamma^* \to \pi\pi$ amplitudes, ...) dispersively

3. **Experimental program:** provide input for model & dispersive approach, e.g. $(\pi^0, \eta, \eta') \to \gamma\gamma^*$ at virtualities $Q^2 \lesssim 3\,$GeV^2; currently active program at BES-III see talk by Y. Guo

4. **Lattice calculations:**
   - RBC-UKQCD T. Blum, N. Christ, T. Izubuchi, L. Jin, Ch. Lehner, ...
   - Mainz N. Asmussen, A. Gérardin, J. Green, HM, A. Nyffeler, H. Wittig...

This talk: how do the findings from different approaches fit together?

Models for a_μ^{HLbL}



Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99 ± 16
axial vectors	2.5±1.0	1.7±1.7	–	22±5	–	15 ± 10	22 ± 5
scalars	−6.8±2.0	–	–	–	–	−7±7	−7±2
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
π, K loops +subl. N_C	–	–	–	0±10	–	–	–
quark loops	21±3	9.7±11.1	–	–	–	2.3 (c-quark)	21±3
Total	83±32	89.6±15.4	80±40	136±25	110±40	105 ± 26	116 ± 39

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = AN '09, JN = Jegerlehner, AN '09

Table from A. Nyffeler, PhiPsi 2017 conference

A recently updated estimate: NB. much smaller axial-vector contribution

$$a_\mu^{\mathrm{HLbL}} = (103 \pm 29) \times 10^{-11}$$ Jegerlehner 1809.07413

Wisdom gained from model calculations Prades, de Rafael, Vainshtein 0901.0306

- heavy (charm) quark loop makes a small contribution

$$a_\mu^{\rm HLbL} = (\frac{\alpha}{\pi})^3 \, N_c \mathcal{Q}_c^4 c_4 \frac{m_\mu^2}{m_c^2}, \qquad c_4 \approx 0.62.$$

- Light-quarks: (A) charged pion loop is negative & *quadratically* divergent:

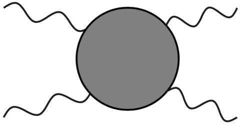
$$a_\mu^{\rm HLbL} \overset{m_\pi\to 0}{=\!=\!=} (\frac{\alpha}{\pi})^3 \, c_2 \frac{m_\mu^2}{m_\pi^2}, \qquad c_2 \approx -0.065.$$

(B) The neutral-pion exchange is positive, $\log^2(m_\pi^{-1})$ divergent:
Knecht, Nyffeler, Perrottet, de Rafael PRL88 (2002) 071802

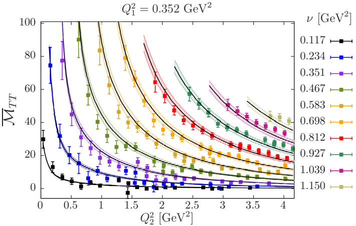
$$a_\mu^{\rm HLbL} = (\frac{\alpha}{\pi})^3 \, N_c \frac{m_\mu^2}{48\pi^2(F_\pi^2/N_c)} \left[\log^2 \frac{m_\rho}{m_\pi} + {\rm O}\Big(\log \frac{m_\rho}{m_\pi}\Big) + {\rm O}(1)\right].$$

- For real-world quark masses: using form factors for the mesons is essential, and resonances up to 1.5 GeV can still be relevant ⇒ medium-energy QCD.
- Two closeby vector currents $V_\mu(x)V_\nu(0) \overset{\rm OPE}{\sim} \epsilon_{\mu\nu\rho\sigma} \frac{x_\rho}{(x^2)^2} A_\sigma + \dots$ 'look like' an axial current from a distance: doubly-virtual transition form factors of 0^{-+} and 1^{++} mesons only fall like $1/Q^2$; but, coupling of axial-vector meson to two real photons forbidden by Yang-Landau theorem.

Test of ‘model wisdom’ via exact dispersive sum rules



$m_\pi = 330\,\text{MeV}$, $96 \cdot 48^3$, $a = 0.063\text{fm}$



$$\text{Lattice}: \ \Pi^E_{\mu_1\mu_2\mu_3\mu_4}(P_4; P_1, P_2) \equiv \int_{X_1,X_2,X_4} e^{-i\sum_a P_a\cdot X_a} \Big\langle J_{\mu_1}(X_1) J_{\mu_2}(X_2) J_{\mu_3}(0) J_{\mu_4}(X_4) \Big\rangle_{\text{E}}$$

$$\mathcal{M}_{\text{TT}}(-Q_1^2, -Q_2^2, -Q_1\cdot Q_2) = \frac{e^4}{4} \underbrace{R^E_{\mu_1\mu_3} R^E_{\mu_2\mu_4}}_{\text{projector}} \Pi^E_{\mu_1\mu_3\mu_4\mu_2}(-Q_2; -Q_1, Q_1),$$

Dispersive sum rule in $\nu = \frac{1}{2}(s + Q_1^2 + Q_2^2)$: [Pascalutsa, Pauk, Vanderhaeghen (2012)]

$$\mathcal{M}_{\text{TT}}(q_1^2, q_2^2, \nu) - \mathcal{M}_{\text{TT}}(q_1^2, q_2^2, 0) = \frac{2\nu^2}{\pi} \int_{\nu_0}^{\infty} d\nu' \, \frac{\sqrt{\nu'^2 - q_1^2 q_2^2}}{\nu'(\nu'^2 - \nu^2 - i\epsilon)} \underbrace{(\sigma_0 + \sigma_2)(\nu')}_{\sigma(\gamma^*\gamma^*\to\text{hadrons})},$$

J. Green et al. PRL115 222003 (2015); A. Gérardin et al. 1712.00421 (PRD).

Model for photon-photon fusion cross-section

Contribution of a narrow meson resonance to a $\gamma^*\gamma^* \to$ hadrons cross-section is

$$\propto \; \delta(s - M^2) \times \Gamma_{\gamma\gamma} \times \left[\frac{F_{M\gamma^*\gamma^*}(Q_1^2, Q_2^2)}{F_{M\gamma^*\gamma^*}(0,0)}\right]^2$$

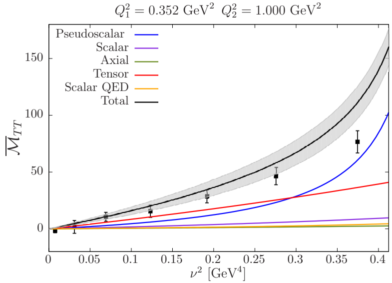
- $\pi^0 \to \gamma^*\gamma^*$ transition form factor $F_{\pi^0\gamma^*\gamma^*}$ determined in dedicated Lat.QCD calculation
- seven other TFFs were parametrized by $1/(1 + Q^2/M^2)^k$ $(k = 1, 2)$ and the parameters M fitted.

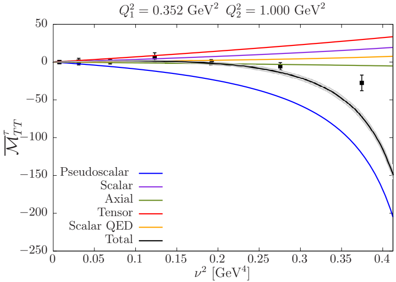
Fitting all eight $\gamma^*\gamma^* \to \gamma^*\gamma^*$ forward amplitudes:

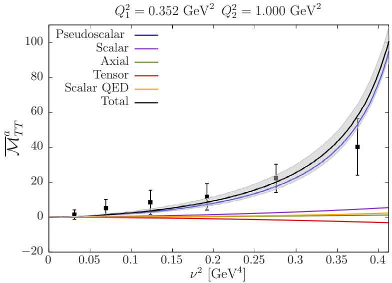
	M_{TT}	M^{τ}_{TT}	M^{a}_{TT}	M_{TL}	M_{LT}	M^{τ}_{TL}	M^{a}_{TL}	M_{LL}
Pseudoscalar	$\sigma_0/2$	$-\sigma_0$	$\sigma_0/2$	$\times$	$\times$	$\times$	$\times$	$\times$
Scalar	$\sigma_0/2$	σ_0	$\sigma_0/2$	$\times$	$\times$	τ_{TL}	τ_{TL}	σ_{LL}
Axial	$\sigma_0/2$	$-\sigma_0$	$\sigma_0/2$	σ_{TL}	σ_{LT}	τ_{TL}	$-\tau_{LT}$	$\times$
Tensor	$\frac{\sigma_0+\sigma_2}{2}$	σ_0	$\frac{\sigma_0-\sigma_2}{2}$	σ_{TL}	σ_{LT}	τ_{TL}	τ^{a}_{TL}	σ_{LL}
Scalar QED	σ_{TT}	τ_{TT}	τ^{a}_{TT}	σ_{TL}	σ_{LT}	τ_{TL}	τ^{a}_{TL}	σ_{LL}

Forward LbL amplitudes: contributions of individual mesons

$N_\mathrm{f} = 2$, $m_\pi = 193$ **MeV**, $128 \cdot 64^3$, $a = 0.063$ **fm, fully connected diagram, in units of** 10^{-6}







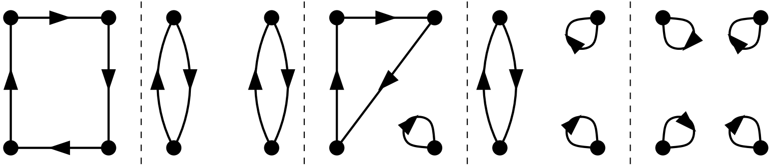
Conclusion:

narrow resonances + $\pi^+\pi^-$ model for $\sigma(\gamma^*\gamma^* \to \text{hadrons})$

provides reasonable description of

$\mathcal{M}_\text{forward}(\gamma^*\gamma^* \to \gamma^*\gamma^*)$ from Lat.QCD.

Quark-line contractions



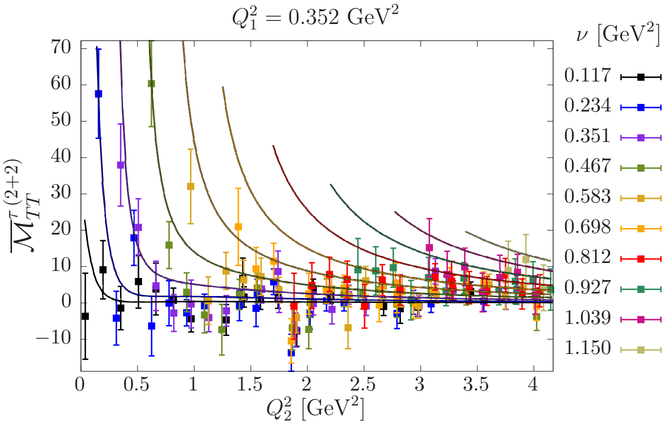
First two classes of diagrams thought to be dominant, with a cancellation between them:

	Weight factor of:	fully connected	(2,2) topology
SU(2)$_\mathrm{f}$:	isovector-meson exchange	$34/9 \approx 3.78$	$-25/9 \approx -2.78$
$m_s = \infty$	isoscalar-meson exchange	0	1
SU(3)$_\mathrm{f}$:	octet-meson exchange	3	-2
$m_s = m_{ud}$	singlet-meson exchange	0	1

Large-N_c argument by J. Bijnens, 1608.01454; $SU(3)_\mathrm{f}$ case in 1712.00421; Fig. by J. Green.

Contribution of (2+2) disconnected diagrams to $\gamma^*\gamma^* \to \gamma^*\gamma^*$

$N_f = 2$, $m_\pi = 193$ **MeV**, $128 \cdot 64^3$, $a = 0.063$ **fm, in units of** 10^{-6}



- large-N_c motivated prediction (no fit): ($\mathcal{M}_{\rm TT}^{\tau}$ determined by $\sigma_{||} - \sigma_\perp$)
  $\mathcal{M}_{\rm TT}^{\tau,(2,2)} = -\frac{25}{9}\mathcal{M}_{\rm TT}^{\tau,(2,2)\,\pi^0} + \mathcal{M}_{\rm TT}^{\tau,(2,2)\,\eta'}$
- agreement at $\sim 30\%$ level for $Q_i^2 \lesssim 1.2\text{GeV}^2$.

Dispersive methods: the Bern approach

Full HLbL tensor:

$$\Pi^{\mu\nu\lambda\sigma}(q_1,q_2,q_3) = i^3 \int_{x,y,z} e^{-i(q_1 x + q_2 y + q_3 z)} \langle 0|T\{j_x^\mu j_y^\nu j_z^\lambda j_0^\sigma\}|0\rangle = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i,$$

e.g. $T_1^{\mu\nu\lambda\sigma} = \epsilon^{\mu\nu\alpha\beta}\,\epsilon^{\lambda\sigma\gamma\delta}\,q_{1\alpha}\,q_{2\beta}\,q_{3\gamma}\,(q_1+q_2+q_3)_\delta$,
where the 54 structures are really **seven** combined with **crossing symmetry**.

Computing $(g-2)_\mu$ using the projection technique (directly at $q=0$):

$$a_\mu^{\rm HLbL} = -e^6 \int \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{\sum_{i=1}^{12}\hat{T}_i(q_1,q_2;p)\hat{\Pi}_i(q_1,q_2,-q_1-q_2)}{q_1^2 q_2^2 (q_1+q_2)^2[(p+q_1)^2-m_\mu^2][(p-q_2)^2-m_\mu^2]}$$

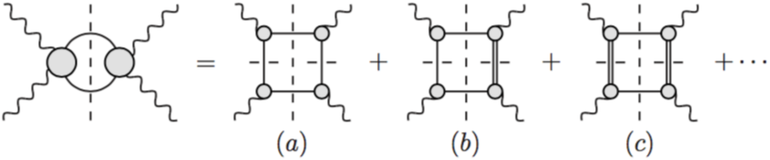
with $\hat{\Pi}_i$ linear combinations of the Π_i.

Performing all "kinematic" integrals using Gegenbauer-polynomial technique after Wick rotation:

$$a_\mu^{\rm HLbL} = \frac{2\alpha^3}{48\pi^2}\int_0^\infty dQ_1^4 \int_0^\infty dQ_2^4 \int_{-1}^1 d\tau \sqrt{1-\tau^2}\sum_{i=1}^{12} T_i(Q_1,Q_2,\tau)\bar{\Pi}_i(Q_1,Q_2,\tau)$$

Colangelo, Hoferichter, Procura, Stoffer (2015)

Dispersive methods (II)



- Charged-pion contributions: Colangelo et al. PRL118, 232001 (2017)

$$a_\mu^{\pi\,\text{box}} + a_{\mu,J=0}^{\pi\pi,\pi-\text{poleLHC}} = -24(1)\cdot 10^{-11}$$

- rescattering effects in $\pi^+\pi^-$ are being worked out for partial waves $\ell \le 2$; first results for the s-wave (presented by Colangelo at $(g-2)$ theory workshop 2018).

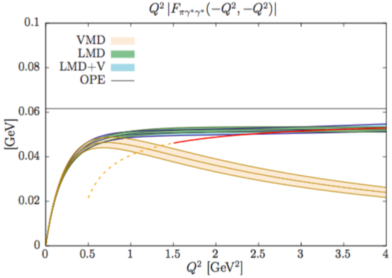
- Dispersive analysis of the $\pi^0 \to \gamma^*\gamma^*$ transition form factor leads to $a_\mu^{\pi^0} = 62.6^{+3.0}_{-2.5}\cdot 10^{-11}$ Kubis et al. PRL121, 112002 (2018)

- Analysis of $\gamma^*\gamma^* \to \pi\pi$
  Danilkin, Deineka & Vanderhaeghen, $(g-2)$ theory workshop, Mainz 2018

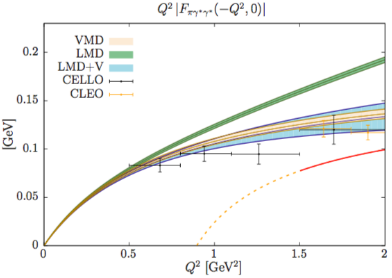
Lattice calculation of $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

$$M_{\mu\nu}(p, q_1) \equiv i \int \mathrm{d}^4x\, e^{iq_1 x}\, \langle\Omega|T\{j_\mu(x)j_\nu(0)\}|\pi^0(p)\rangle = \epsilon_{\mu\nu\alpha\beta}\, q_1^\alpha\, q_2^\beta\, \mathcal{F}_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2)\,,$$

Double-virtual



Single-virtual



Contribution to the $(g-2)_\mu$: using a conformal-mapping parametrization of $\mathcal{F}(Q_1^2, Q_2^2)$ in each virtuality, obtain

$$a_\mu^{\mathrm{HLbL}}|_{\pi^0} = (60.4 \pm 3.6)\cdot 10^{-11} \qquad \text{(preliminary)}.$$

Compatible (and competitive) with the dispersive result of Kubis et al.

Gérardin et al 1607.08174 (PRD); $(g-2)$ Theory workshop, Mainz 2018.

Direct lattice calculation of HLbL in $(g-2)_\mu$

At first, this was thought of as a QED+QCD calculation
[pioneered in Hayakawa et al., hep-lat/0509016].

Today's viewpoint: the calculation is considered a QCD four-point Green's function, to be integrated over with a weighting kernel which contains all the QED parts.

RBC-UKQCD: calculation of $a_\mu^{\rm HLbL}$ using coordinate-space method in muon rest-frame; photon+muon propagators:

- either on the $L \times L \times L$ torus (QED$_L$) (1510.07100–present)
- or in infinite volume (QED$_\infty$) (1705.01067–present).

T. Blum, N. Christ, T. Izubuchi, L. Jin, Ch. Lehner, . . .

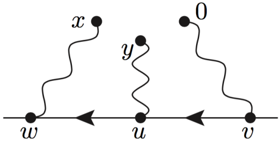
Mainz:

- manifestly covariant QED$_\infty$ coordinate-space approach, averaging over muon momentum using the Gegenbauer polynomial technique (1510.08384–present).

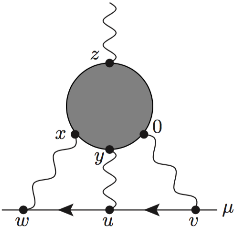
N. Asmussen, A. Gérardin, J. Green, HM, A. Nyffeler, . . .

Coordinate-space approach to $a_\mu^{\rm HLbL}$, Mainz version

QED kernel $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$



$\Rightarrow$



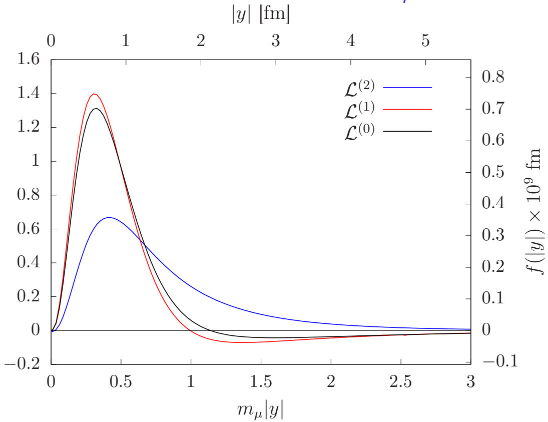
$$a_\mu^{\rm HLbL} = \frac{me^6}{3} \underbrace{\int d^4y}_{=2\pi^2|y|^3 d|y|} \Big[\int d^4x\, \underbrace{\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\rm QED}\; \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{\rm =QCD\ blob}\Big].$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z\; z_\rho \Big\langle\, j_\mu(x)\, j_\nu(y)\, j_\sigma(z)\, j_\lambda(0) \Big\rangle.$$

- $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ computed in the continuum & infinite-volume
- no power-law finite-volume effects & only a 1d integral to sample the integrand in $|y|$.

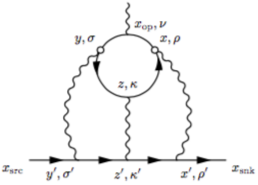
[Asmussen, Gérardin, Green, HM, Nyffeler 1510.08384, 1609.08454]

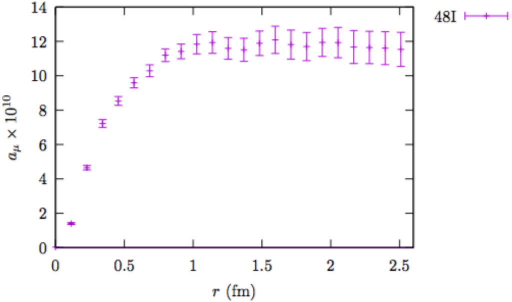
What to expect: contribution of the π^0 to a_μ^{HLbL} (physical pion mass)



- Even more freedom in choosing best lattice implementation than in HVP.
- The form of the $|y|$-integrand depends on the precise QED kernel used: can perform subtractions (Blum et al. 1705.01067; $\mathcal{L} \to \mathcal{L}^{(2)}$), impose Bose symmetries on $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ or add a longitudinal piece $\partial_\mu^{(x)} f_{\rho;\nu\lambda\sigma}(x,y)$.

RBC-UKQCD: quark-connected diagram using QED$_L$

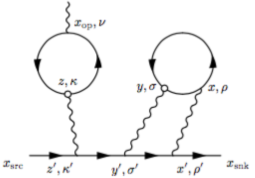




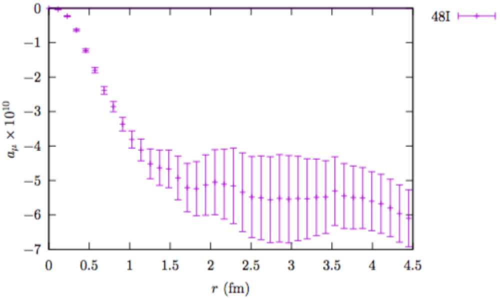
- $48^3 \times 96,\ m_\pi = 139\,\mathrm{MeV},\ a^{-1} = 1.73\,\mathrm{GeV},\ L = 5.47\mathrm{fm}$
- $a_\mu^{\mathrm{HLbL}}(\mathrm{connected}) = (116.0 \pm 9.6) \times 10^{-11}$

T. Blum et al, PRL118 (2017) no.2, 022005

RBC-UKQCD: (2,2)-disconnected diagram using QED$_L$



$M = 1024$ propagators/config
M^2 trick to reduce noise.



- $48^3 \times 96,\ m_\pi = 139\,\mathrm{MeV},\ a^{-1} = 1.73\,\mathrm{GeV},\ L = 5.47\mathrm{fm}$
- $a_\mu^{\rm HLbL}((2,2)) = (-62.5 \pm 8.0) \times 10^{-11}$
- together: $a_\mu^{\rm HLbL} = (53.5 \pm 13.5) \cdot 10^{-11}$

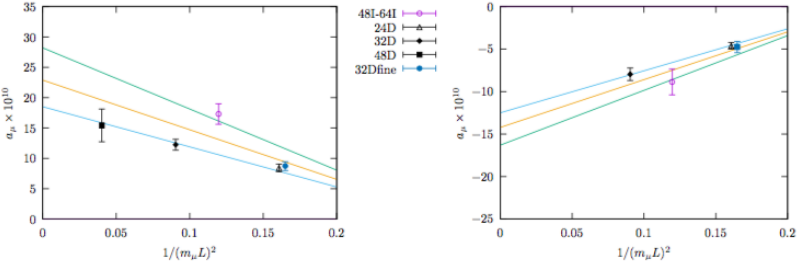
T. Blum et al, PRL118 (2017) no.2, 022005

Comments [1712.00421]:

- Total is about a factor 2 lower than model estimates.
- This method has O$(1/L^2)$ finite-size effects. Or model missing something?
- Based on the model and large-N_c-based argument, one would expect $a_\mu^{\rm HLbL}((2,2)) \approx -150 \cdot 10^{-11}$, dominated by (π^0, η, η') exchange.

Update by RBC-UKQCD: continuum, infinite-volume extrapol.

$$F_2(a, L) = F_2\left(1 - \frac{c_1}{(m_\mu L)^2}\right)(1 - c_2 a^2)$$



Connected diagrams

Disconnected diagrams

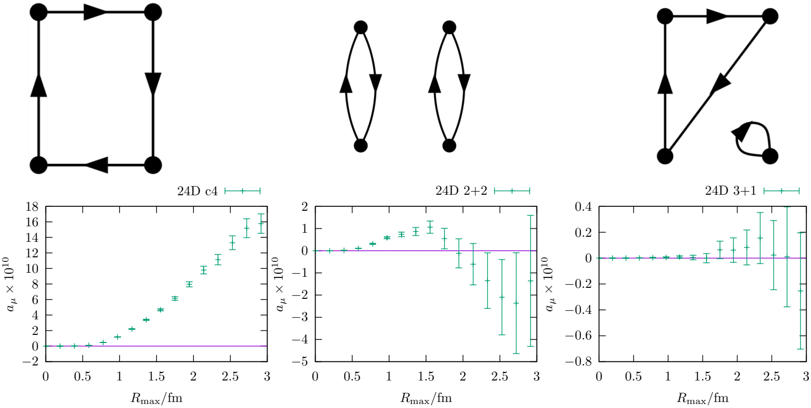
$$a_\mu^{\rm cHLbL} = (28.2 \pm 4.0) \times 10^{-10} \qquad a_\mu^{\rm dHLbL} = (-16.3 \pm 3.4) \times 10^{-10}$$

$$a_\mu^{\rm HLbL} = (11.9 \pm 5.3) \times 10^{-10}$$

L. Jin @ Lattice 2018

My comment: the central value is much more in line with model expectation; uncertainty still large.

RBC-UKQCD first results for (3,1) diagram topology

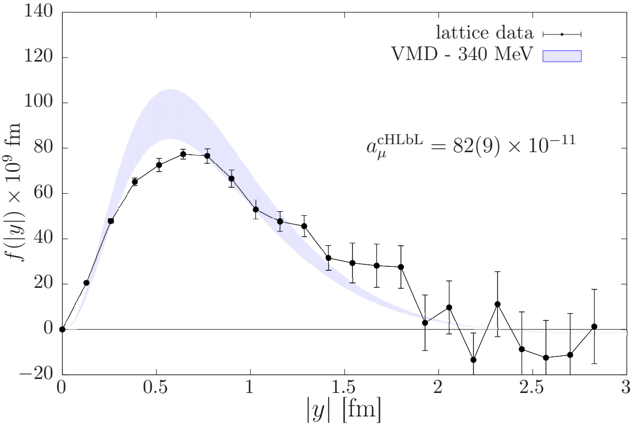


$24^3 \times 64,\ m_\pi = 141\,\mathrm{MeV},\ a^{-1} = 1.015\,\mathrm{GeV}$

- calculation on coarse lattice strongly suggests the $(3,1)$ topology is negligible.

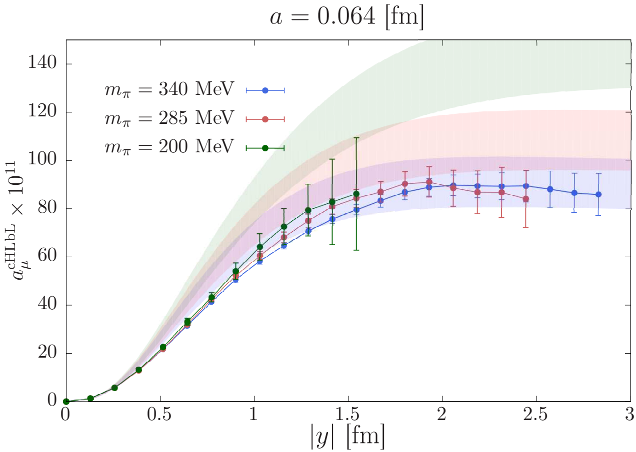
L. Jin, private communication

Mainz: integrand of a_μ^{cHLbL} with $\mathcal{L}^{(2)}$, $m_\pi = 340\,\text{MeV}$, $a = 0.064\,\text{fm}$, $96 \cdot 48^3$



- fully connected diagram only
- the π^0 exchange with VMD form factor provides a decent approximation to the full QCD computation.

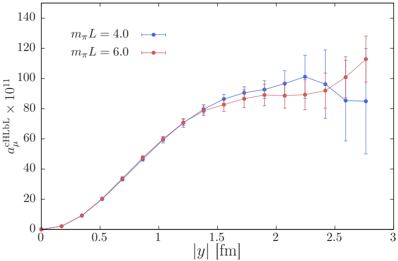
Mainz: pion mass dependence of a_μ^{cHLbL}



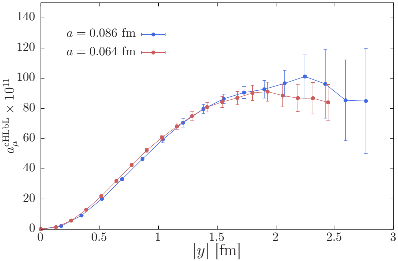
- bands = π^0 contributions (band-width is difference between factor 3 and 34/9)
- upward trend for decreasing pion mass? needs more statistics.

Mainz: investigating systematic effects at $m_\pi = 285\,\mathrm{MeV}$

Check of finite-size effects



Check of discretization effects



- finite size and discretisation effects appear to be under control.

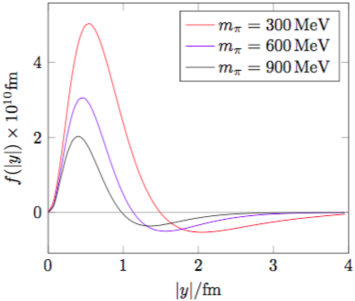
Mainz, A. Gérardin et al.

Conclusion

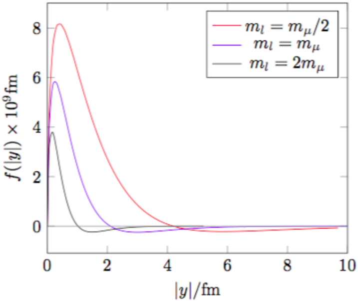
- Model approach to hadronic light-by-light scattering in $(g-2)_\mu$ is gradually getting superseded by lattice and dispersive approach.
- Significant progress in the Bern dispersive framework.
- Lattice QCD now has a well-established method to handle $a_\mu^{\rm HLbL}$.
- So far, lattice results (the Mainz forward scattering amplitudes and RBC-UKQCD $a_\mu^{\rm HLbL}$ results extrapolated to infinite volume) are in line with model expectations.
- Could $a_\mu^{\rm HLbL}$ explain the tension between the SM prediction and the experimental value of a_μ? It does not look like it, but the effort to reduce uncertainties is worthwhile.

Backup Slides

Continuum tests: contribution of the π^0 and lepton loop to a_μ^{HLbL}



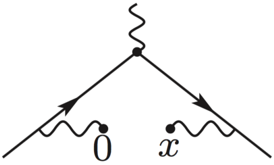
Integrand of the pion-pole contribution with VMD transition form factor.



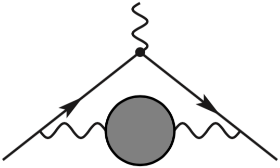
Integrand of the lepton-loop contribution.

- Even more freedom in choosing best lattice implementation than in HVP.
- The form of the $|y|$-integrand depends on the precise QED kernel used: can perform subtractions (Blum et al. 1705.01067; $\mathcal{L} \to \mathcal{L}^{(2)}$), impose Bose symmetries on $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ or add a longitudinal piece $\partial_\mu^{(x)} f_{\rho;\nu\lambda\sigma}(x,y)$.

Hadronic vacuum polarization in x-space HM 1706.01139



QED kernel $H_{\mu\nu}(x)$



$a_\mu^{\rm hvp}$

$$a_\mu^{\rm hvp} = \int d^4x\; H_{\mu\nu}(x)\; \Big\langle j_\mu(x) j_\nu(0) \Big\rangle_{\rm QCD},$$

$$j_\mu = \frac{2}{3}\bar u\gamma_\mu u - \frac{1}{3}\bar d\gamma_\mu d - \frac{1}{3}\bar s\gamma_\mu s + \dots; \qquad H_{\mu\nu}(x) = -\delta_{\mu\nu}\mathcal{H}_1(|x|) + \frac{x_\mu x_\nu}{x^2}\,\mathcal{H}_2(|x|)$$

a transverse tensor known analytically in terms of Meijer's functions, $\mathcal{H}_i(|x|) = \frac{8\alpha^2}{3m_\mu^2} f_i(m_\mu |x|)$ and

$$f_2(z) = \frac{G^{2,2}_{2,4}\left(z^2 \,\middle|\, \begin{matrix} \frac{7}{2}, 4 \\ 4, 5, 1, 1 \end{matrix}\right) - G^{2,2}_{2,4}\left(z^2 \,\middle|\, \begin{matrix} \frac{7}{2}, 4 \\ 4, 5, 0, 2 \end{matrix}\right)}{8\sqrt{\pi} z^4},$$

$$f_1(z) = f_2(z) - \frac{3}{16\sqrt{\pi}} \cdot \left[G^{2,3}_{3,5}\left(z^2 \,\middle|\, \begin{matrix} 1, \frac{3}{2}, 2 \\ 2, 3, -2, 0, 0 \end{matrix}\right) - G^{2,3}_{3,5}\left(z^2 \,\middle|\, \begin{matrix} 1, \frac{3}{2}, 2 \\ 2, 3, -1, -1, 0 \end{matrix}\right) \right].$$

Explicit form of the QED kernel

$$\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) = \sum_{A=\mathrm{I,II,III}} \mathcal{G}^{A}_{\delta[\rho\sigma]\mu\alpha\nu\beta\lambda} T^{(A)}_{\alpha\beta\delta}(x,y),$$

with e.g.

$$\mathcal{G}^{\mathrm{I}}_{\delta[\rho\sigma]\mu\alpha\nu\beta\lambda} \equiv \frac{1}{8}\mathrm{Tr}\Big\{\Big(\gamma_\delta[\gamma_\rho,\gamma_\sigma] + 2(\delta_{\delta\sigma}\gamma_\rho - \delta_{\delta\rho}\gamma_\sigma)\Big)\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\lambda\Big\},$$

$$\begin{aligned}
T^{(\mathrm{I})}_{\alpha\beta\delta}(x,y) &= \partial^{(x)}_\alpha(\partial^{(x)}_\beta + \partial^{(y)}_\beta)V_\delta(x,y),\\
T^{(\mathrm{II})}_{\alpha\beta\delta}(x,y) &= m\partial^{(x)}_\alpha\Big(T_{\beta\delta}(x,y) + \frac{1}{4}\delta_{\beta\delta}S(x,y)\Big)\\
T^{(\mathrm{III})}_{\alpha\beta\delta}(x,y) &= m(\partial^{(x)}_\beta + \partial^{(y)}_\beta)\Big(T_{\alpha\delta}(x,y) + \frac{1}{4}\delta_{\alpha\delta}S(x,y)\Big),
\end{aligned}$$

$$\begin{aligned}
S(x,y) &= \int_u G_{m_\gamma}(u-y)\Big\langle J(\hat\epsilon,u)J(\hat\epsilon,x-u)\Big\rangle_{\hat\epsilon}, \quad J(\hat\epsilon,y) \equiv \int_u G_0(y-u)\, e^{m\hat\epsilon\cdot u} G_m(u)\\
V_\delta(x,y) &= x_\delta \bar{\mathfrak{g}}^{(1)}(|x|,\hat x\cdot\hat y,|y|) + y_\delta \bar{\mathfrak{g}}^{(2)}(|x|,\hat x\cdot\hat y,|y|),\\
T_{\alpha\beta}(x,y) &= (x_\alpha x_\beta - \frac{x^2}{4}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(1)} + (y_\alpha y_\beta - \frac{y^2}{4}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(2)} + (x_\alpha y_\beta + y_\alpha x_\beta - \frac{x\cdot y}{2}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(3)}.
\end{aligned}$$

The QED kernel $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ is parametrized by six weight functions.

$(g-2)_\mu$: a reminder

$$\boldsymbol{\mu} = g\,\mu_B \boldsymbol{s}, \qquad \mu_B = \frac{e}{2m_\mu}$$

- $g = 2$ in Dirac's theory
- $a_\mu \equiv (g-2)/2 = F_2(0) = \frac{\alpha}{2\pi}$ (Schwinger 1948)
- direct measurement (BNL): $a_\mu = (11659208.9 \pm 6.3) \cdot 10^{-10}$
- Standard Model prediction $a_\mu = (11659182.8 \pm 4.9) \cdot 10^{-10}$.
- $a_\mu^{\rm exp} - a_\mu^{\rm th} = (26.1 \pm 8.0) \cdot 10^{-10}$.

Numbers from 1105.3149 Hagiwara et al.