Probing the photon emission rate of quark-gluon plasma in lattice QCD

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Cluster of Excellence

Outline

Part 1: numerics

- Probing the photon rate: a calculation in two-flavor lattice QCD.

Part 2: theory development

- Dispersion relation of momentum-space Euclidean correlators at fixed, vanishing photon virtuality.

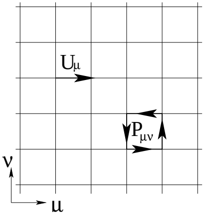
List of coauthors & References

- Bastian B. Brandt, Marco Cè, Anthony Francis, Tim Harris, HM, Aman Steinberg, 1710.07050 (LAT2017)

Lattice papers on the photon rate:

- Karsch, Laermann, Petreczky, Stickan, Wetzorke 2002; S. Gupta 2004; Aarts, Allton, Foley, Hands, Kim 2007: quenched calculations, $k = 0$.
- hep-lat/0610061 (LAT06): Aarts, Allton, Foley, Hands: quenched, $k \neq 0$
- 1012.4963: Ding, Francis, Kaczmarek, Karsch, Laermann, Soeldner, quenched calculation with continuum limit, $k = 0$.
- 1212.4200 (JHEP): Brandt, Francis, HM, Wittig: $N_f = 2$, $N_t = 16$, $k = 0$, $m_\pi = 270$, $T = 250$MeV.
- 1307.6763 (PRL), 1412.6411 (JHEP): Aarts, Allton, Amato,Giudice, Hands, Skullerud: $N_f = 2 + 1$, $k = 0$, anisotropic, fixed-scale temperature scan, $m_\pi = 384$ MeV
- 1512.07249 (PRD): Brandt, Francis, Jäger, HM, $N_f = 2$, $k = 0$, $N_t = 12 \to 24$, $m_\pi = 270$, fixed-scale scan across the phase transition.
- 1604.07544 (PRD): Ghiglieri, Kaczmarek, Laine, F. Meyer: quenched calculation with continuum limit, $k \neq 0$.
- here: $N_f = 2$ calculation with continuum limit at $T = 250$MeV, $k \neq 0$.

Regularization of QCD on a lattice



Gluons: $U_\mu(x) = e^{iag_0 A_\mu(x)} \in SU(3)$
'link variables'

Quarks: $\psi(x)$ 'on site', Grassmann

Gauge-invariance exactly preserved; no gauge-fixing required.

Imaginary-time path-integral representation of QFT (Matsubara formalism):

- Starting point for **Monte-Carlo simulations** using importance sampling.
- Representation of the Euclidean correlator as a Fourier series:

$$G_E^{AB}(x) = T \sum_{\ell \in \mathbb{Z}} e^{-i\omega_\ell x_0}\ \tilde{G}_E^{AB}(\omega_\ell, \boldsymbol{x}), \qquad \omega_\ell = 2\pi \ell T.$$

Definitions

Euclidean-time vector correlators ($\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} = 2\mathrm{diag}(1,-1,-1,-1)$),

$$G^{\mu\nu}(x_0,\boldsymbol{k}) = \int d^3x\; e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \Big\langle j^\mu(x)\, j^\nu(y) \Big\rangle, \qquad j^\mu = \sum_f Q_f\, \bar\psi_f \gamma^\mu \psi_f$$

- all diagonal components of $G^{\mu\nu}$ are positive; spectral representation:

$$G^{\mu\nu}(x_0,\boldsymbol{k}) \overset{\mu=\nu}{=} \int_0^\infty \frac{d\omega}{2\pi}\, \rho^{\mu\nu}(\omega,\boldsymbol{k})\, \frac{\cosh[\omega(\beta/2-x_0)]}{\sinh(\beta\omega/2)}.$$

- $\rho^{\mu=\nu}(\omega,\boldsymbol{k})/\omega$ is even in ω and non-negative.
- from current conservation: $\omega^2\rho^{00}(\omega,k) = k^i k^j \rho^{ij}(\omega,k)$.
- consequence: $(\hat k^i \hat k^j \rho^{ij} - \rho^{00})/\omega$ has the same sign as $\mathcal{K}^2 \equiv \omega^2 - k^2$, and vanishes at $\omega = k$.
- consider the linear combination

$$\begin{aligned} \rho(\omega,k,\lambda) &= (\delta^{ij} - \hat k^i \hat k^j)\rho^{ij} + \lambda\,(\hat k^i \hat k^j \rho^{ij} - \rho^{00}) \qquad k \equiv |\boldsymbol{k}|, \quad \hat k^i = k^i/k, \\ \text{e.g. } \rho(\omega,k,1) &= \rho^{ii} - \rho^{00} = -\rho^\mu{}_\mu(\omega,k) \quad \rightsquigarrow \quad \text{relevant for the dilepton rate.} \end{aligned}$$

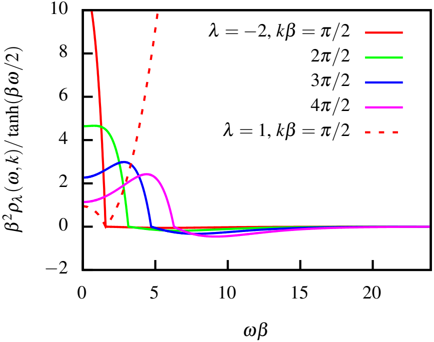
- The differential photon rate per unit volume of plasma:

$$d\Gamma_\lambda(\boldsymbol{k}) = e^2\, \frac{d^3k}{(2\pi)^3\, 2k}\, \frac{\rho(k,k,\lambda)}{e^{\beta k}-1} \quad \text{is independent of } \lambda.$$

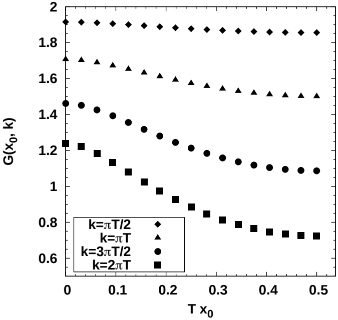
Non-interacting fermions

$$\rho(\omega,k,\lambda) = \begin{cases} -\rho^{\mu}{}_{\mu}(\omega,k) = 2\rho_T + \rho_L & \lambda = 1 \\ (\delta^{ij} - 3\hat{k}^i\hat{k}^j)\rho^{ij} + 2\rho^{00} = 2(\rho_T - \rho_L) & \lambda = -2. \end{cases}$$

Spectral function



Euclidean correlator with $\lambda = -2$

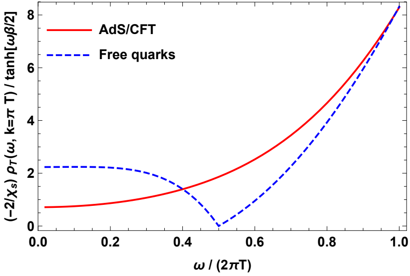


- We choose $\lambda = -2$ from now on: UV-finite correlator even at $x_0 = 0$.
- for $k = \mathrm{O}(\pi T)$, $\rho(k,k,\lambda) = \mathrm{O}(\alpha_s \log \alpha_s)$ in perturbation theory.

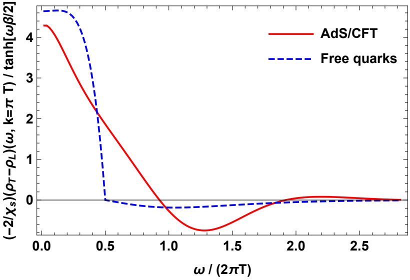
Qualitative form of the spectral functions: weak and strong coupling

Spatial momentum $k = \pi T$:

ρ_T



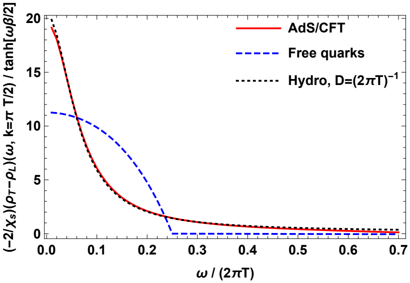
$\rho_T - \rho_L$



Spatial momentum $k = \pi T/2$:
At strong coupling, hydro works:

$$-2(\rho_T - \rho_L)(\omega, k)/\omega \approx \frac{4\chi_s\, Dk^2}{\omega^2 + (Dk^2)^2},$$

Refs: hep-th/0607237 and 1310.0164.



A sum rule for $\rho \equiv \rho_{\lambda=-2}$

i. Lorentz invariance and transversity $\Rightarrow \tilde{G}_{\rm E}(\omega_n, k) = 0$ in vacuum and UV finite at $T > 0$

ii. OPE: from power-counting one expects $\tilde{G}_{\rm E}(\omega_n, k) \sim \frac{\langle \mathcal{O}_4 \rangle}{\omega_n^2}$

Furthermore, charge conservation demands $\tilde{G}_{\rm E}(\omega_n, k) \to 0$ as $k \to 0$ and $\omega_n \neq 0$, so actually

$$\tilde{G}_{\rm E}(\omega_n, k) \sim \frac{k^2 \langle \mathcal{O}_4 \rangle}{\omega_n^4}$$

iii. From the dispersive representation:

$$\tilde{G}_{\rm E}(\omega_n, k) = \int_0^\infty \frac{{\rm d}\omega}{\pi} \omega \frac{\rho(\omega, k)}{\omega^2 + \omega_n^2} \stackrel{\omega_n \to \infty}{\longrightarrow} \frac{1}{\pi \omega_n^2} \int_0^\infty {\rm d}\omega\, \omega\, \rho(\omega, {\rm k})$$

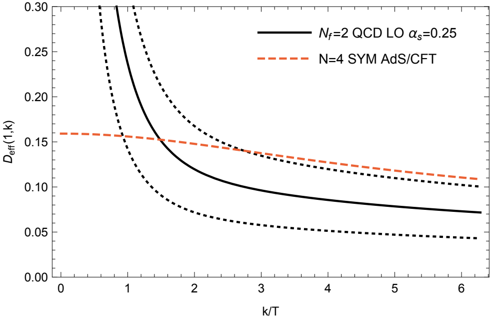
The two expressions are only compatible if the super-convergent sum rule

$$\boxed{\int_0^\infty {\rm d}\omega\, \omega \rho(\omega, {\rm k}) = 0}$$

holds.

Summary: properties of $\rho(\omega, k) \equiv \rho(\omega, k, -2)$

- non-negative for $\omega \leq k$
- $\rho(\omega, k) \overset{\omega\to\infty}{\sim} k^2/\omega^4$
- sum rule: $\int_0^\infty d\omega\, \omega \rho(\omega, \mathrm{k}) = 0$ (so $\rho(\omega, k)$ must go negative somewhere for $\omega > k$)
- define $D_{\rm eff}(\xi, k) \equiv \frac{\xi\rho(\xi k, k)}{4\chi_s k}$ which tends to D in the limit $k \to 0$ at fixed $\xi = \omega/k$ (inspired by Ghilghieri, Kaczmarek, Laine, F. Meyer 1604.07544).
- $D_{\rm eff}(1, k) \propto$ photon rate.

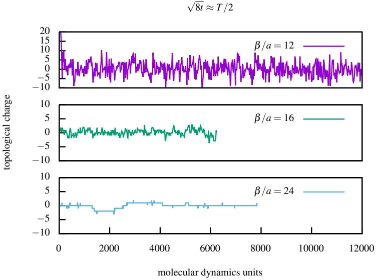


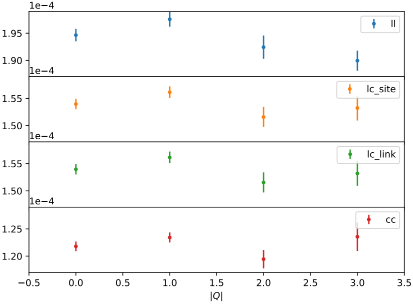
Results from Arnold, Moore, Yaffe hep-ph/0111107 (JHEP); AdS/CFT from hep-ph/0607237.

Lattice set-up with $N_f = 2$ $O(a)$-improved Wilson fermions

T (MeV)	T/T_c	β_{LAT}	β/a	L/a	$m_{\overline{MS}(2\,GeV)}$ (MeV)	N_{meas}
250	1.2	5.3	12	48	12	8256
"	"	5.5	16	64	"	4880
"	"	5.83	24	96	"	9600
500	2.4	6.04	16	64	"	8064

- enables continuum limit at $T = 250$ MeV





- only weak dependence of observable on topological charge
- impact of long autocorrelation time on vector correlator under control.

Continuum limit 1/3

There are four independent discretizations of the $\lambda = -2$ isovector vector correlator

$$G^{\lambda=-2}(\tau, \boldsymbol{k}) = \left(\delta^{ij} - \frac{3k^i k^j}{k^2}\right) G^{ij}(\tau, \boldsymbol{k}) + 2G^{00}(\tau, \boldsymbol{k})$$

where $G^{\mu\nu}(\tau, \boldsymbol{k}) = \int \mathrm{d^3 x e^{-ikx}} \langle \mathrm{J}^\mu(\tau, \mathrm{x}) \mathrm{J}^\nu(0) \rangle$ using both the local or exactly-conserved lattice vector current

In the local-conserved case, there are two discretizations possible by defining the local current on the link, or the conserved current on the site

$$G^{ij}(\tau + a/2, \boldsymbol{k}) = \frac{1}{2}\Big(G^{ij}(\tau, \boldsymbol{k}) + G^{ij}(\tau + a, \boldsymbol{k})\Big)$$

$$G^{00}(\tau, \boldsymbol{k}) = \frac{1}{2}\left(G^{00}(\tau - a/2, \boldsymbol{k}) + G^{00}(\tau + a/2, \boldsymbol{k})\right)$$

Project to all spatial momenta, on and off-axis, with $k\beta \leq 2\pi$

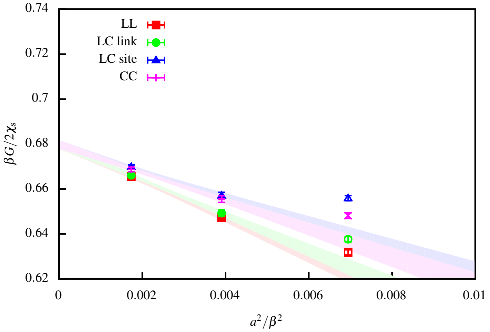
Continuum limit 2/3

In the chirally-symmetric phase, the matrix-elements of the $\mathrm{O}(a)$-improvement counterterms are suppressed, so we perform a continuum limit in a^2/β^2

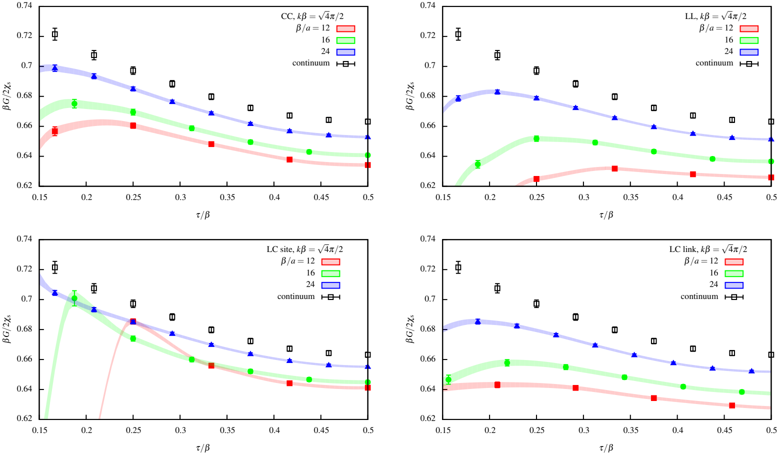
Instead we perform tree-level improvement, defined via

$$G^{\lambda=-2}(\tau,\boldsymbol{k}) \to \frac{G^{\lambda=-2}_{\rm cont.t.l.}(\tau,\boldsymbol{k})}{G^{\lambda=-2}_{\rm lat.t.l}(\tau,\boldsymbol{k})} G^{\lambda=-2}(\tau,\boldsymbol{k})$$

A piecewise spline interpolation is used before taking the combined continuum limit of the four discretizations of $\beta G^{\lambda=-2}(\tau,\boldsymbol{k})/\chi_{\rm s}$. For $x_0 = \beta/3$:



Continuum limit 3/3 using tree-level improved at $k = \pi T$



- Coarsest ensemble $N_t = 12$ is not included in the continuum extrapolation.
- In the subsequent analysis, we use the continuum-extrapolated correlator at $x_0 \geq \beta/4$.

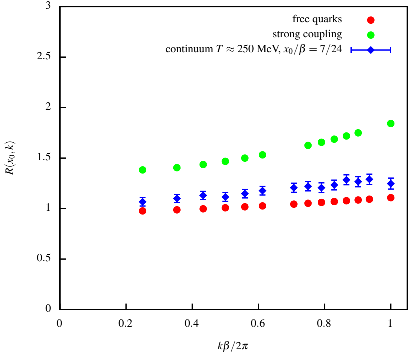
Can the lattice distinguish a weak- from a strong-coupling $\rho(\omega, k)$?

In the "transverse minus longitudinal" channel, consider the ratio

$$
\begin{aligned}
R(x_0, k) &\equiv \frac{16\pi}{(\beta - 2x_0)^2 k^2} \left[\frac{G(x_0, k)}{G(\beta/2, k)} - 1 \right] \\
&= \frac{16\pi}{(\beta - 2x_0)^2 k^2} \frac{\int_0^\infty d\omega\ \rho(\omega, k)(\cosh[\omega(\beta/2 - x_0)] - 1)/\sinh(\omega\beta/2)}{\int_0^\infty d\omega\ \rho(\omega, k)/\sinh(\omega\beta/2)}.
\end{aligned}
$$

This observable differs by a factor ~ 1.5 between the extreme cases of AdS/CFT and non-interacting quarks.

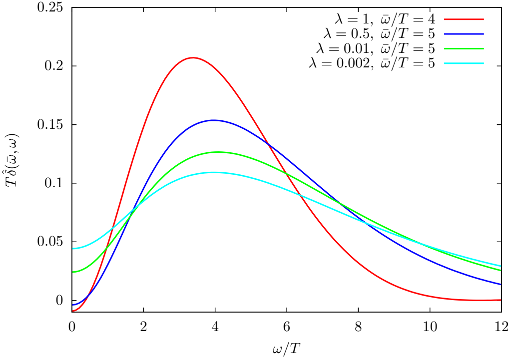
The continuum-extrapolated data lies between AdS/CFT and the free-quark prediction. Can the $O(\alpha_s)$ corrections account for the lattice data?



Gross features of the spectral function: the Backus-Gilbert method

$$\text{Linearity:} \quad \sum_{i=1}^{n} c_i(\bar\omega)\, G(t_i) = \int_0^\infty \frac{d\omega}{2\pi}\, \rho(\omega) \underbrace{\sum_{i=1}^{n} c_i(\bar\omega)\, \frac{\cosh[\omega(\beta/2 - t_i)]}{\sinh[\omega\beta/2]}}_{\widehat\delta(\bar\omega,\omega)}$$

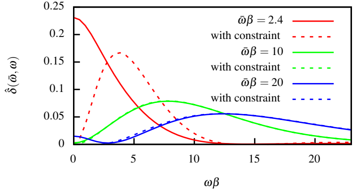
- choose the coefficients $c_i(\bar\omega)$ so that the 'resolution function' $\widehat\delta(\bar\omega,\omega)$ is as narrowly peaked around a given frequency $\bar\omega$ as possible
  (idea behind the Backus-Gilbert method, [used in Robaina et al. PRD 92 (2015) 094510.])



Resolution function at $\bar\omega = 4T$ for $N_t = 24$, $t_i/a = 5, \ldots 12$.

- Resolution only improves slowly with increasing n
- Large, sign-alternating coefficients $\Rightarrow$ need for ultra-precise input data.

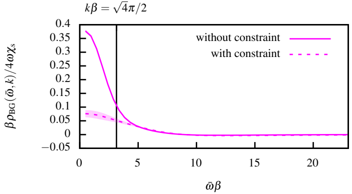
Backus-Gilbert method 2/2



← resolution function $\hat{\delta}(\bar{\omega},\omega)$

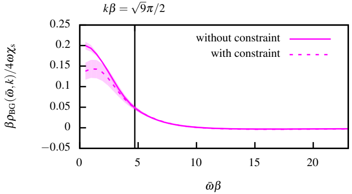
acts like a smearing kernel

a linear constraint $\hat{\delta}(\bar{\omega}=0,\omega)=0$ removes contributions from $\rho(\omega=0,k)$



← spectral function $\rho_{\rm BG}(\bar{\omega},k)$

The general morphology of the spectral function is confirmed: UV contribution is very small.



Padé fit ansatz for the spectral function

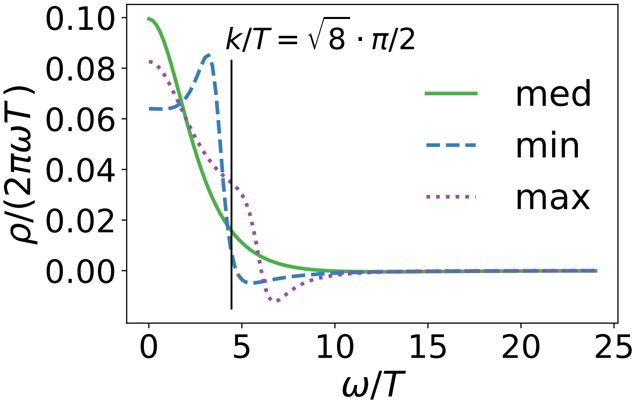
$$\frac{\rho(\omega,k)}{\tanh[\omega\beta/2]} = \frac{A(1+B\omega^2)}{[(\omega-\omega_0)^2+b^2][(\omega+\omega_0)^2+b^2][\omega^2+a^2]},$$

- $\rho(\omega,k) \sim 1/\omega^4$ at large ω (consistent with OPE);
- sum rule $\Rightarrow B = B(a,b,\omega_0)$;
- four-parameter fit (one linear, three non-linear);
- at small k, expect $a = Dk^2$ and $(\omega_0, b) = \mathrm{O}(T)$;
- it turns out that the χ^2 has a flat valley $\Rightarrow$ scan in the non-linear parameters (a, b, ω_0).
- accept all solutions that satisfy:
    1. $\rho(\omega,k) \geq 0$ for $\omega \leq k$;
    2. $\chi^2/\mathrm{d.o.f.} \leq 1$ (keeping only diagonal part of covariance matrix)
    3. "there can be no arbitrarily long relaxation times": $\min(a,b) > \min(D_{\mathrm{AdS/CFT}}k^2, D_{\mathrm{pert}}^{-1})$

$$D_{\mathrm{AdS/CFT}} = \frac{1}{2\pi T}, \qquad D_{\mathrm{pert}}^{-1} = \mathrm{O}(\alpha_s^2)T = 0.46T, \qquad \alpha_s = 0.25.$$

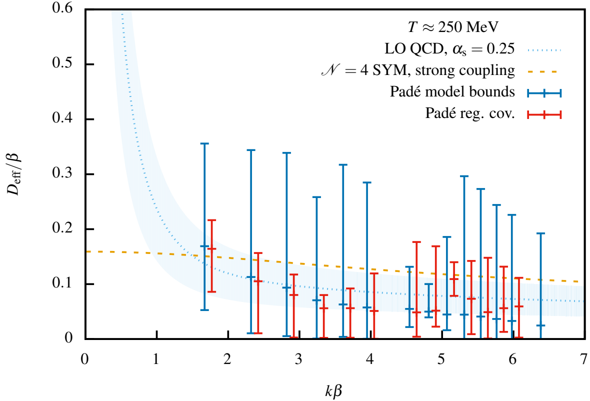
↑ Arnold, Moore, Yaffe hep-ph/0302165

Typical spectral functions resulting from the Padé fit



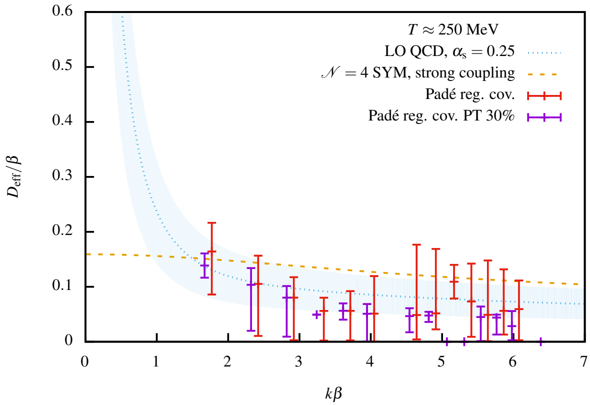
- All three describe the lattice data, fullfill the positivity requirement and do not have singularities too close to the real axis.

Result at $T = 250\,$MeV



- Final result: red points. Used covariance matrix C with off-diagonal elements multiplied by $x = 0.80$ (regularization).
- Regularization is necessary not because C^{-1} is poorly determined, but because the most accurate modes still suffer from cutoff effects.
- Influence of using the information in the covariance matrix is strong: only keeping the diagonal elements of the covariance matrix leads to much looser constraints on $D_{\rm eff}(k)$ (blue intervals).

Result assuming additional perturbative behavior at small x_0



- assuming $\frac{\partial^2}{\partial x_0^2} G(x_0 = 0, k)$ is given by the leading perturbative prediction within 30%, the intervals can be reduced substantially.
- Incentive to calculate O(α_s) corrections or go to finer lattice spacings.

A dispersion relation for a Euclidean correlator at zero virtuality

- Let $\sigma(\omega) \equiv \rho_T(\omega, |\boldsymbol{k}| = \omega)$ be the relevant spectral function proportional to the photon emission rate;
- let $H_E(\omega_n) \equiv G_E(\omega_n, k = i\omega_n)$ the momentum-space Euclidean correlator with Matsubara frequency ω_n and *imaginary* spatial momentum $k = i\omega_n$;
- once-subtracted dispersion relation: $(\sigma(\omega) \sim \omega^{1/2}$ at weak coupling)

$$H_E(\omega_n) - H_E(\omega_r) = \int_0^\infty \frac{d\omega}{\pi}\, \omega\, \sigma(\omega) \Big[\frac{1}{\omega^2 + \omega_n^2} - \frac{1}{\omega^2 + \omega_r^2}\Big].$$

Representation through non-static screening masses

$$\tilde{G}_E(\omega_r, x_3) = -2\int_0^\beta dx_0\; e^{i\omega_r x_0} \int dx_1 dx_2\; \langle J_1(x) J_1(0)\rangle = \sum_n |A_n^{(r)}|^2 e^{-E_n^{(r)}|x_3|}$$

$$\Rightarrow \underbrace{H_E(\omega_r)}_{=\mathrm{O}(g^2)} \equiv \int_{-\infty}^{\infty} dx_3\; \tilde{G}_E(\omega_r, x_3)\; e^{\omega_r x_3} = 2\omega_r^2 \sum_{n=0}^{\infty} \underbrace{|A_n|^2}_{=\mathrm{O}(g^4)} \underbrace{\frac{1}{E_n^{(r)}\,(E_n^{(r)2} - \omega_r^2)}}_{=\mathrm{O}(g^{-2})}.$$

This helps explain the connection observed in [Brandt et al, 1404.2404] between non-static screening masses and the LPM-resummation contributions to the photon emission rate [Aurenche et al, hep-ph/0211036].

In lattice regularization, Lorentz symmetry is absent $\Rightarrow H_E(\omega_r)$ does not vanish in vacuum as it does in the continuum. Explicitly subtracting the *in vacuo* $H_E(\omega_r)$ from the thermal $H_E(\omega_r)$ may be necessary.

Sketch of the (standard) derivation of the dispersion relation

$G_R(\omega,k) = i(\delta_{il} - \frac{k_i k_l}{k^2}) \int d^4x\, e^{i\mathcal{K}\cdot x}\theta(x^0) \left\langle [\mathrm{j}^i(x), \mathrm{j}^l(0)] \right\rangle$. But

$$[j^\mu(x), j^\nu(0)] = 0 \quad \text{for} \quad x^2 < 0,$$

$\Rightarrow$ the retarded correlator $H_R(\omega) \equiv G_R(\omega, k=\omega)$ at lightlike momentum is analytic for $\mathrm{Im}\,(\omega) > 0$. Similarly, the advanced correlator $H_A(\omega)$ is analytic for $\mathrm{Im}\,(\omega) < 0$.

Define the function $H(\omega) = \begin{cases} H_R(\omega) & \mathrm{Im}\,(\omega) > 0 \\ H_A(\omega) & \mathrm{Im}\,(\omega) < 0 \end{cases}$.

It is analytic everywhere, except for a discontinuity on the real axis:

$$H(\omega + i\epsilon) - H(\omega - i\epsilon) = H_R(\omega) - H_A(\omega) = i\sigma(\omega),$$

Write a Cauchy contour-integral representation (using two half-circles) of $H(\omega)$ just above the real axis, where it coincides with $H_R(\omega)$:

$$H_R(\omega) = H_R(\omega_r) + \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\, \sigma(\omega') \Big[\frac{1}{\omega' - \omega - i\epsilon} - \frac{1}{\omega' - \omega_r - i\epsilon} \Big].$$

The dispersion relation for the Euclidean correlator follows from the observation $G_E(\omega_n, k^2) = G_R(i\omega_n, k^2), \qquad n > 0.$

Conclusion

- Photon rate: first lattice calculation in dynamical QCD with continuum limit.
- The transverse-minus-longitudinal combination cancels a large ultraviolet and admits a super-convergent sum rule.
- Results for $k > \pi T/2$ still compatible with weak-coupling prediction and AdS/CFT.
- Dispersion relation at fixed photon virtuality $q^2 = 0$ can be used to probe exclusively the photon rate (rather than the full (ω, k) dependence).