Finite-size effects in lattice calculations of hadronic-light-by-light scattering in $(g-2)_\mu$

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Cluster of Excellence

THE LOW-ENERGY FRONTIER
OF THE STANDARD MODEL

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My work in collaboration with:

- N. Asmussen
- A. Gérardin
- A. Nyffeler

Closely related: the presentations by RBC/UKQCD collaboration at UConn workshop, 12-14 March 2018 by

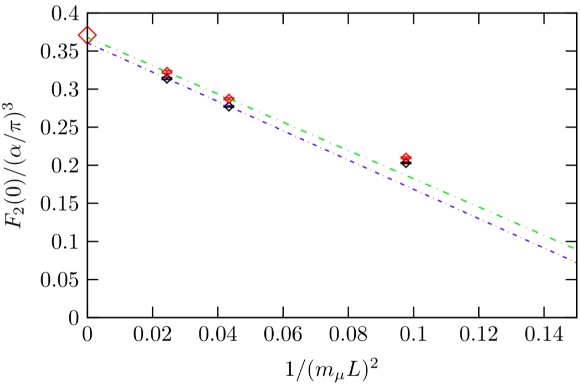
- Ch. Lehner, *HLbL contribution to* $(g-2)_\mu$ *on the lattice: finite-volume effects*
- N. Christ, *Determining the long-distance contribution to the HLbL portion of* $g-2$ *in position space from the* π^0 *pole*

+

- J. Bijnens, *Models and HLBL: disconnected contributions and first steps towards finite-volume corrections*.

Finite-size effects (FSE): general remarks

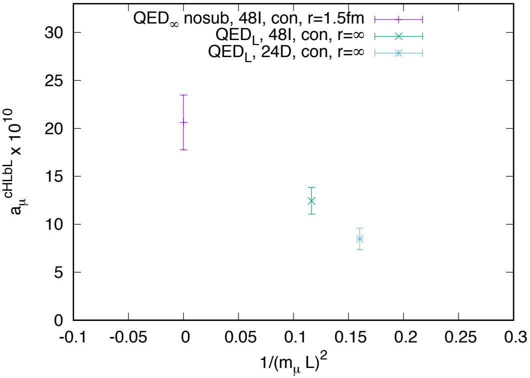
- highly dependent on the formulation;
- one method pursued by RBC/UKQCD: **finite-volume QCD and QED** ($\boldsymbol{k}_\gamma \neq 0$). Asymptotically leading effect is O$(1/L^2)$. Seen explicitly in extrapolating the **lepton loop** to infinite-volume: [1510.07100 (PRD)]



- NB. $1/(m_\mu L)^2 = 0.04 \longleftrightarrow L = 9.3\,$fm;
  of course, the coefficient of the $1/L^2$ can be quite different in QCD.

Numerical study: RBC/UKQCD

Consistency of QED_L and QED_∞:

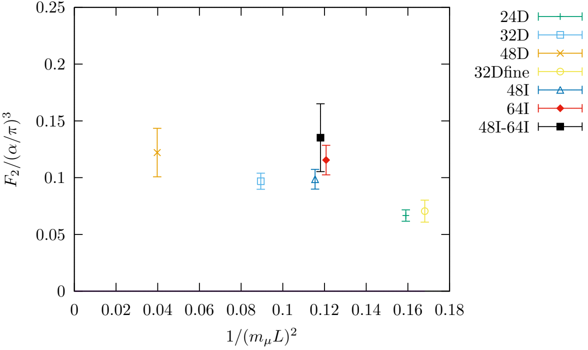


This plot is preliminary and needs to be refined with a proper continuum limit since 24D and 48I have different lattice cutoff.

Slide from talk by Ch. Lehner at UConn 2018.

Numerical study: RBC/UKQCD

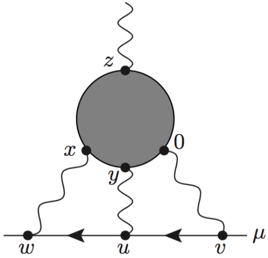
QED$_L$, connected diagram



(all particles with physical masses)

Slide from talk by T. Blum this morning.

Coordinate-space approach to $a_\mu^{\rm HLbL}$ using infinite-volume QED



$$a_\mu^{\rm HLbL} = \frac{me^6}{3}\int d^4y \Big[\int d^4x \underbrace{\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\rm QED}\ \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{\rm QCD}\Big].$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z\ z_\rho \Big\langle\, j_\mu(x)\, j_\nu(y)\, j_\sigma(z)\, j_\lambda(0)\Big\rangle.$$

[this version: N. Asmussen, J. Green, HM, A. Nyffeler 1510.08384 (LAT2015);
see RBC/UKQCD 1705.01067 (PRD) for a slightly different version]

- $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ computed in the continuum & infinite-volume;
- no power-law finite-volume effects. But what is the asymptotically leading effect?
- 1st goal: understand FSE on $i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)$.

Dominant finite-volume effects on $i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)$ (I)

HLbL scattering amplitude (momentum space):

$$\Pi_{\mu\nu\sigma\lambda}(q_1,q_2,q_3) = \int_{x,y,z} \Big\langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \Big\rangle e^{-i(q_1 x + q_2 y + q_3 z)}.$$

On the torus, this quantity receives only corrections of order $e^{-m_\pi L}$ for Euclidean momenta, which I will neglect in the following.

For instance, the pion-pole contribution reads
($\mathcal{F}(q_1^2, q_2^2)$ = transition form factor $\pi^0 \to \gamma^*\gamma^*$)

$$\Pi_{\mu\nu\sigma\lambda}(q_1,q_2,q_3)\Big|_{\pi_0} =$$

$$\frac{\mathcal{F}(-q_1^2,-q_2^2)\mathcal{F}(-q_3^2,-(q_1+q_2+q_3)^2)}{(q_1+q_2)^2+m_\pi^2} \epsilon_{\mu\nu\alpha\beta}\, q_{1\alpha}\, q_{2\beta}\, \epsilon_{\sigma\lambda\gamma\delta}\, q_{3\gamma}\, (q_1+q_2)_\delta$$

$$+ \frac{\mathcal{F}(-q_1^2,-(q_1+q_2+q_3)^2)\mathcal{F}(-q_2^2,-q_3^2)}{(q_2+q_3)^2+m_\pi^2} \epsilon_{\mu\lambda\alpha\beta}\, q_{1\alpha}\, (q_2+q_3)_\beta\, \epsilon_{\nu\sigma\gamma\delta}\, q_{2\gamma}\, q_{3\delta}$$

$$+ \frac{\mathcal{F}(-q_1^2,-q_3^2)\mathcal{F}(-q_2^2,-(q_1+q_2+q_3)^2)}{(q_1+q_3)^2+m_\pi^2} \epsilon_{\mu\sigma\alpha\beta}\, q_{1\alpha}\, q_{3\beta}\, \epsilon_{\nu\lambda\gamma\delta}\, q_{2\gamma}\, (q_1+q_3)_\delta.$$

Dominant finite-volume effects on $i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)$ (II)

But, what is computed on the lattice,

$$i\widehat{\Pi}^{(L)}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int_{z\in[0,L]^4} [z_\rho] \left\langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \right\rangle_{\rm torus},$$

$$[z_\rho] = \begin{cases} z_\rho & 0 \le z_\rho \le L_\rho/2 \\ (z_\rho - L_\rho) & L_\rho/2 \le z_\rho \le L_\rho \end{cases}$$

has finite-volume effects of order $L\, e^{-m_\pi L/2}$ [HM, ETHZ workshop, 9 March 2018].

Derivation: In 1d-Fourier transformation of a periodic function $f(z)$,

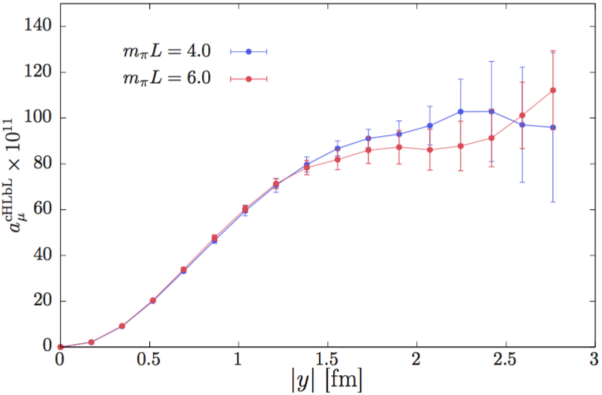
$$A \equiv \int_0^L [z]\; f(z) = -i \sum_{q\neq 0} \frac{\tilde f(q)}{q}\, e^{iqL/2}.$$

where $\tilde f(q) = \int_0^L dz\; e^{-iqz}\; f(z)$, $q = \frac{2\pi n}{L}$, $n \in \mathbb{Z}$.
$\Rightarrow$ if $\tilde f(q) = q\,\tilde\phi(q) + \tilde f(0)$, with $\tilde\phi(q)$ finite at $q = 0$,

$$A = i\tilde f'(q=0) - iL\;\phi(z = L/2),$$

where $\phi(z) = \frac{1}{L}\sum_q \tilde\phi(q)\, e^{iqz} = \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{dq}{2\pi}\; \tilde\phi(q)\; e^{i(qz+nL)}$.
$\rightsquigarrow$ as compared to the infinite-volume expression, we pick up the second term, which is of order $L\, \exp(-m_\pi L/2)$.

A numerical study: Mainz group (fully conn. diag., $m_\pi = 280\,$MeV)



- no statistically significant finite-size effect is seen, for now;
- the infinite-volume prediction for the π^0 $(\times \frac{34}{9})$ describes the integrand well at large $|y|$ (see N. Asmussen's talk);
- further checks at smaller m_π desirable.

To-do list

- quantitatively confront the analytically calculated finite-size correction with the lattice data;
- if consistent, use the analytic prediction to:
  - correct the lattice data at the *integrand* level for the leading finite-size effect;
  - guide the extrapolation of the integrand to $|x|, |y|, |z| \to \infty$
- as in the HVP, the two are closely related.