Cosmology and General Relativity: HW 4

Turn in to "Wittig" (Hausaufgaben) mailbox in KPH by **noon, 13 June 2019**

Problem 1 Commutators and Jacobi identity:

a. Show that equipping the space of vector fields on manifold M with the commutator $[X, Y] = XY - YX$ forms an algebra by showing that the commutator is bilinear and closed.[1] *2pt.*

b. For vector fields X, Y, and Z (all smooth) on manifold M. Verify the Jacobi identity: *1pt.*

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \ .$$

c. Let $V_1, \ldots, V_n$ be a set of vector fields on the n-dimensional manifold M such that they form a basis of the tangent space at all points on M. We can define the *structure constants* with respect to this basis

$$[V_i, V_j] \equiv f_{ij}{}^k V_k$$

with the symmetry condition $f_{ij}{}^k = -f_{ji}{}^k$. Express the Jacobi identity in terms of the structure constants. *1pt.*

Problem 2 Properties of the covariant derivative:

A covariant derivative (also called an *affine connection*) ∇ is defined as a map from two vector fields to one vector field,

$$\begin{aligned} \nabla : \mathrm{Vect}(M) \times \mathrm{Vect}(M) &\to \mathrm{Vect}(M) \\ (X, Y) &\mapsto \nabla_X Y, \end{aligned}$$

which satisfies the following rules,

$$\begin{aligned} \nabla_X(Y + Z) &= \nabla_X Y + \nabla_X Z, & \nabla_{(fX)} Y &= f \nabla_X Y, \\ \nabla_{(X+Y)} Z &= \nabla_X Z + \nabla_Y Z, & \nabla_X(fY) &= X[f]Y + f\nabla_X Y \ \text{(Leibniz)}, \end{aligned}$$

[1] Algebraic aside: the tangent space at a point $p \in M$ forms a *vector* space over the field $\mathbb{R}$, and equipping a vector space with a bilinear operator defines an *algebra* (over a field). However, the space of vector fields actually forms a *module* over real-valued smooth functions. A module is like a vector space except over a *ring* (the functions form a ring because there is no multiplicative inverse for any function that has a zero). So vector fields equipped with the commutator form an *algebra over a ring*.

where $\nabla_X f \equiv X[f] \equiv df(X) = X^\mu \frac{\partial f}{\partial x^\mu}$ for a smooth map $f : M \to \mathbb{R}$. Here, we are considering the chart (U, φ) such that for $p \in M$, $\varphi(p) = x$. The components of the *Christoffel symbols* $\Gamma^\lambda_{\mu\nu}$ are given by:

$$\nabla_{\partial_\mu}\partial_\nu \equiv \nabla_\mu \partial_\nu = \Gamma^\lambda_{\mu\nu}\partial_\lambda \quad \text{with } \partial_\mu \equiv \frac{\partial}{\partial x^\mu}.$$

For two vector fields $X = X^\mu \partial_\mu$ and $Y = Y^\mu \partial_\mu$, one obtains:

$$(\nabla_X Y)^\nu = X^\mu \partial_\mu Y^\nu + X^\mu \Gamma^\nu_{\mu\rho} Y^\rho.$$

a. Show that for a vector field $X = X^\mu \partial_\mu$ the action of the covariant derivative ∇ on a 1-form $\omega = \omega_\nu dx^\nu$ is given by: *1pt.*

$$(\nabla_X \omega)_\nu = X^\mu \partial_\mu \omega_\nu - X^\mu \Gamma^\lambda_{\mu\nu} \omega_\lambda.$$

b. Consider another chart (V, ψ) where $y = \psi(p)$. In the region V, the Christoffel symbols can be written in y-coordinate components as $\tilde{\Gamma}^\gamma_{\alpha\beta}$. Show that in the intersection $U \cap V$, this symbol can be written in terms of Γ as:

$$\tilde{\Gamma}^\gamma_{\alpha\beta} = \frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial x^\mu}{\partial y^\beta}\frac{\partial y^\gamma}{\partial x^\nu}\Gamma^\nu_{\lambda\mu} + \frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta}\frac{\partial y^\gamma}{\partial x^\nu}.$$

Observe that Γ does not transform as a tensor due to the second derivative term. *2pt.*

c. Show that the two-index object,

$$(\nabla_\mu \omega)_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu}\omega_\lambda,$$

transforms as components of a tensor. Here, $\omega = \omega_\nu dx^\nu$ is a 1-form. *1pt.*

Problem 3 Geodesics on S^2: The geodesic differential equation has the form:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \ ,$$

Here, we consider S^2 with the metric $ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta d\varphi^2$, for polar angle θ, azimuthal angle φ, and constant $a > 0$.

a. Calculate the Christoffel symbols to show that the above equation takes the following form on S^2: *2pt.*

$$\frac{d^2\theta}{d\tau^2} - \sin\theta\cos\theta\left(\frac{d\varphi}{d\tau}\right)^2 = 0, \qquad \frac{d^2\varphi}{d\tau^2} + 2\cot\theta\frac{d\theta}{d\tau}\frac{d\varphi}{d\tau} = 0 \ .$$

b. Now show that if $\theta = \theta(\varphi)$ is the equation of a geodesic, then the two equations can be combined into a single equation: *1pt.*

$$\frac{d^2\theta}{d\varphi^2} - 2\cot\theta\left(\frac{d\theta}{d\varphi}\right)^2 - \sin\theta\cos\theta = 0.$$

c. Now define $f(\theta) = \cot\theta$. Show that f satisfies the differential equation:

$$\frac{d^2 f}{d\varphi^2} + f = 0 \ .$$

Give the general solution. What do the geodesics on S^2 look like? *2pt.*

Hint: it may help to consider f in the standard embedding of the sphere in $\mathbb{R}^3$.

Problem 4 Curvature of the sphere:
The Riemann tensor is a (1,3)-tensor given by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \ ,$$

for $X, Y, Z \in TM$.

a. Write the components of the Riemann tensor

$$R^k{}_{\ell ij} \equiv dx^k\left(R(\partial_i,\partial_j)\partial_\ell\right)$$

in terms of partial derivatives and Christoffel symbols. *1.5pt.*

b. Show that the Riemann tensor vanishes for flat Euclidean space $ds^2 = dx^2 + dy^2$. *1pt.*

c. Calculate the Riemann tensor for the sphere S^2, where $ds^2 = a^2 d\theta^2 + a^2\sin^2\theta d\varphi^2$. What is the a-dependence for $R^k{}_{\ell ij}$? The Riemann tensor with only lower indices is defined as: $R_{k\ell ij} = g_{mk}R^m{}_{\ell ij}$. What is the a-dependence of $R_{k\ell ij}$? *2.5pt.*

Hint: there should only be one independent, non-zero term after considering symmetries in the components of the Riemann tensor.

d. Calculate the Ricci tensor $R_{ij} = R^k{}_{ikj}$ and the Ricci scalar $R = g^{ij}R_{ij}$. What is the a-dependence of the Ricci scalar and how does this scalar vary over the sphere? *2pt.*