Chapter 3

Top-Down Approach

Let us assume the following physics setting

- One heavy mass M
- Look at processes well below energies of the heavy mass particle, i.e. $E \ll M$
- Scattering matrix expressed via path integral, i.e.

$$\langle 0|T\Big[\phi(x_1)\dots\phi(x_n)\Big]|0\rangle = \frac{1}{Z}\int \underbrace{\mathcal{D}\phi}_{\prod_x d\phi(x)} e^{iS}\phi(x_1)\dots\phi(x_n) \tag{3.1}$$

with

$$Z = e^{iS}. \tag{3.2}$$

3.1 Effective action and power counting

Let us split the field into low and high energy modes

$$\phi(x) = \phi_L(x) + \phi_H(x) \tag{3.3}$$

where the separation is meant with respect to fourier modes. Since we are only interested in physics at $E \ll M$, we will only look at

$$\langle 0|T\Big[\phi_L(x_1)\dots\phi_L(x_n)\Big]|0\rangle \tag{3.4}$$

that is Green's functions of the low energy modes of the fields ϕ. We can calculate these in the full theory

$$\begin{aligned}\langle 0|T\Big[\phi_L(x_1)\dots\phi_L(x_n)\Big]|0\rangle &= \int \mathcal{D}\phi_L \int \mathcal{D}\phi_H e^{iS(\phi_L+\phi_H)}\phi_L(x_1)\dots\phi_L(x_n) \\ &= \int \mathcal{D}\phi_L e^{iS_\Lambda(\phi_L)}\phi_L(x_1)\dots\phi_L(x_n) \end{aligned} \tag{3.5}$$

$S_\Lambda(\phi_L)$ is called the Wilsonian effective action. This explains the notion of *Integrating out* heavy degrees of freedom. In S_Λ all physics related to energies at the heavy scale $\Lambda \sim M$ is included. For scales of $\Delta x \approx \frac{1}{\Lambda}$ the effective action S_Λ is non-local. For energies well below Λ we can however expand the non-local action in a series of local operators

$$S_\Lambda(\phi_L) = \int d^4x \mathcal{L}^{\text{eff}}_\Lambda(x) \tag{3.6}$$

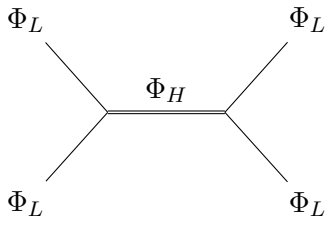
where

$$\mathcal{L}^{\text{eff}}_\Lambda(x) = \sum g_i \mathcal{O}_i(x) \tag{3.7}$$

The effective Lagrangian contains an infinite number of local operators, the coefficients of which are called Wilson coefficients.

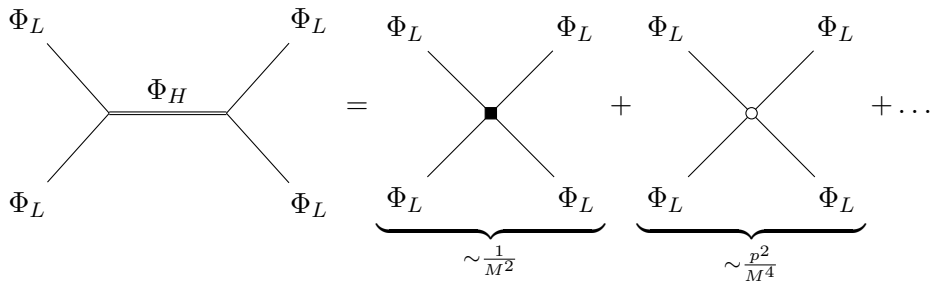
How do we handle in infinite number of terms?

We will need some criterion to tell which operators are important! Let us look at the dimension of the operators and let us revisit the example from chapter 1, i.e. a theory with one heavy particle. Let us again look at the $2 \to 2$ scattering of the light particle



$$\sim \frac{1}{s - M^2} = \frac{1}{(p_1 + p_2)^2 - M^2} \tag{3.8}$$

Since all momenta are much smaller than M, i.e. $p_1, p_2 \ll M$ we can expand the propagator



(3.9)

where $\mathcal{L}_{\text{eff}}$ will contain terms like

$$\Phi_L^4, \quad (\partial_\mu \Phi_L \partial^\mu \Phi_L)^2, \quad \partial_\mu \Phi_L \partial^\mu \Phi_L \Phi_L^2 \tag{3.10}$$

which can be matched to the above expansion. In general the g_i from Eq. (3.7) are not easily calculated. Let us discuss the problem of the infinite tower of local interactions in the next section.

3.2 Power counting & loop expansion

The expansion of the non local action in terms of local operators leads to termsof the form

$$\frac{1}{M^2}, \quad \frac{p^2}{M^4}, \ldots, \quad \text{etc.} \tag{3.11}$$

A natural way of organizing these terms is to use the dimension of the operator. To that end let us define the negative mass dimension γ_i of the Wilson coefficient g_i,

$$g_i = C_i M^{-\gamma_i}, \tag{3.12}$$

$$C_i \equiv \text{dimensionless coefficient.} \tag{3.13}$$

The dimensionless coefficient is expected to be of natural size, i.e. $C_i \sim \mathcal{O}(1)$. Let us look at a dimensionless variable f

$$[f(g_i, E)] \sim \left(\frac{E}{M}\right)^{\gamma_i} = \begin{cases} \mathcal{O}(1) & \gamma_i = 0 \\ \gg 1 & \gamma_i < 0 \\ \ll 1 & \gamma_i > 0 \end{cases} \tag{3.14}$$

The contributions to f can be split into important and less important parts, according to their mass dimension, where contributions for $\gamma_i < 0$ are important at low energies. This establishes a power counting where if we are interested in a quantity to precision Δ we include all operators to a finite order in their mass dimension, i.e.

$$\Delta \sim \Big(\frac{E}{M}\Big)^n, \tag{3.15}$$

$$\ln \Delta = n \ln\Big(\frac{E}{M}\Big), \tag{3.16}$$

$$n = \frac{\ln \Delta}{\ln\Big(\frac{E}{M}\Big)}. \tag{3.17}$$

How do we determine the mass dimensions of the operators and their Wilson coefficients? We use the fact that the action, which gives the weight in the path integral, has to be a pure number. Since $x \sim \frac{1}{M}$ the mass dimension of the Lagrangian $\mathcal{L}$ in D dimensions is $[\mathcal{L}] = D$. Furthermore let us assume the theory is weakly coupled, why this is necessary we will see later. To make the discussion more explicit let us look at a scalar theory in D dimensions, i.e. the free part is given by

$$S_D = \int d^D x \frac{1}{2}\Big(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\Big) \tag{3.18}$$

where we can fix the dimension of the field ϕ using the term proportional to m^2,

$$[\mathcal{L}] = D = [m^2][\phi^2] \tag{3.19}$$

$$D = 2 + 2[\phi] \tag{3.20}$$

$$[\phi] = \frac{D}{2} - 1. \tag{3.21}$$

Since the ingredients of the Lagrangian are the fields, masses and momenta we can use these to fix the mass dimension of the Wilson coefficients by first fixing the dimension of the operators, i.e.

$$[\mathcal{L}] = D = [g_i \mathcal{O}_i] \tag{3.22}$$

$$D = [g_i] + [\mathcal{O}_i] \tag{3.23}$$

$$D = -\gamma_i + \delta_i, \tag{3.24}$$

where δ_i denotes the dimension of the operator $\mathcal{O}_i$. The mass dimensions for the operators and Wilson coefficients for some operatore are given in In

Operator	δ_i	γ_i	g_i
$\partial_\mu\phi\partial^\mu\phi$	D	0	~ 1
ϕ^2	$D-2$	-2	$\sim M^2$
ϕ^4	$2D-4$	$D-4$	$\sim M^{4-D}$

Table 3.1: Mass dimension of operators of scalar fields ϕ in D dimensions.

general operators of scalar fields ϕ with n derivatives and m fields have the mass dimension

$$\delta_i = n\Big(\frac{D}{2}-1\Big)+m, \tag{3.25}$$

$$\gamma_i = \delta_i - D \tag{3.26}$$

$$= (n-2)\Big(\frac{D}{2}-1\Big)+(m-2). \tag{3.27}$$

One observation is that for $D>2$ only few operators have negative γ_i. The usual terminology in the context of power counting is given in table 3.2. The mass dimension of the operators is also referred to as the scaling dimension of the operator. To understand this name and also to deal with cases where the operators are not organized according to their mass dimension we discuss the scaling properties of the action. Again consider a scalar theory, this time including interactions

$$S[\phi] = \int d^Dx \left[\frac{1}{2}\Big(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\Big) - \frac{\lambda}{4!}\phi^4 - \frac{\tau}{6!}\phi^6\right]. \tag{3.28}$$

The dimensions are

$$[S]=0, \qquad [\phi]=\frac{D-2}{2}, \qquad [m^2]=2 \tag{3.29}$$

$$[\lambda]=4-D, \qquad [\tau]=6-2D. \tag{3.30}$$

We want to study the behavior of Green's functions

$$\langle T(\phi(x_1)\dots\phi(x_n))\rangle$$

Dimension	Importance at $E \ll M$	Calling Convetion
$\delta_i < D$ &$\gamma_i < 0$	grows	relevant (super-renormalizable)
$\delta_i = D$ &$\gamma_i = 0$	const	marginal (renormalizable)
$\delta_i > D$ &$\gamma_i < 0$	decreases	irrelevant (non-renormalizable)

Table 3.2: Calling convention of different dimension operators in the EFT (QFT) language.

at long distances (low energies). To that end we analyse the Green's functions change under rescaling of the space time coordinates $x \to sx$. This rescaling amounts to an inverse rescaling of the derivative, i.e. $\partial \to \frac{1}{s}\partial$. Let us rescale Eq. (3.28)

$$S[\phi(sx)] = \int d^D x s^D \left[\frac{1}{2}\Big(\frac{1}{s^2}\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\Big) - \frac{\lambda}{4!}\phi^4 - \frac{\tau}{6!}\phi^6\right] \tag{3.31}$$

and *canonicalize* the free kinetic part

$$= \int d^D x s^{D-2} \left[\frac{1}{2}\Big(\partial_\mu\phi\partial^\mu\phi - s^2m^2\phi^2\Big) - s^2\frac{\lambda}{4!}\phi^4 - s^2\frac{\tau}{6!}\phi^6\right] \tag{3.32}$$

introduce redefinition $\phi \to s^{(2-D)/2}\phi'$

$$= \int d^D x \left[\frac{1}{2}\Big(\partial_\mu\phi'\partial^\mu\phi' - s^2m^2\phi'^2\Big) - s^{4-D}\frac{\lambda}{4!}\phi'^4 - s^{6-2D}\frac{\tau}{6!}\phi'^6\right] \tag{3.33}$$

For the Green's function the rescaling results in

$$\langle T(\phi(sx_1)\dots\phi(sx_n))\rangle = s^{n(2-D)/2}\langle T(\phi'(x_1)\dots\phi'(x_n))\rangle. \tag{3.34}$$

This explains the word scaling dimension. At $D = 4$ we see that $s \to \infty$

- m^2 term gets more and more important
- τ term is less important
- λ term is equally important at all scales.

Identify the mass scale with the scale of new physics

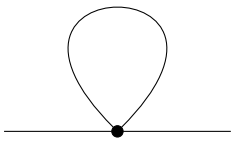
$$\Rightarrow m^2 \sim \Lambda^2_{\text{new}}, \quad \lambda \sim \Lambda^0_{\text{new}}, \quad \tau \sim \Lambda^{-2}_{\text{new}}. \tag{3.35}$$

This explains why renormalizability is not an issue, as we would expect new physics to set in before the nonrenormalizable interaction can do any harm. Note that relevant operators can spoil power counting and we will discuss in some detail what the problem is. The term irrelevant is a misnomer, as the contributions are only suppressed, i.e. in precise enough measurements effects of these *irrelevant* operators can be sizable.

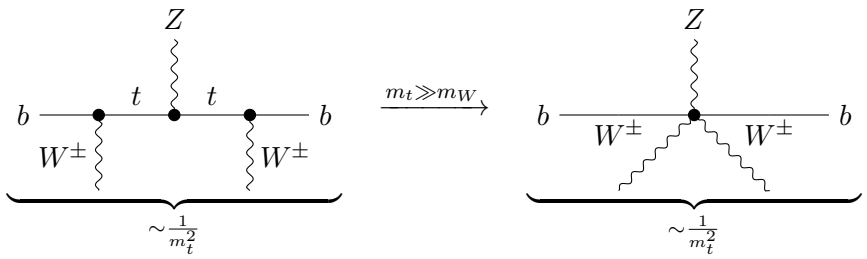
How about that relevant coupling, i.e. the operator ϕ^2 would give rise to terms

$$\phi^2 \sim \Lambda^2_{\text{new}}\sigma \tag{3.36}$$

where σ is of natural size. The problem with terms like these is that they contribute to the mass of the particle, e.g. in the Standard Model there is a contribution to the Higgs mass of a quartic coupling, i.e.

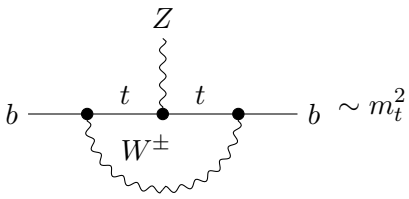

$\sim \Lambda^2$ (3.37)

however Λ is supposed to be much higher than m_H which seems to be a contradiction. Another example



(3.38)

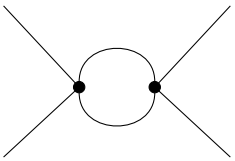
However if one closes the $W^\pm$ lines to form a loop the result is proportional to m_t^2, i.e.



$\sim m_t^2$ (3.39)

i.e. the loop expansion and power counting seem to give incompatible results, at least using a cut off as a UV regulator. We will see that in an appropriately chosen regulator the *naive* dimensional analysis also holds for

loop diagrams. Let us look at what kind of contributions one loop diagrams generate

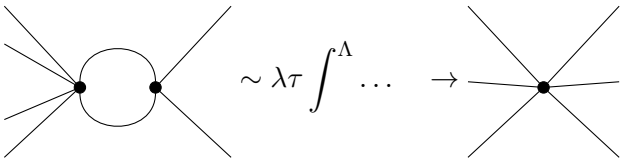


$$\sim \lambda^2 \int d^D k \frac{1}{k^2 - m^2} \frac{1}{(p+k)^2 - m^2} \tag{3.40}$$

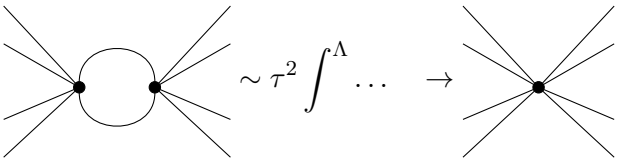
For large k

$$\lambda^2 \int d^D k \frac{1}{k^2 - m^2} \frac{1}{(p+k)^2 - m^2} \to \int^{\Lambda} \frac{dk}{k} \sim \ln \Lambda \tag{3.41}$$

the above diagram renormalizes the ϕ^4 vertex. Similarly



$$\sim \lambda\tau \int^{\Lambda} \dots \quad \to \tag{3.42}$$



$$\sim \tau^2 \int^{\Lambda} \dots \quad \to \tag{3.43}$$

where we see in the last line that the one loop calculation including non-renormalizable interactions generates divergences for which a counter term of higher order in the number of fields, i.e. ϕ^8 is necessary. This interaction is missing in the action, hence the name *nonrenormalizable*. However if dimensional counting were to hold, than we would expand the amplitudes in powers of $1/\Lambda$, which means that in the EFT one needs to include all interaction up to that order, i.e. including the ϕ^8 interaction, in the theory. However these contributions would be suppressed, on dimensional ground with respect to the ϕ^4 and ϕ^6 interactions. In this sense renormalizability is overrated.

Example:
Weak interaction

$$\overline{O} = \overline{u}\gamma_\mu(1-\gamma_5)d\overline{e}\gamma^\mu(1-\gamma_5)\nu. \tag{3.44}$$

The dimensions of the fermion fields is $[u] = [d] = [e] = [\nu] = 3/2$, i.e. the operator is of dimension 6, hence the Wilson coefficient will have -2 mass

dimensions, i.e. a suppression of $1/M^2$ is to be expected, where M is some heavy mass of the theory. Using this EFT one can guess what the scale of new physics, i.e. the mass of the heavy particle would be. Including all group theoretical factors one can infer from the experimental data, that new physics would enter at $M \geq 60\text{GeV}$.

Let us look at four fermion theory, i.e.

$$\mathcal{L}_{\text{free}} = \overline{\Psi}(i\not{\partial} - m)\Psi \tag{3.45}$$

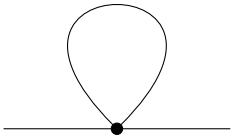
$$\mathcal{L}_{\text{int}} = \overline{\Psi} \cdot \Psi\overline{\Psi}\Psi\tilde{a} \tag{3.46}$$

The dimensions are

$$[\overline{\Psi}] = \frac{3}{2}, \quad [\Psi] = \frac{3}{2}, \quad [\tilde{a}] = -2 \tag{3.47}$$

$$\tilde{a} = \frac{a}{M^2} \tag{3.48}$$

where a is now dimensionless. The one loop contribution to the mass is given by



$$\sim \frac{a}{M^2}\int d^4k \frac{\not{k} + m}{k^2 - m^2}. \tag{3.49}$$

We can use the fact that *odd* integrals over the loop momenta vanish

$$I = \frac{a}{M^2}\int d^4k \frac{\not{k} + m}{k^2 - m^2} \tag{3.50}$$

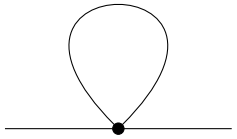
$$= \frac{am}{M^2}\int d^4k \frac{1}{k^2 - m^2} \tag{3.51}$$

Let us look at the different regions of momenta for the loop integral. First $k \sim m$

$$\int d^4k \frac{1}{k^2 - m^2} \sim m^2 \tag{3.52}$$

which can be seen by arguments of dimensionality. For the low energy part we see the expected behavior of the one loop amplitude with respect to the

power counting



$$\sim \frac{am^3}{M^2}. \tag{3.53}$$

This is a small contribution to m suppressed by $1/M$. For the region $k \gg m$ let us do the full integral in cutoff theory. To that end we will switch to euclidean momenta

$$I_1 = \frac{am}{M^2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 - m^2} \tag{3.54}$$

$$= i \frac{am}{M^2} \int \frac{d^D k_E}{(2\pi)^D} \frac{1}{-k^2 - m^2} \tag{3.55}$$

$$= -i \frac{am}{M^2} \frac{2}{(4\pi)^2} \int_0^{\Lambda_{\mathrm{UV}}} dk \frac{k_E^3}{k_E^2 + m^2} \tag{3.56}$$

$$= -i \frac{am}{M^2} \frac{1}{(4\pi)^2} \left(m^2 \left(\log\left(m^2\right) - \log\left(m^2 + \Lambda_{\mathrm{UV}}^2\right) \right) + \Lambda_{\mathrm{UV}}^2 \right) \tag{3.57}$$

$$= -i \frac{am}{M^2} \frac{1}{(4\pi)^2} \left(-\frac{m^4}{\Lambda_{\mathrm{UV}}^2} + 2m^2 \log\left(\frac{m}{\Lambda_{\mathrm{UV}}}\right) + \Lambda_{\mathrm{UV}}^2 + \dots \right). \tag{3.58}$$

The first and second term in the paranthesis are suppressed however the last term is proportional to positive powers of the UV-scale. It is a general feature that relevant terms may spoil power counting in loop calculations if a cutoff regukator is used.
Ways out of this problem

- Modify high energy behavior of LET such that power counting holds $\rightarrow$ change couplings (become Λ dependent) to absorb power counting violating terms
- Use regularization that respects dimensional analysis $\rightarrow$ dimensional rgularization DimReg

DimReg is sometimes referred to as a mass independent schemes. The most general definitioin of a mass independent scheme is (Georgi)

> "All evolution functions (β-functions) should be independent of renormalization scale and only depend on physical parameters."

The self energy diagram in DimReg reads

$$-i\frac{am}{M^2}\int\frac{d^Dk_E}{(2\pi)^D}\frac{1}{k_E^2+m^2}=-i\frac{am}{M^2}\frac{2\Big(\mu^2e^{\gamma_e}/4\pi\Big)^{\varepsilon}}{(4\pi)^{D/2}\Gamma(D/2)}\int_0^\infty dk_E\frac{k_E^{D-1}}{k_E^2+m^2} \tag{3.59}$$

$$=-i\frac{am}{M^2}\frac{\pi\csc(\pi D/2)e^{\varepsilon\gamma_E}}{\Gamma(2-\varepsilon)(4\pi)^2}\Big(\frac{\mu^2}{m^2}\Big)^{\varepsilon}m^2 \tag{3.60}$$

$$=i\frac{am}{(4\pi)^2}\frac{m^2}{M^2}\Big(\frac{1}{2\varepsilon}+1-\ln\Big(\frac{m^2}{\mu^2}\Big)+\mathcal{O}(\varepsilon)\Big), \tag{3.61}$$

where we use $\varepsilon=(4-D)/2$. We see that the expression in Eq. (3.61)

- respects the power counting,
- has the same log as in Eq. (3.58) with the replacement $\Lambda\leftrightarrow\mu$.

A simple way to renormalize this expression is to introduce counterterms that cancel the divergent piece, i.e. $\sim 1/\varepsilon$. This prescription is called minimal subtraction, the term we have shown in the above expression already subtracts an additional $(4\pi)^{\varepsilon}=1+\varepsilon\ln 4\pi+\mathcal{O}(\varepsilon^2)$ term and is called $\overline{\text{MS}}$. Dimensional regularization does not break power counting and one can count the loop momentum as a small quantity. Let us stress that the power counting only applies to renormalized couplings. We could therefore add counterterms also to the cutoff regulated theory such that all power counting violating terms would be absorbed in thee redefinition of the Wilson coefficients. However dimensional regularization is an easier way of performing loop calculations!

3.3 Dimensional Regularization

We are interested in calculating integrals of the form Eq. (3.59)

$$\int\frac{d^Dk_E}{(2\pi)^D}\frac{1}{k_E^2+m^2}=\Omega_D\int_0^\infty dk_E\frac{k_E^{D-1}}{k_E^2+m^2} \tag{3.62}$$

$$=\Omega_D\frac{1}{2}\pi\left(\frac{1}{m^2}\right)^{1-\frac{D}{2}}\csc\left(\frac{\pi D}{2}\right)\qquad,m\in\mathbb{R}\ \wedge\ 0<D<2. \tag{3.63}$$

In the following we will use the abbreviation

$$\frac{d^D k}{(2\pi)^D} = đ^D k. \tag{3.64}$$

The following Axioms define dimensional regularization

- Linearity:

$$\int đ^D k \ [af(k) + bg(k)] = a \int đ^D k \ f(k) + b \int đ^D k \ g(k) \tag{3.65}$$

- Translational invariance:

$$\int đ^D k \ f(a + k) = \int đ^D k \ f(k) \tag{3.66}$$

- Scaling:

$$\int đ^D k \ f(sk) = s^{-D} \int đ^D \tilde{k} \ f(\tilde{k}) \qquad \text{, with } k = s^{-1}\tilde{k} \tag{3.67}$$

In the Euclidean the D-dimensional integral can be split using spherical coordinates, where the integral over the angular part is given as

$$d^D k = dk d^{D-1} d\Omega_D \tag{3.68}$$

$$\int d\Omega)_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}. \tag{3.69}$$

The UV-divergence is in the one dimensional integral over the radius. For spherically symmetric integrands we have

$$đ^D k = \frac{1}{(2\pi)^D} d^D k = \frac{2\pi^{D/2}}{\Gamma(D/2)} dk k^{D-1} \tag{3.70}$$

$$= \frac{2}{(4\pi)^{D/2}\Gamma(D/2)} dk k^{D-1}. \tag{3.71}$$

Some (for later usefull) results read

$$\int đ^D k \frac{(k^2)^\alpha}{(k^2 + A)^\beta)} = \frac{1}{(4\pi)^{D/2}} A^{D/2+\alpha-\beta} \frac{\Gamma(\beta - \alpha - D/2)}{\Gamma(\beta)} \frac{\Gamma(\alpha + D/2)}{\Gamma(D/2)}. \tag{3.72}$$

We use

$$D = 4 - 2\varepsilon \tag{3.73}$$

where

$$\varepsilon = \begin{cases} > 0 \text{ renders UV finite} \\ < 0 \text{ renders IR finite} \end{cases} \tag{3.74}$$

Based on dimensional arguments scaleless integrals vanish

$$\int đ^D k (k^2)^\alpha = 0 \qquad , \forall \alpha \in \mathbb{R}. \tag{3.75}$$

Dimensional regularization is well defined even in the case of UV and IR divergent integrals. The idea is to calculate the integral in a region of D where it is convergent and the analytically continue to other D.

$$\int đ^D k f(k^2) \qquad \text{exists for } 0 < D < D_{\max}. \tag{3.76}$$

To obtain a solution in the range $2 < D < D_{\max}$ rewrite

$$\int đ^D k f(k^2) = \frac{2}{(4\pi)^{D/2}\Gamma(D/2)} \left\{ \int_c^\infty dk k^{D-1} f(k^2) + \int_0^c dk d^{D-1} (f(k^2) - f(0)) + \frac{f(0)\ c^D}{D} \right\} \tag{3.77}$$

LHS independet of c. Perform limit $c \to \infty$

$$\int đ^D k f(k^2) = \frac{2}{(4\pi)^{D/2}\Gamma(D/2)} \int_0^\infty dk d^{D-1} (f(k^2) - f(0)). \tag{3.78}$$

Why is this integral better behaved in the IR than the original one? Taylor expansion of the integrand near zero

$$f(k)^2 = f(0) + k^2 f'(0) + \dots \tag{3.79}$$

$$f(k^2) - f(0) = \underbrace{k^2 f'(0)}_{\text{less IR divergent by 2}} \tag{3.80}$$

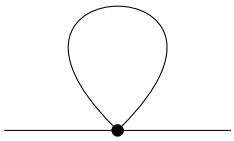
Repeat the above to extend the solution in D.

3.3.1 MS-Scheme

One way to renormalize the amplitudes in dimensionally regularized theories is the minimal subtraction scheme (MS-scheme). Here the idea is to rewrite the bare couplings in terms of renormalized ones

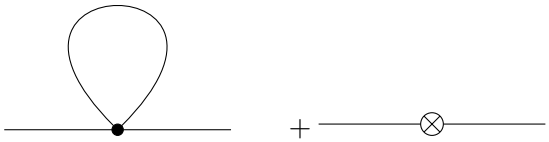
$$m^{\text{bare}} = Z_m m = m + \hbar\delta m,$$

and redefine the renormalized couplings in such a way that only the divergent terms are absorbed, e.g. Eq. (3.61) where the divergent part of the self energy reads



$$= i\frac{am}{(4\pi)^2}\frac{m^2}{M^2}\Big(\frac{1}{2\varepsilon}\Big) \tag{3.81}$$

and after subtraction the divergent part should be cancelled



$$= 0 \tag{3.82}$$

$$i\frac{am}{(4\pi)^2}\frac{m^2}{M^2}\Big(\frac{1}{2\varepsilon}\Big) + \delta m = 0 \tag{3.83}$$

In dimensional regularization we furthermore introduce a scale μ such that the couplings have the same dimension irrespective of D, e.g.

$$g^{\text{bare}}\overline{\Psi}\not{A}\Psi, \qquad [g^{\text{bare}}] = \frac{4-D}{2} = \varepsilon \tag{3.84}$$

$$g^{\text{bare}} = Z_g\mu^{2\varepsilon}g(\mu), \tag{3.85}$$

where in the last line we have introduced the renormalized coupling $g^{\text{bare}} = Z_g g$ and extracted the dependence on the regularization scale μ. μ is not a property of loop integration itself, but rather keeps the dimension of our coupling the same in all dimensions. In loop calculations this scale μ will normalize the other scales of the integrand in the logs that appear after the expansion of D around 4.

In principle the definition of the scale is not unique as one can introduce

$$\mu'^2 = \frac{\mu^2 e^{\gamma_E}}{4\pi} \tag{3.86}$$

with

$$\gamma_E = -\Gamma'(1). \tag{3.87}$$

The easiest way to see all this at work is to look at the integral

$$\mu^{2\varepsilon} \int \bar{d}^D k \frac{1}{k_E^2 + m^2} = \frac{m^2}{16\pi^2} \Big[\underbrace{\underbrace{-\frac{1}{\varepsilon}}_{\text{MS}} + \gamma_E - \ln 4\pi}_{\overline{\text{MS}}} - 1 - \ln\Big(\frac{\mu^2}{m^2}\Big) \Big] \tag{3.88}$$

$$\Big(\mu^2 e^{\gamma_E}\Big)^{\varepsilon} \int \bar{d}^D k \frac{1}{k_E^2 + m^2} = -\frac{m^2}{16\pi^2 \varepsilon} + \frac{m^2}{16\pi^2} \Big[-1 - \ln\Big(\frac{\mu^2}{m^2}\Big) \Big]. \tag{3.89}$$

where we have indicated the terms that are dropped in the different schemes in the first line. The second line shows that $\overline{\text{MS}}$ can be implemented via a different choice of μ.

Summary dimension regularization

- Good
  - Preserves symmetries
  - easy to use in calculations
  - often gives manifest power counting
- Bad
  - physcial picture less clear
  - decoupling theorem does not apply

3.4 Decoupling Theorem

At the foundations of the EFT approach is the fact that we can compute low energy observable without knowing the high energy part of the theory. There is a more formal statement by Appelquist & Carazone

> "If the remaining low energy effective theory is renormalizable, and we use a physical renormalization scheme, then all effects due to heavy particles appear as changes in the coupling conmstants if the LET or are suppressed by $1/\Lambda$. "

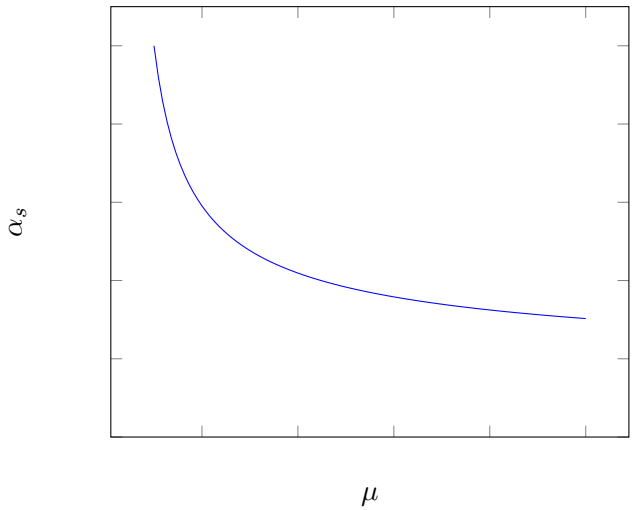
An example of a physical scheme is the off-shell momentum subtraction scheme. $\overline{\text{MS}}$ is not a physical scheme, especially it does not see mass thresholds of particles. The running QCD-coupling in $\overline{\text{MS}}$ reads

$$\beta(g) = \mu \frac{d}{d\mu} g(\mu) = -\frac{g^3}{16\pi^2}\Big(\frac{11}{3}C_A - \frac{4}{3}T_f n_F\Big). \tag{3.90}$$

where C_A is the Casimir operator of the gauge group (SU(N)_C), $T_f = 1/2$ and n_F is the number of flavors. One can solve this differential equation for the strong coupling

$$\alpha_s(\mu) = \frac{g^2(\mu)}{4\pi} \tag{3.91}$$

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \alpha_s(\mu_0)\frac{b_0}{2\pi}\ln\frac{\mu}{\mu_0}}. \tag{3.92}$$



The problem with Eq. (3.92) is that is does not depend on the masses of say the top or bottom quark, i.e. it has no knowledge of the particle production thresholds. As is, the top quark contributes at any μ to the running of α, i.e. also in the LET. This happens only in an unphysical scheme and we have to be careful here. The way out is to put in the decoupling by hand, that is once we pass the threshold of a particle we calculate the running

of the coupling not in the full theory but rather in the theory where that particle has been integrated out. For the calculation of the β function this means

$$b_0 = \begin{cases} \frac{11}{3}C_A - \frac{4}{3}T_F 6 & m_t < \mu \\ \frac{11}{3}C_A - \frac{4}{3}T_F 5 & m_b < \mu < m_t \\ \vdots & \end{cases} \tag{3.93}$$

This will lead us to the general discussion of matching and the renormalization group equations.

3.5 Renormalization Group

So far we have always completely integrated out heavy particles and expanded the resulting non local interaction in an infinite tower of local interactions. In this section we will see what happens if we only integrate out a momentum slice $\delta\Lambda$ and look at the changed that comes from this. This means that in going from a theory with a cutoff of Λ to one with a cutoff $\Lambda - \delta\Lambda$

- Particle content is unchanged
- Action is unchanged, only couplings g_i change

For each cutoff we get a set of Wilson coefficients g_i fixed either via matching to the full theory or to experiment.

$$\left.\begin{array}{c} \{g_i(\Lambda)\} \\ \{g_i(\Lambda - \delta\Lambda)\} \\ \vdots \\ \{g_(\Lambda - n\delta\Lambda)\} \end{array}\right\} \text{repeated n times} \tag{3.94}$$

The procedure of subsequent matching of a theory at infinitesimally changed cutoff is called Renormalization Group an it relates the coupling of a theory at different scales

$$\Lambda \frac{dg_i}{d\Lambda} = f(\{g_i\}). \tag{3.95}$$

As an example let us at a toy model

$$\mathcal{L} = \frac{1}{2}\phi(x)\Big[-m^2 - \Box + c\Box^2 + \dots\Big]\phi(x), \tag{3.96}$$

where again we have normalized the Lagrangian to the kinetic term. This interaction is quadratic in the fields an we can solve the generating functional analytically.

$$\phi(x) = \int \widetilde{dk} e^{-ikx} \tilde{\phi}(k). \tag{3.97}$$

The action in D dimensions reads

$$S = \frac{1}{2} \int d^D x \int \widetilde{dp} \int \widetilde{dk} \tilde{\phi}(p) \Big[-m^2 + k^2 + ck^4 + \dots \Big] \tilde{\phi}(k) e^{i(p+k)x} \tag{3.98}$$

$$= \frac{1}{2} \int \widetilde{dk} \tilde{\phi}(-k) \Big[-m^2 + k^2 + ck^4 + \dots \Big] \tilde{\phi}(k). \tag{3.99}$$

We are working in a UV-cutoff theory, i.e.

$$\int \widetilde{dk} \to \int_{-\Lambda}^{\Lambda}. \tag{3.100}$$

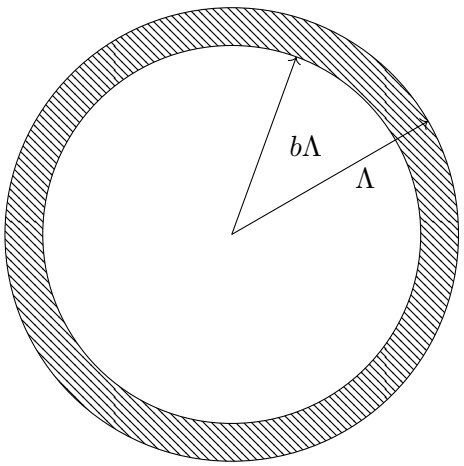
We split the fields into high and low modes

$$\phi = \phi_L + \phi_H, \tag{3.101}$$

in the following way

$$\tilde{\phi}(k) = \tilde{\phi}_L(k) + \tilde{\phi}_H(k) \tag{3.102}$$

$$= \begin{cases} \tilde{\phi}_L(k) & |k| < b\Lambda \\ \tilde{\phi}_H(k) & \Lambda > |k| > b\Lambda \end{cases}. \tag{3.103}$$



For the quadratic interaction the action splits into low and high modes without mixing of the two

$$S = S_L + S_H \tag{3.104}$$

$$= \frac{1}{2}\int_{-b\Lambda}^{b\Lambda} \tilde{\phi}_L(k)[\dots]\tilde{\phi}_L(k) + \frac{1}{2}\int_{-\Lambda}^{\Lambda} \tilde{\phi}_H(k)[\dots]\tilde{\phi}_H(k). \tag{3.105}$$

The Green's functions of the low modes can be written in terms of the generating functional

$$\langle 0|T(\phi_L(x_1)\dots\phi_L(x_n))|0\rangle = \frac{1}{Z}\int \mathcal{D}\phi_L \int \mathcal{D}\phi_H e^{iS_H} e^{iS_L}\phi_L(x_1)\dots\phi_L(x_n), \tag{3.106}$$

$$= \frac{1}{Z_L}\int \mathcal{D}\phi_L e^{iS_L}\phi_L(x_1)\dots\phi_L(x_n). \tag{3.107}$$

Comparing the low energy effective action with the one of the original theory wee see that the transformations

$$k' = \frac{k}{b}, \quad x' = xb \tag{3.108}$$

bring the effective action to the original form with the cutoff at Λ instead of $b\Lambda$, i.e.

$$S_L = \int_{-\Lambda}^{\Lambda} \widetilde{dk'} b^D \tilde{\phi}(-k')[m^2 + b^2 k'^2 + b^4 c k'^4 + \dots]\tilde{\phi}(k'). \tag{3.109}$$

Rescaling the fields

$$\tilde{\phi} \to \tilde{\phi}' b^{-(D+2)/2} \tag{3.110}$$

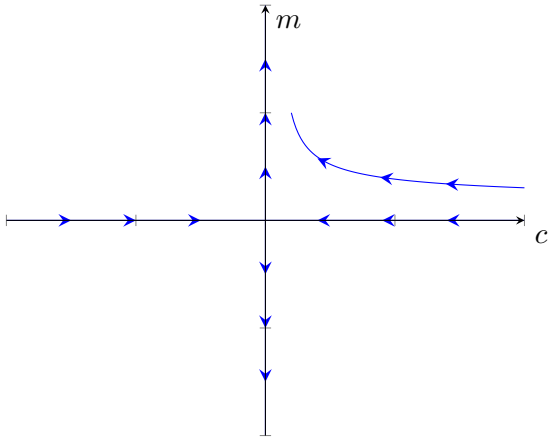
to again normalize the kinetic term, the low energy effective action reads

$$S_L = \int_{-\Lambda}^{\Lambda} \tilde{\phi}'(-k)[\frac{m^2}{b^2} + k'^2 + b^2 c k'^4 + \dots]\tilde{\phi}(k'). \tag{3.111}$$

We can iterate this transformation to establish the renormalization group flow. In Eq. (3.111) we can see as we lower b

- the mass term becomes more important
- the term proportional to c becomes less important.

The flow of the coupling is controlled by b



The point $m = c = 0$, i.e. the free massless scalar field theory has a (*Gaussian*) fixed point. When extending the analysis to the case of weakly coupled theories the results stay predominantly the same, i.e.

- relevant operators stay relevant
- irrelevant operators stay irrelevant

in going from a theory cut off at $\Lambda_1 \to \Lambda_2$. The case of marginal operators is more interesting. Let us look at ϕ^4 theory (this time in euclidean space)

$$Z = \int \mathcal{D}\phi e^{-\int d^D x \left[\frac{1}{2}(\partial_\mu \phi \partial^\mu \phi + m^2\phi^2) + \frac{\lambda}{4!}\phi^4\right]} \tag{3.112}$$

where the measure is understood to be

$$\mathcal{D}\phi = \prod_{|k|<\Lambda} d\tilde{\phi}(k), \tag{3.113}$$

i.e. a cutoff regulated generating functional. We again split the field into low and high modes

$$\tilde{\phi}(k) = \tilde{\phi}_L(k) + \tilde{\phi}_H(k) \tag{3.114}$$

$$\tilde{\phi}_H(k) = \tilde{\phi}(k)\Theta(|k| < \Lambda)\Theta(|k| > b\Lambda). \tag{3.115}$$

For the quadratic part this splitting leads to a simple split in the action, i.e.

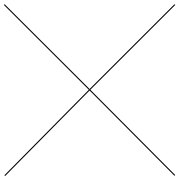
$$S = S_L + S_H \tag{3.116}$$

the interaction part however contains admixtures of low and high modes.

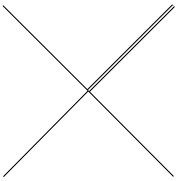
$$\begin{aligned} S(\phi_L + \phi_H) = S(\phi_L) + S(\phi_H) + \int d^4x \lambda \frac{\phi_L^4}{4!} \\ + \int d^4x \lambda \Big[\frac{\phi_L \phi_H^3}{3!} + \frac{\phi_L^2 \phi_H^2}{2!2!} + \frac{\phi_L^3 \phi_H}{3!} \Big] \end{aligned} \tag{3.117}$$

We can derive Feynmanrules for this action

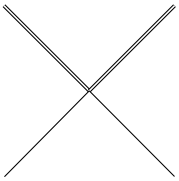
$$\begin{aligned} \Phi_H \quad &= \Delta_H = \langle 0|T(\phi_H(x_1)\phi_H(x_2))|0\rangle \\ &= \int^{\Lambda} \widetilde{dk} e^{ikx} \frac{1}{k^2 + m^2} \Theta(|k| > b\Lambda) \\ \Phi_L \quad &= \Delta_L = \langle 0|T(\phi_L(x_1)\phi_L(x_2))|0\rangle \\ &= \int^{\Lambda} \widetilde{dk} e^{ikx} \frac{1}{k^2 + m^2} \end{aligned} \tag{3.118}$$



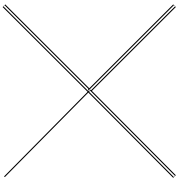
$$= i\lambda \tag{3.119}$$



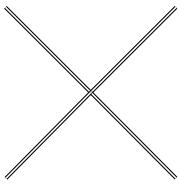
$$= i\lambda \tag{3.120}$$



$$= i\lambda \tag{3.121}$$

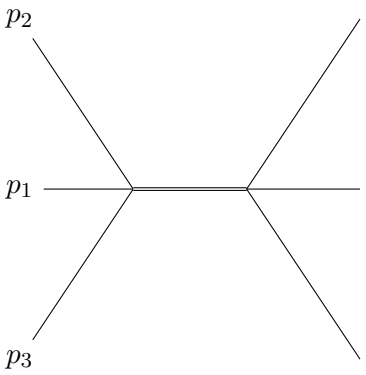


$$= i\lambda \tag{3.122}$$



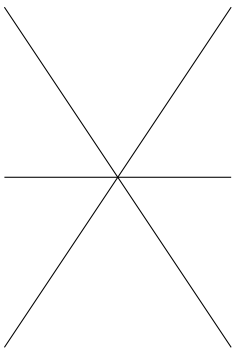
$$= i\lambda \tag{3.123}$$

There is no propagator mixing low and high energy modes. We will calculate the contribution of the high modes in n-point Green's functions of the low modes. At tree level we have



$$= \frac{1}{(p_1+p_2+p_3)^2+m^2)}\Theta(|p_1+p_2+p_3|<b\Lambda) \tag{3.124}$$

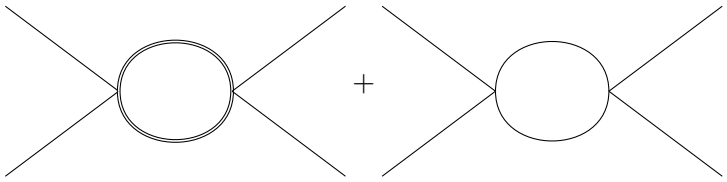
Integrating the high modes leads to a six point interaction



$$\sim \phi_L^6 \sim \frac{1}{(\Lambda b)^2} \tag{3.125}$$

If we are interested in the corrections to the four point function only one

loop corrections arise



(3.126)

Additionally one loop corrections to the external legs arise. We only need to look at the contributions with heavy degrees of freedom. For the first diagram we obtain

$$A = \frac{\lambda^2}{2}\int \frac{d^D k}{(2\pi)^D}\frac{1}{k^2+m^2}\frac{1}{(k+p_1+p_2)^2+m^2}\Theta(|k|>b\Lambda)\Theta(|k|<\Lambda)\Theta(|k+p_1+p_2|>b\Lambda)\Theta(|k+p_1+p_2|< \tag{3.127}$$

We will use $m \ll b\Lambda$ and $p_i \ll b\Lambda$ and Taylor expand the integrand keeping only th eleading order contributions

$$\begin{aligned} A &= \frac{\lambda^2}{2}\int \frac{d^D k}{(2\pi)^D}\frac{1}{(k^2)^2}\Theta(|k|>b\Lambda\Theta(k|<\Lambda) \\ &= \frac{\lambda^2}{2}\frac{\Omega_D}{(2\pi)^D}\int_{b\Lambda}^{\Lambda} k^{D-5} \\ &= \frac{\lambda^2}{2}\frac{\Omega_D}{(2\pi)^D}\underbrace{\frac{\Lambda^{D-4}-(b\Lambda)^{D-4}}{D-4}}_{\ln\frac{1}{b}} \qquad (3.128) \\ &= \frac{\lambda^2}{16\pi^2}\ln\frac{1}{b} \qquad (3.129) \end{aligned}$$

The same contribution comes from the t- and u-channel diagram thus the total contribution is

$$D_{\text{Full}} = \frac{3\lambda^2}{16\pi^2}\ln\frac{1}{b}. \tag{3.130}$$

Now we perform the matching of the two theories, i.e. we want

$$D_{\text{Full}} - D_{\text{LET}} = 0 \tag{3.131}$$

$$\lambda' = \lambda - \frac{3\lambda^2}{16\pi^2}\ln\frac{1}{b} \tag{3.132}$$

The coupling gets smaller when integrating out the high eenergy modes. Performing the matching at decreasing energies

$$d\lambda = \frac{3\lambda^2}{16\pi^2} d\ln b \tag{3.133}$$

$$\int_{\lambda(1)}^{\lambda(b)} \frac{d\lambda}{\lambda^2} = \frac{3\lambda^2}{16\pi^2}[\ln b - \ln 1] \tag{3.134}$$

$$\frac{1}{\lambda(b)} - \frac{1}{\lambda(1)} = \frac{3\lambda^2}{16\pi^2} \ln b \tag{3.135}$$

$$\lambda(b) = \frac{\lambda(1)}{1 + \frac{3\lambda^2}{16\pi^2}\lambda(1) \ln 1/b} \tag{3.136}$$

To see how this integral representation arises let us define $\Delta \equiv \Delta(\ln \mu) = \frac{1}{N} \ln \frac{\mu_2}{\mu_1}$ that is the energy step in going from μ_2 to μ_1 in N matching steps

$$\lambda(\mu_1 e^{\Delta}) = \lambda(\mu_1) + B\lambda^2(\mu_1)\Delta + \mathcal{O}(\lambda^3) \tag{3.137}$$

$$\lambda(\mu_1 e^{2\Delta}) = \lambda(\mu_1 e^{\Delta}) + B\lambda^2(\mu_1 e^{\Delta})\Delta + \mathcal{O}(\lambda^3) \tag{3.138}$$

$$\vdots = \vdots \tag{3.139}$$

$$\lambda(\mu_2) = \lambda(\mu_1) + \sum_{j=0}^{N-1} B\lambda^2(\mu_1 e^{j\Delta}) \tag{3.140}$$

For $N \to \infty$

$$\lambda(\mu_2) = \lambda(\mu_1) + \int_{\mu_1}^{\mu_2} B\lambda^2(\mu) \frac{d\mu}{\mu} \tag{3.141}$$

$$= \lambda(\mu_1) + \int_{\mu_1}^{\mu_2} B\lambda^2(\mu) d\ln \mu \tag{3.142}$$

This is the integral form of the renormalization group equation.

3.6 Naturalness

In or discussion so far we have tacitly assumed that the dimensionless couplings are of natural size. We have seen that mass terms generate problems

in the EFT picture, as they have positive mass dimensions and by dimensional arguments we have related the mass of the light particle to the scale of new physics Eq. (3.36) if we maintain the naturalness of the coupling. There are some caveats to this naive arguments and it is not clear whether or not this constitutes a real problem. I will try to lay out what the main issues are. Let me start by a definition of naturalness

> "The naturalness criterion states that a parameter is allowed to be much smaller than unity only if setting it to zero increases the symmetry of the theory. If this does not happen, the theory is unnatural." t'Hooft 1979

We would need a symmetry that *protects* our particles from acquiring a mass on the order of the cutoff scale. For fermions there is such a symmetry, as we will later see, however for scalars there is not. The naturalness problem is present for scalar particles, e.g. in the SM the Higgs particle is affected by this. Here the problem is called the hierarchy problem and it relates to the question that if the SM is an EFT, why is the Higgs so much lighter then the expected scale of new physics.