## Chapter 4

## Bottom-Up Approach

The construction of the Wilsonian effective action is very intuitive, however it might be very complicated to perform the explicit derivation of the action. The good news is we do not need to go through all the steps. Instead of introducing a cutoff and integrating out heavy degrees of freedom wee treat the effective theory as a normal continuum effective theory. Why the notion of continuum? The hard UV cutoff can be regarded as introducing a lattice to regularize the theory, with finite lattice spacing.

How do we formulate the effective theory

- First fix the particle content of the theory, i.e. the relevant degrees of freedom. So far we looked at scenarios of heavy particle that are weakly coupled, where the relevant degrees of freedoms have been all particles with masses below the heavy particle's mass. There are nontrivial examples like QCD, where at low energies the degrees of freedomare the hadrons, i.e. $\pi, K, \rho, \Delta$, protons NOT quarks,
- Construct the most general Lagrangian consistent with the symmetries of the theory. Give a power counting scheme ordering the interaction terms, e.g. dimension of operators,
- Matching (possibly) where we determine the coupling constants of the low energy n-point Green's function in the full and low energy theory. We expand the full theory result according to the rules of power counting and match the coefficients such that results agree, or
Fix the unknown low energy constants (LEC) to physical observables and use the so obtained values in other amplitudes.
- Use renormalization group techniques to resum large logs. We will see that it can be problematic to directly match at a given scale the expressions obtained from perturbation theory. The idea is to match at a scale where perturbation theory works and run the coupling down to a lower scale using renormalization group equations.

In the following we will use a mass independent renormalization scheme, i.e. MS. It is counterintuitive to not use a cutoff and integrate over momenta for which the effective theory is not even valid. However we have seen that this still can be done, as changing the UV should not affect the low energy, i.e. the only changes will be to the renormalized couplings absorbing potential power counting violating terms.

Correct low energy behavior $\leftrightarrow$ absolutely wrong high energy behavior

### 4.1 Construction of EFTs

Let us start wit a toy model, the scalar theory of the last chapter

$$
\begin{align*}
\mathcal{L} & =\frac{1}{2} \partial_{\mu} \phi_{L} \partial^{\mu} \phi_{L}-\frac{m^{2}}{2} \phi_{L}^{2}+\frac{1}{2} \partial_{\mu} \phi_{H} \partial^{\mu} \phi_{H}-\frac{M^{2}}{2} \phi_{H}^{2} \\
& -\frac{\lambda_{L}}{4!} \phi_{L}^{4}-\frac{\lambda_{H}}{4!} \phi_{H}^{4}-\frac{\lambda_{H L}}{2!2!} \phi_{L}^{2} \phi_{H}^{2}-\frac{g}{2!} \phi_{H} \phi_{L}^{2} . \tag{4.1}
\end{align*}
$$

Eq. (4.1) has a $\mathbb{Z}_{2}$-symmetry for the low energy degrees of freedom, i.e. $\phi_{L} \rightarrow-\phi_{L}$, which leads to only an even number of light fields. Let us construct the EFT given the outline above.

- Identify the degrees of freedomat low energies, $\Rightarrow \phi_{L}$
- Write the effective Lagrangian incorporating $\mathbb{Z}_{2}$-symmetry

$$
\begin{align*}
\mathcal{L}_{\text {eff }} & =\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi-\frac{\tilde{m}^{2}}{2} \phi^{2}-\frac{\tilde{\lambda}}{4!} \phi^{4} \\
& -\frac{c}{M^{2}} \phi \square^{2} \phi-\frac{1}{6!} \frac{\tilde{c}}{M^{2}} \phi^{6}-\frac{1}{4!} \frac{c^{\prime}}{M^{2}} \phi^{2} \square \phi^{2}+\mathcal{O}\left(\frac{1}{M^{4}}\right) \tag{4.2}
\end{align*}
$$

For power counting we will use the dimension of the operators.

- Matching will be done using the 2 -, 4 - and 6 -point functions.

Matching at tree level.

## EFT:

$$
\begin{align*}
i \Gamma_{2} & =-\left(p^{2}-\tilde{m}^{2}\right)  \tag{4.3}\\
i \Gamma_{4} & =\tilde{\lambda}-\frac{c^{\prime}}{M^{2}} \frac{1}{3}\left(\left(p_{1}+p_{2}\right)^{2}+\left(p_{1}-p_{3}\right)^{2}\left(p_{1}-p_{4}\right)^{2}\right)  \tag{4.4}\\
i \Gamma_{6} & =\frac{\tilde{c}}{M^{2}}+\ldots \tag{4.5}
\end{align*}
$$

Full theory:

$$
\begin{align*}
i \Gamma_{2} & =p^{2}-m^{2} \Rightarrow \tilde{m}=m+\mathcal{O}(\lambda)  \tag{4.0}\\
i \Gamma_{4} & =0 \Rightarrow c=0+\mathcal{O}(\lambda)  \tag{4.7}\\
i \Gamma_{6} & =\lambda_{L}+i(i g)^{2}\left(\frac{i}{\left(p_{1}+p_{2}\right)^{2}-M^{2}}+\frac{i}{\left(p_{1}-p_{3}\right)^{2}-M^{2}}+\frac{i}{\left(p_{1}-p_{4}\right)^{2}-M^{2}}\right) \tag{4.8}
\end{align*}
$$

$$
\begin{equation*}
=\lambda_{L}-\frac{3 g^{2}}{M^{2}}-\frac{g^{2}}{M^{4}}\left[\left(p_{1}+p_{2}\right)^{2}+\left(p_{1}-p_{3}\right)^{2}+\left(p_{1}-p_{4}\right)^{2}\right]+\ldots \tag{4.9}
\end{equation*}
$$

$$
\begin{equation*}
\tilde{\lambda}=\lambda_{L}-\frac{3 g^{2}}{M^{2}} \tag{4.10}
\end{equation*}
$$

$$
\begin{equation*}
c^{\prime}=\frac{3 g^{2}}{M^{2}} \tag{4.11}
\end{equation*}
$$

where we used the Taylor expansion for the heavy propagator. For the 6 -point function at tree level we have


The first line contains one light particle reducible diagrams only, i.e. cutting through the propagator of a light degrees of freedomgenerates diagrams of light degrees of freedomwith less external legs. In the above example cutting a light degrees of freedomline generates $2 \phi_{L}^{4}$ vertices. The contribution of
the second line contribute to the matching of the six point function. Since there are not derivatives in the Lagrangian for $\tilde{c}$ we can set the momenta to zero in the matching calculation. We obtain

$$
\frac{\tilde{c}}{M^{2}}=i(-i g)^{2}\left(-i \lambda_{H L}\right)\left(\frac{i}{-M^{2}}\right)^{2} \cdot 45 \Rightarrow \tilde{c}=45 \lambda_{H L} \frac{g^{2}}{M^{2}}
$$

This completes the matching at tree level. The factor 45 is combinatorial factor and is to do with the possible contractions leading to the same diagram. Looking at the operators

$$
\begin{equation*}
\phi_{L}^{2}(x) \phi_{H}(x), \quad \phi_{L}^{2}(y) \phi_{H}^{2}(y), \quad \phi_{L}^{2}(z) \phi_{H}(z) \tag{4.13}
\end{equation*}
$$

thus the total number of contractions with external light degrees of freedomis 6 !. Out of these 6 ! contractions $4 \times 2 \times 2 \times 2$ give the same contribution.

### 4.2 Representation Independence

After matching is performed all off-shell n-point Green's functions of the full and effective theory are exactly the same at low energies. However only physical matrix elements are unambiguously defined, i.e. only the on-shell Green's functions. There is an ambiguity in the off-shell Green's functions, which we will exploit to simplify the Lagrangian.
Example

$$
\begin{align*}
\phi^{\prime} & =\left(1+\frac{\alpha}{M^{2}} \square\right) \phi \\
\mathcal{L}_{\mathrm{Eff}} & =\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi-\frac{m^{2}}{2} \phi^{2}-\frac{\tilde{\lambda}}{4!} \phi^{4}-\frac{c}{2 M^{2}} \phi \square^{2} \phi \\
& -\frac{1}{6!} \frac{\tilde{c}}{M^{2}} \phi^{6}-\frac{c^{\prime}}{4!M^{2}} \phi^{2} \square \phi^{2}, \\
\mathcal{L}_{\mathrm{Eff}}^{\prime} & =\mathcal{L}_{\mathrm{Eff}}-\frac{\alpha}{M^{2}} \phi \square\left(\square+m^{2}+\frac{\tilde{\lambda}}{3!} \phi^{2}\right) \phi, \tag{4.14}
\end{align*}
$$

where in the last line we have only kept terms up to order $\alpha$. If we compare the two Lagrangians we see that for the choice $\alpha=-c / 2$ we effectively cancel the term proportional to $c$, i.e. $\mathcal{L}_{\text {Eff }}^{\prime}$ free of $-\frac{c}{2 M^{2}} \phi \square^{2} \phi$.

$$
\begin{equation*}
\mathcal{L}_{\mathrm{Eff}}^{\prime}=\left.\mathcal{L}_{\mathrm{Eff}}\right|_{c=0}+\frac{c}{2 M^{2}} \phi\left(m^{2}+\frac{\tilde{\lambda}}{3!} \phi^{2}\right) \phi \tag{4.15}
\end{equation*}
$$

Let us look at what we would have obtained by using the leading order equation of motion, i.e.

$$
\begin{equation*}
\left(\square+m^{2}+\frac{\tilde{\lambda}}{3!} \phi^{2}\right) \phi=0 \tag{4.16}
\end{equation*}
$$

Using Eq. 4.16) in Eq. (4.14) we arrive at the same solution Eq. 4.15). In general a field redefinition of the form

$$
\begin{equation*}
\phi^{\prime}=\phi+\left(\frac{1}{M^{2}}\right)^{b} f(\phi)=\phi+\delta \phi \tag{4.17}
\end{equation*}
$$

leads to

$$
\begin{equation*}
\mathcal{L}_{\mathrm{Eff}}^{\prime}=\mathcal{L}_{\mathrm{Eff}}+\left(\frac{1}{M^{2}}\right)^{n} f(\phi)[\underbrace{\square \phi+m^{2}+\frac{\lambda}{3!} \phi^{2}}_{\mathrm{EOM}}]+\mathcal{O}\left(\left(\frac{1}{M^{2}}\right)^{n+1}\right) \tag{4.18}
\end{equation*}
$$

This means that one can use classical equations of motion to reduce the number of independent terms in the effective Lagrangian. In general the EOM only hold on the classical level however we can show that the effect of EOM is achieved by field redefinitions. Under certain assumptions this redefinition leaves physics untouched and is justified.

Generalized theorem
Field redefinitions that preserve symmetries and have the same one particle states allow classical EOM to be used to simplify local effective field theories without changing physics.

In the following we consider the following local EFT Lagrangian

$$
\begin{equation*}
\mathcal{L}_{\mathrm{Eff}}=\sum_{n=0}^{\infty} c_{n} \mathcal{L}^{(n)} \tag{4.19}
\end{equation*}
$$

which is a function of the complex scalar field $\phi$, i.e.

$$
\begin{equation*}
\mathcal{L}^{(0)}=\left(\partial_{\mu} \phi\right)^{\dagger} \partial^{\mu} \phi-m^{2} \phi^{\dagger} \phi+\text { h.c. } \tag{4.20}
\end{equation*}
$$

The goal is to remove a generic term like $c_{1} f(\phi) \square \phi$ from $\mathcal{L}^{(1)}$. The theory is described by the generating functional

$$
\begin{align*}
Z[j] & =\int \prod_{i} d \phi_{i} \exp \left\{i \int d ^ { D } x \left[\mathcal{L}^{(0)}+c_{1}\left(\mathcal{L}^{(1)}-f(\phi) \square \phi\right)+c_{1} f(\phi) \square \phi\right.\right. \\
& \left.\left.+\sum_{k} j_{k} \phi_{k}+\ldots\right]\right\} \tag{4.21}
\end{align*}
$$

where we have just added a zero. All Green's functions can be obtained as functional derivatives from this generating functional. Let's rewrite the generating functional using the field redefinition

$$
\begin{align*}
& \phi^{\dagger}=\phi^{\prime \dagger}+c_{1} f(\phi)  \tag{4.22}\\
& Z[j]=\int \prod_{i} d \phi_{i}^{\prime}\left|\frac{\delta \phi^{\dagger}}{\delta \phi^{\dagger \dagger}}\right| \exp \left\{i \int d ^ { D } x \left[\mathcal{L}^{(0)}+c_{1} f(\phi)\left[\frac{\partial \mathcal{L}^{(0)}}{\partial \phi^{\dagger}}-\partial_{\mu} \frac{\partial \mathcal{L}^{(0)}}{\partial \partial_{\mu} \phi^{\dagger}}\right]\right.\right. \\
&\left.\left.c_{1}\left(\mathcal{L}^{(1)}-f(\phi) \square \phi^{\prime}\right)+c_{1} f(\phi) \square \phi^{\prime}+\sum_{k} j_{k} \phi_{k}+j_{\phi^{\dagger}} c_{1} f(\phi) \ldots\right]\right\} \tag{4.23}
\end{align*}
$$

The field redefinitions gives rise to a Jacobian and a new source term. We will analyze these two contributions in the following. First let us look at the change of the Lagrangian

$$
\begin{align*}
\mathcal{L}^{(n)} & =\left(\partial_{\mu} \phi\right)^{\dagger} \partial^{\mu} \phi-m^{2} \phi^{\dagger} \phi+(\ldots)  \tag{4.24}\\
\mathcal{L}^{(n)} & =\left(\partial_{\mu} \phi^{\prime}\right)^{\dagger} \partial^{\mu} \phi^{\prime}-m^{2} \phi^{\prime \dagger} \phi^{\prime}+c_{1} f\left(-\square \phi^{\prime}-m^{2} \phi^{\prime}\right)+(\ldots)^{\prime} \tag{4.25}
\end{align*}
$$

Since the transformation is assumed to preserve symmetries terms that are in $(\ldots)^{\prime}$ are already present in (...). Next we look at the Jacobian where we will use the trick

$$
\begin{equation*}
\operatorname{det} \partial^{\mu} D_{\mu}=\int \mathcal{D} c \mathcal{D} \bar{c} \exp \left\{i \int d^{4} x \bar{c}\left[-\partial^{\mu} D_{\mu}\right] c\right\} \tag{4.26}
\end{equation*}
$$

which relates positive power of the determinant of an operator to the gaussian integral over Grassmann valued fields. These fields are bosons with the wrong statistics, they are called Ghosts. The procedure was first introduced by Fadeev and Popov. Using

$$
\begin{equation*}
\frac{\delta \phi^{\dagger}}{\delta \phi^{\prime \dagger}}=1+c_{1} \frac{\delta f}{\delta \phi^{\prime \dagger}} \tag{4.27}
\end{equation*}
$$

and rewriting the determinant using Ghost degrees of freedomwe obtain the following contribution to the Lagrangian

$$
\begin{equation*}
\mathcal{L}_{\text {Ghost }}=\bar{c} c+c_{1} \bar{c} \frac{\delta f}{\delta \phi^{\prime \dagger}} c . \tag{4.28}
\end{equation*}
$$

Recall that the EFT is only valid for $p^{2} \ll \Lambda_{\text {New }}^{2}$ which leads to

$$
\begin{equation*}
\Lambda_{\mathrm{New}} \approx \frac{1}{\sqrt{c_{1}}} \tag{4.29}
\end{equation*}
$$

this means the ghosts will have masses of the order of the scale of new physics. Essentially the ghosts decouple just as the other heavy degrees of freedom. We can see this for an explicit transformation

$$
\begin{align*}
& f(\phi)=\square \phi^{\dagger}+\lambda \phi^{\dagger} \phi^{\dagger} \phi  \tag{4.30}\\
& \Rightarrow \bar{c}\left(1+c_{1} \square+2 c_{1} \lambda \phi^{\dagger} \phi\right) c \tag{4.31}
\end{align*}
$$

we will rescale the ghost fields to normalize the kinetic term, i.e. $c^{\prime}=\frac{c}{\sqrt{c_{1}}}$

$$
\begin{equation*}
\bar{c}^{\prime}\left(\Lambda^{2}+\square+2 \lambda \phi^{\dagger} \phi\right) c^{\prime} \tag{4.32}
\end{equation*}
$$

For this argument to hold we need at least one $\phi^{\dagger}$ in the transformation $f$. Another point of view is that the ghosts will only appear in loops, where if we use a generic transformation

$$
\begin{equation*}
\phi=\phi^{\prime}+\left(\frac{1}{M^{2}}\right)^{n} f\left(\phi^{\prime}\right) \tag{4.33}
\end{equation*}
$$

then we can rewrite

$$
\begin{align*}
\int \mathcal{D} \phi & =\int \mathcal{D} \phi^{\prime} \operatorname{det} \frac{\delta \phi}{\delta \phi^{\prime}}  \tag{4.34}\\
\frac{\delta \phi(x)}{\delta \phi^{\prime}\left(x^{\prime}\right)} & =\delta\left(x-x^{\prime}\right)+\left(\frac{1}{M^{2}}\right)^{n} f^{\prime}\left(\phi^{\prime}(x)\right) \delta\left(x-x^{\prime}\right)+\ldots  \tag{4.35}\\
\operatorname{det} \frac{\delta \phi(x)}{\delta \phi^{\prime}\left(x^{\prime}\right)} & =\int \mathcal{D} \bar{c} \int \mathcal{D} c \exp \{i \int d^{D} x \bar{c}(x)[1+\underbrace{\left(\frac{1}{M^{2}}\right)^{n} f^{\prime}\left(\phi^{\prime}\right)}_{\text {small perturbation }}] c(x)\} \tag{4.36}
\end{align*}
$$

This means the ghosts enter in the calculations with propagators of the form

$$
\begin{align*}
D_{\text {Ghost }} & =\frac{i}{1}  \tag{4.37}\\
\int d^{D} k\left(\frac{i}{1}\right)^{n} & =0 \tag{4.38}
\end{align*}
$$

where in the last line we have calculated the contribution of $n$ ghost field propagators in dimensional regularization to be zero (property of scaleless integrals). Next we will address the additional source term. We consider the $n$-point Green's functions

$$
\begin{equation*}
G^{(n)}=\langle 0| T\left(\phi\left(x_{1}\right) \ldots \phi\left(x_{n}\right)\right)|0\rangle . \tag{4.39}
\end{equation*}
$$

We will assume real fields right now however everything applies to complex fields as well. The Green's function under the field redefinition takes the form

$$
\begin{equation*}
G^{(n)}=\langle 0| T\left(\left(\phi\left(x_{1}\right)+c_{1} f^{x_{1}}\right) \ldots\left(\phi\left(x_{n}\right)+c_{1} f^{x_{n}}\right)\right)|0\rangle . \tag{4.40}
\end{equation*}
$$

As a reminder recall the LSZ expression

$$
\begin{array}{r}
\int d^{4} x_{i} e^{ \pm i p_{i} x_{i}}\langle 0| T\left(\phi\left(x_{1}\right) \ldots \phi\left(x_{n}\right)\right)|0\rangle \\
\sim\left(\prod_{i} \frac{\sqrt{Z} i}{p_{i}^{2}-m_{i}^{2}+i 0}\right) \underbrace{\left\langle p_{1}, p_{2} \ldots\right| S\left|p_{j}, p_{j}+1 \ldots\right\rangle}_{\text {Observable }} \tag{4.41}
\end{array}
$$

The statement that physics ought to be unchanged means that the observables may not be affected by the field redefinitions. Let us look at some examples

$$
\begin{equation*}
\phi=\phi+c_{1} \phi=\left(1+c_{1}\right) \phi \tag{4.42}
\end{equation*}
$$

For the four point functions this means

$$
\begin{equation*}
G^{\prime}=\left(1+c_{1}\right)^{4}\langle 0| T\left(\phi\left(x_{1}\right) \phi\left(x_{2}\right) \phi\left(x_{3}\right) \phi\left(x_{4}\right)\right)|0\rangle . \tag{4.43}
\end{equation*}
$$

This change in the Green's functions under the field redefinition is compensated by the accompanying shift in the Wavefunction renormalization. Let us look at a more complicated example,

$$
\begin{equation*}
\phi \rightarrow\left(1+\frac{\alpha}{M^{2}} \square\right) \phi \tag{4.44}
\end{equation*}
$$

The physical matrix element in the original theory reads

$$
\begin{equation*}
\mathcal{G}=-\lambda \tag{4.45}
\end{equation*}
$$

After the field redefinition the change in the Lagrangian is

$$
\begin{equation*}
\delta \mathcal{L}=-\frac{\alpha}{M^{2}} \phi \square\left(\square+m^{2} \frac{\lambda}{3!} \phi^{2}\right) \phi \tag{4.46}
\end{equation*}
$$

The matrix element with the new Lagrangian reads

$$
\begin{align*}
\mathcal{G} & =-\lambda\left(1-\frac{\alpha}{M^{2}} \frac{1}{3!} 3!\sum_{i=1}^{4} p_{i}^{2}\right)\left(Z^{1 / 2}\right)^{4}  \tag{4.47}\\
& =-\lambda\left(1-\frac{\alpha}{M^{2}} 4 m^{2}\right)\left(Z^{1 / 2}\right)^{4} \tag{4.48}
\end{align*}
$$

where the last term comes from LSZ. The wavefunction renormalization must cancel the change for physics to be unchanged. Let us calculate the wavefunction renormalization, that is the residue of the dressed propagator.

$$
\begin{align*}
D & \frac{i}{p^{2}-m^{2}}+\frac{2 \alpha}{M^{2}} \frac{i}{p^{2}-m^{2}}\left(-i p^{2}\right)\left(p^{2}-m^{2}\right) \frac{i}{p^{2}-m^{2}}  \tag{4.49}\\
& =\frac{i}{p^{2}-m^{2}}\left(1+\frac{2 \alpha}{M^{2}} p^{2}\right)  \tag{4.50}\\
& =\frac{i}{p^{2}-m^{2}}\left(1+\frac{2 \alpha}{M^{2}} m^{2}\right)+\text { non-pole-terms }  \tag{4.51}\\
Z & =\left(1+\frac{2 \alpha}{M^{2}} m^{2}\right)  \tag{4.52}\\
Z^{2} & =\left(1+\frac{2 \alpha}{M^{2}} m^{2}\right)^{2}=1+\frac{4 \alpha}{M^{2}} m^{2}+\mathcal{O}\left(1 / M^{4}\right) \tag{4.53}
\end{align*}
$$

Insert this WFR constant into $\mathcal{G}$ gives

$$
\begin{equation*}
\mathcal{G}=-\lambda+\mathcal{O}\left(1 / M^{4}\right) \tag{4.54}
\end{equation*}
$$

which is indeed the same matrix element.

### 4.3 Matching \& Loops

Let us recall the matching procedure, which comprises of calculating the n-point Green's functions of the degrees of freedomin the full and effective theory and then demanding that these are the same at low energies. For tree level it is intuitive this procedure works, as everything in the full theory is polynomial in external momenta and the heavy degrees of freedomare always far from being on shell. The EFT is local i.e. all polynomial terms, that respect the symmetries,, are present therefor matching will always work. However this is not so clear with loops? We have several problems to address here

- The loop momentum $k$ can be such that heavy degrees of freedomare close to on-shell
- UV-divergences need to be dealt with (both in full theory and EFT)
- Loop contributions are not polynomial in momenta. $\mathcal{L}_{\text {eff }}$ is local, i.e. Wilson coefficients can only absorb polynomial dependence. This means the nonanalytic parts have to be the same in EFT and full theory.
- Due to renormalization Wilson coefficients will depend on renormalization scale $\mu$. This will lead to problems with logarithms becoming large at low energies.

We will address the above points in the following, where we perform matching at one loop. Let us compute the two point function to one loop, i.e. in the full theory we have the following diagrams
(a)
(b)
(c)
(d)


In the EFT the only diagram contributing at one loop is


Let us calculate the full theory result first. Diagram (a)

$$
\begin{align*}
\Sigma^{(a)} & =i\left(-i \lambda_{L}\right) \frac{1}{2} \int \frac{d^{D} k}{(2 \pi)^{D}} \frac{i}{k^{2}-m^{2}} \mu^{2 \varepsilon}  \tag{4.57}\\
& =\frac{\lambda_{L}}{2}(4 \pi)^{-D / 2} \Gamma\left(1-\frac{D}{2}\right)\left(m^{2}\right)^{D / 2-1} \mu^{2 \varepsilon}  \tag{4.58}\\
& =\frac{m^{2} \lambda_{L}}{32 \pi^{2}}[\underbrace{-\frac{1}{\varepsilon}+\gamma_{\mathrm{E}}-\ln 4 \pi}_{\overline{\mathrm{MS}} \equiv 0}-1+\ln \frac{m^{2}}{\mu^{2}}]+\mathcal{O}(\varepsilon) \tag{4.59}
\end{align*}
$$

One can redefine $\mu$ to absorb the additional finite (nonpole in $\varepsilon$ ) parts

$$
\begin{equation*}
\tilde{\mu}^{2}=\mu^{2} e^{\gamma_{\mathrm{E}}} 4 \pi \tag{4.61}
\end{equation*}
$$

This leads to

$$
\begin{equation*}
\Sigma^{(a)}=\frac{m^{2} \lambda_{L}}{32 \pi^{2}}\left[-\frac{1}{\varepsilon}-1+\ln \frac{m^{2}}{\tilde{\mu}^{2}}\right] \tag{4.62}
\end{equation*}
$$

Diagram (b)

$$
\begin{align*}
& =i \Sigma^{(b)}  \tag{4.63}\\
\Sigma^{(b)} & =\frac{\lambda_{H L}}{32 \pi^{2}} M^{2}\left[-\frac{1}{\varepsilon}-1+\ln \frac{M^{2}}{\tilde{\mu}^{2}}\right] \tag{4.64}
\end{align*}
$$

Diagram (c)

$$
\begin{align*}
\Sigma^{(c)} & =\frac{i}{2}\left(-i g^{2}\right) \frac{i}{-M^{2}} \int \frac{d^{D} k}{(2 \pi)^{D}} \frac{i}{k^{2}-m^{2}} \tilde{\mu}^{2 \varepsilon}  \tag{4.65}\\
& =\frac{g^{2}}{32 \pi^{2}} \frac{m^{2}}{M^{2}}\left[+\frac{1}{\varepsilon}+1-\ln \frac{m^{2}}{\tilde{\mu}^{2}}\right] \tag{4.66}
\end{align*}
$$

Diagram (d) is the first non-trivial one

$$
\begin{equation*}
\Sigma^{(d)}=i\left(i g^{2}\right) \int \frac{d^{D} k}{(2 \pi)^{D}} \frac{i^{2} \tilde{\mu}^{2}}{\left[(k+p)^{2}-m^{2}\right]\left[k^{2}-M^{2}\right]} \tag{4.67}
\end{equation*}
$$

For the purpose of matching we do not need the complete expression for the above integral, we only need the parts polynomial in external momenta and the light mass. This is what makes matching worthwhile.

$$
\begin{equation*}
\Sigma^{(d)}=\frac{g^{2}}{16 \pi^{2}}\left[-\frac{1}{\varepsilon}-1+\ln \frac{M^{2}}{\tilde{\mu}^{2}}\right]+\frac{g^{2}}{16 \pi^{2}}\left[-\frac{p^{2}}{2 M^{2}}-\frac{m^{2}}{M^{2}} \ln \frac{m^{2}}{M^{2}}+\mathcal{O}\left(1 / M^{4}\right)\right] \tag{4.68}
\end{equation*}
$$

The diagram in the EFT is

$$
\begin{equation*}
\Sigma^{\mathrm{EFT}}=\frac{\tilde{\lambda}}{32 \pi^{2}} \tilde{m}^{2}\left[-\frac{1}{\varepsilon}-1+\ln \frac{\tilde{m}^{2}}{\tilde{\mu}^{2}}\right] \tag{4.69}
\end{equation*}
$$

Now we need to deal with the divergences in the full and the EFT. To that end we use $\overline{\mathrm{MS}}$ prescription and drop the divergent parts, i.e. $\frac{1}{\varepsilon}$. This amounts to a redefinition of the appropriate bare couplings in the Lagrangian. We will use the same approach in the full and effective theory.

After the Green's functions have been rendered finite we demand that the two expressions coincide, i.e.

$$
\begin{align*}
\Delta & =\Sigma_{\text {full }}-\Sigma_{\text {eft }}  \tag{4.70}\\
& =0 \tag{4.71}
\end{align*}
$$

which gives a condition on the coupling in the full and effective theory. More exactly we do not demand that the amputated Green's functions coincide but rather the physical matrix elements, i.e. we still need to apply the LSZ formalism. For the effective Lagrangian we can write

$$
\begin{equation*}
\mathcal{L}_{\text {eff }}=\frac{1}{2} \partial_{\mu} \phi^{0} \partial^{\mu} \phi^{0}-\frac{\tilde{m}^{0}}{2}\left(\phi^{0}\right)^{2}-\frac{\tilde{\lambda}^{0}}{4!}\left(\phi^{0}\right)^{4} \tag{4.72}
\end{equation*}
$$

where we have indicated the bare coupling and fields with an index 0 . We will replace the bare fields and couplings with renormalized expressions

$$
\begin{align*}
\phi^{0} & =\phi Z_{\phi}^{1 / 2}  \tag{4.73}\\
\tilde{m}^{0} & =\tilde{m} Z_{m}  \tag{4.74}\\
\tilde{\lambda}^{0} & =\tilde{\lambda} Z_{\lambda} \tag{4.75}
\end{align*}
$$

Inserting these into the effective Lagrangian

$$
\begin{equation*}
\mathcal{L}_{\text {eff }}=\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi Z_{\phi}-\frac{\tilde{m}}{2} Z_{\phi} Z_{m}(\phi)^{2}-\frac{\tilde{\lambda}}{4!} Z_{\lambda} Z_{\phi}^{2}(\phi)^{4} \tag{4.76}
\end{equation*}
$$

Expanding everything to one loop order, i.e.

$$
\begin{align*}
Z_{m} & =1+\hbar \delta m  \tag{4.77}\\
Z_{\lambda} & =1+\hbar \delta \lambda  \tag{4.78}\\
Z_{\phi} & =1+\hbar \delta Z  \tag{4.79}\\
Z_{\phi}^{2} & =1+2 \hbar \delta Z \tag{4.80}
\end{align*}
$$

The WFR $Z_{\phi}$ happens to be 1 to one loop order in the EFT, that is the self energy is independent of $p^{2}$. For the two point function we have

$$
\begin{align*}
\Delta & =-\tilde{m}^{2}+p^{2}+(1+\delta Z)\left(m^{2}-p^{2}\right)+\Sigma_{\text {full }}-\Sigma_{\text {eft }}  \tag{4.81}\\
& =0 \tag{4.82}
\end{align*}
$$

An important part is that the difference of the one loop expression has to be polynomial in the external momenta and masses, i.e. the nonanalytic parts,
e.g. $\ln m$ have to cancel between full and effective theory. We can check that this is true, i.e. the coefficient of $\Delta$ w.r.t. $\ln m$ reads

$$
\begin{equation*}
\Delta_{\ln }=\frac{m^{2}}{16 \pi^{2}} \ln m\left[\lambda_{L}-\frac{3 g^{2}}{M^{2}}-\tilde{\lambda}\right] \ldots \tag{4.83}
\end{equation*}
$$

This term has to vanish and indeed if we use the relation we found at tree order, $\tilde{\lambda}=\lambda_{L}-\frac{3 g^{2}}{M^{2}}$ the nonanalytic terms vanish. Let us look at the rest

$$
\begin{align*}
\Delta & =-\tilde{m}^{2}+(1+\delta Z) m^{2}+\frac{1}{16 \pi^{2}}\left\{\frac{m^{2}}{2} \lambda_{L}\left(-1+\ln \frac{m^{2}}{\tilde{\mu}^{2}}\right)+\frac{M^{2}}{2} \lambda_{H L}\left(-1+\ln \frac{M^{2}}{\tilde{\mu}^{2}}\right)\right. \\
& +\frac{g^{2}}{2 M^{2}} m^{2}\left(1-\ln \frac{m^{2}}{\tilde{\mu}^{2}}\right)+g^{2}\left(-1+\ln \frac{M^{2}}{\tilde{\mu}^{2}}\right)+\frac{g^{2} m^{2}}{M^{2}}\left(-1-\ln \frac{m^{2}}{M^{2}}\right) \\
& \left.-\frac{\tilde{\lambda}}{2} \tilde{m}^{2}\left(-1+\ln \frac{\tilde{m}^{2}}{\tilde{\mu}^{2}}\right)\right\} \\
& +p^{2}-(1+\delta Z) p^{2}-\frac{g^{2}}{32 \pi^{2}} \frac{p^{2}}{M^{2}} \tag{4.84}
\end{align*}
$$

From the last line we can infer

$$
\begin{equation*}
\delta Z=-\frac{1}{32 \pi^{2}} \frac{g^{2}}{M^{2}} \tag{4.85}
\end{equation*}
$$

which in principle can be computed from the self energy.

$$
\begin{equation*}
\left.\frac{d \Sigma_{\text {full }}}{d p^{2}}\right|_{p^{2}=m^{2}}=-\frac{g^{2}}{32 \pi^{2}} \frac{1}{M^{2}} \tag{4.86}
\end{equation*}
$$

As we have already discussed the terms proportionl to $\ln m$ vanish thus

$$
\begin{align*}
0 & =-\tilde{m}^{2}+\left(1-\frac{1}{32 \pi^{2}} \frac{g^{2}}{M^{2}}\right) m^{2}+\frac{1}{16 \pi^{2}}\left\{\frac{m^{2}}{2} \lambda_{L}(-1)+\frac{M^{2}}{2} \lambda_{H L}\left(-1+\ln \frac{M^{2}}{\tilde{\mu}^{2}}\right)\right. \\
& +\frac{g^{2}}{2 M^{2}} m^{2}(1)+g^{2}\left(-1+\ln \frac{M^{2}}{\tilde{\mu}^{2}}\right)+\frac{g^{2} m^{2}}{M^{2}}\left(-1+1-\ln \frac{\tilde{\mu}^{2}}{M^{2}}\right) \\
& \left.-\frac{\lambda_{L}-3 g^{2} / M^{2}}{2} \tilde{m}^{2}(-1)\right\} \\
\tilde{m}^{2} & =\left(1-\frac{1}{32 \pi^{2}} \frac{g^{2}}{M^{2}}\right) m^{2}+\frac{1}{16 \pi^{2}}\left\{g^{2}\left(1+\frac{m^{2}}{M^{2}}\right)+\frac{M^{2}}{2} \lambda_{H L}\right\}\left[-1+\ln \frac{M^{2}}{\tilde{\mu}^{2}}\right] \tag{4.87}
\end{align*}
$$

When expanding to higher orders in $p^{2}$ one obtains more mathcing conditions for the Wilson coefficients of higher order, e.g. $\phi_{L} \square^{2} \phi_{L}$. The number of diagrams to consider rises considerably though. Let us stress again that for purposes of matching we do not need the full expression for the one loop integrals but rather the expansion in the external momenta.

Comment on power counting in cutoff scheme.

### 4.4 Renormalization Group Improved Perturbation Theory

The Wilson coefficient after one loop matching depend on the full theory coupling and on the renormalization scale. The renormalization scale dependence can be generically written as

$$
\begin{align*}
C_{i}(\mu, \lambda, m) & =C_{i}^{(0,0)}+\lambda(\mu)\left[C_{i}^{(1,1)} \ln \frac{m^{2}}{\mu^{2}}+C_{i}^{(1,0)}\right] \\
& +\lambda^{2}(\mu)\left[C_{i}^{(2,2)} \ln ^{2} \frac{m^{2}}{\mu^{2}}+C_{i}^{(2,1)} \ln \frac{m^{2}}{\mu^{2}}+C_{i}^{(2,0)}\right]+\ldots \tag{4.88}
\end{align*}
$$

where $C^{(n, m)}$ denotes the Wilson coefficient of order $n$ in the power counting to loop order $m$, i.e. $m \leq n$. Instead of looking at the two point function we will look at the one loop matching of the $\phi^{4}$ coupling


Matching the finite parts

$$
\begin{equation*}
\tilde{\lambda}(\mu)=\lambda(\mu)+\frac{3 \lambda_{H L}}{32 \pi^{2}}\left[\ln \frac{M^{2}}{\mu^{2}}+c\right] \tag{4.90}
\end{equation*}
$$

A convenient choice for the scale is $\mu \sim M$, since otherwise the log-term can become large. The one loop 4 point function in the EFT is given by

where now $\tilde{\lambda}(\mu)$ is the $\overline{\mathrm{MS}}$ renormalized coupling in the EFT. In the last line we see that the scale in this amplitude should be of order $\mu \sim m$ such that the logs are kept small.
Problem

- Matching requires $\mu \sim M$
- EFT matrix element requires $\mu \sim m$
- but $m \ll M$

This problem appears in any one loop calculation with two widely separated scales $m$ and $M$. Trace of this is terms proportional to

$$
\begin{equation*}
\ln \frac{m^{2}}{M^{2}}=\ln \frac{m^{2}}{\mu^{2}}+\ln \frac{\mu^{2}}{M^{2}} \tag{4.93}
\end{equation*}
$$

which lead to a breakdown of perturbation theory for $m \ll M$ even for small coupling $\lambda$. The way out is the Renormalization Group Equation (RGE)

$$
\begin{equation*}
\frac{d \tilde{\lambda}(\mu)}{d \ln \mu}=\mu \frac{d \tilde{\lambda}(\mu)}{d \mu}=\beta(\lambda(\mu)) \tag{4.94}
\end{equation*}
$$

for the coupling in the effective theory. In general the equation for any Wilson coefficient is

$$
\begin{equation*}
\frac{d C_{i}(\mu)}{d \ln \mu}=\gamma_{i j}(\mu) C_{j}(\mu), \tag{4.95}
\end{equation*}
$$

This is a matrix equation and accounts for possible mixing of the operators with the same quantum numbers (via loop effects).

In general the strategy can be depicted in the following way

## Match at scale $\mu \sim M$

 Renormalization GroupMatch at scale $\mu \sim M$ with rundown coupling

The bare Green's functions should not depend on the renormalization scale $\mu$, which lead to the following differential RGE

$$
\begin{align*}
\frac{d}{d \ln \mu} \Gamma & =0  \tag{4.96}\\
\Rightarrow \frac{d}{d \ln \mu} \tilde{\lambda}(\mu) & =\beta(\lambda)=\frac{3 \tilde{\lambda}(\mu)}{16 \pi^{2}}+\ldots \tag{4.97}
\end{align*}
$$

The solution (via separation of variable) reads

$$
\begin{align*}
\int_{\tilde{\lambda}\left(\mu_{0}\right)}^{\tilde{\lambda}(\mu)} \frac{d \tilde{\lambda}}{\tilde{\lambda}^{2}} & =\frac{3}{16 \pi^{2}} \int_{\ln \mu(0)}^{\ln \mu} d \ln \mu^{\prime} \\
\frac{1}{\tilde{\lambda}(\mu)}-\frac{1}{\tilde{\lambda}\left(\mu_{0}\right)} & =\frac{3}{16 \pi^{2}} \ln \frac{\mu}{\mu_{0}} \\
\tilde{\lambda}(\mu) & =\frac{\tilde{\lambda}\left(\mu_{0}\right)}{1-\frac{3}{16 \pi^{2}} \tilde{\lambda}\left(\mu_{0}\right) \ln \frac{\mu}{\mu_{0}}} \tag{4.98}
\end{align*}
$$

This equation is only accurate up to order $\lambda$, if we want to improve we need $\beta$ to oder $\lambda^{3}$

$$
\begin{align*}
\mu \frac{d \tilde{\lambda}}{d \mu} & =\beta(\tilde{\lambda})=\tilde{\lambda}\left[3 \frac{\tilde{\lambda}}{16 \pi^{2}}-\frac{17}{3}\left(\frac{\tilde{\lambda}}{16 \pi^{2}}\right)^{2}+\ldots\right] \\
\Rightarrow \frac{3}{16 \pi^{2}} \ln \frac{\mu}{\mu_{0}} & =\frac{1}{\tilde{\lambda}\left(\mu_{0}\right)}-\frac{1}{\tilde{\lambda}(\mu)}+\frac{17}{9} \frac{1}{16 \pi^{2}} \ln \frac{\tilde{\lambda}(\mu)}{\tilde{\lambda}\left(\mu_{0}\right)}+\mathcal{O}(\tilde{\lambda}) \tag{4.99}
\end{align*}
$$

We will use Eq. (4.98) to iteratively get from $\mu \sim M$ to $\mu \sim m$. This means that we can treat

- $\tilde{\lambda}\left(\mu_{0}\right)$ as a small quantity
- $\tilde{\lambda}(\mu)$ as a small quantity
- $\ln \frac{\mu}{\mu_{0}}$ is count as $\frac{1}{\lambda}$

Let us stress again that Eq. (4.98) is accurate to $\mathcal{O}(\tilde{\lambda})$. In general we need the $\beta$ functions one order higher than the matching caluclation was performed in.

The bottom up procedure is as follows

- Identify relevant degrees of freedomat low energies
- Construct the most general Lagrangian consistent with symmetries
- Dimesion of operators as power counting

$$
\begin{equation*}
\mathcal{L}_{\text {eff }}=\sum \mathcal{L}_{\text {eff }}^{(n)} \tag{4.100}
\end{equation*}
$$

where terms from $\mathcal{L}_{\text {eff }}^{(n)}$ contribute

$$
\begin{equation*}
\left(\frac{E}{\Lambda}\right)^{n} \tag{4.101}
\end{equation*}
$$

This means for a given accuracy $\Delta$

$$
\begin{align*}
\Delta & =\left(\frac{E}{\Lambda}\right)^{n}  \tag{4.102}\\
\ln \Delta & =n \ln \frac{E}{\Lambda}  \tag{4.103}\\
\Rightarrow b & =\frac{\ln \Delta}{\ln E / \Lambda} \tag{4.104}
\end{align*}
$$

- Use field redifinitions to elimiate redundant terms in $\mathcal{L}_{\text {eff }}$.
- Perform matching (if possible) to the full theory to fix Wilson coefficients
- Perfrom RGI, i.e. compute the anomalius dimensions and $\beta$ functions.

