Cosmology and General Relativity: HW 5

Turn in to "Wittig" (Hausaufgaben) mailbox in KPH by **noon, 27 June 2019**

Problem 1 Maximally symmetric spaces: The easiest example of a maximally symmetric space is $\mathbb{R}^n$ with the flat Euclidean metric. Its isometries are the rotation and translations in n dimensions. In the neighborhood of some fixed point p the translations move this point along an axis, and since there are n independent axes, there are n possible translations. Rotations leave p invariant, but transform one axis through p and into another one. Since there are n axes, we can move each into $n-1$ other axes. After considering double-counting, there are $\frac{1}{2}n(n-1)$ independent rotations. In total, there are $\frac{1}{2}n(n+1)$ independent symmetries in $\mathbb{R}^n$. Because we considered the local effect of the symmetries these arguments should also apply for curved spaces. If the manifold is pseudo-Riemannian, some rotations will be boosts, but the counting is the same. Since the number of isometries is the number of linearly independent Killing Vector fields, we can call an n-dimensional manifold with $\frac{1}{2}n(n+1)$ Killing vectors *maximally symmetric*. For a maximally symmetric manifold, the curvature is the same everywhere and in every direction. This we can classify maximally symmetric manifolds by their Ricci curvature scalar R, their dimension n, their metric signature, and perhaps some additional information on the global topology (e.g., the size of the manifold). For a given dimension and signature, we can classify maximally symmetric spaces (ignoring global topology) into the basic categories of R being positive, negative, or zero; since the magnitude of R corresponds to an overall scaling of the space. For Riemannian manifolds, the flat (i.e., $R = 0$) maximally symmetric spaces correspond to planes (or their high dimensional generalizations).
We want to consider the two-dimensional hyperboloid H^2. There are several ways to represent H^2 (which has the same topology as $\mathbb{R}^2$). We choose the *Poincaré half-plane*, being defined on the upper half-plane of $\mathbb{R}^2$ with the metric

$$ds^2 = \frac{a^2}{y^2}\left(dx^2 + dy^2\right) \ .$$

a. Compute the lengths of vertical line segments ($x = \text{cnst}$) from y_1 to y_2. What happens for $y \to 0$? *1pt*

b. Calculate the Christoffel symbols and show that the geodesics satisfy

$$(x - x_0)^2 + y^2 = \ell^2 \ ,$$

for some constants x_0 and ℓ. What do these geodesics look like? *2pt*

c. Calculate the Riemannian tensor and Ricci tensor. Show that the Ricci curvature scalar is constant on the manifold. *2.5pt*

Problem 2 Killing vector fields on the 2-sphere: Consider the 2-sphere S^2 with the usual induced metric, $ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$.

a. Write down the three independent Killing equations

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0 \ ,$$

for a vector field $X = X^\theta \partial_\theta + X^\varphi \partial_\varphi$. *1pt*

b. Show that X_θ is independent of θ, i.e. that you can also write $X_\theta(\theta, \varphi) = f(\varphi)$. Perform this substitution in the Killing equations and show that:

$$X_\varphi = -F(\varphi) \sin\theta \cos\theta + g(\theta) \ ,$$

where $F(\varphi)$ is the primitive of $f(\varphi)$, so $F'(\varphi) = f(\varphi)$, and $g(\varphi)$ is an integration constant. *1.5pt*

c. Insert the result of the previous item in the last Killing equation and — using separation of variables — show that one obtains:

$$\frac{dg}{d\theta} - 2g(\theta) \cot\theta = C \ , \quad (1)$$

$$\frac{df}{d\varphi} + F(\varphi) = -C \ , \quad (2)$$

where C is a constant. *1.5pt*

d. Determine $g(\theta)$ by integrating equation (1). Differentiate equation (2) and show that f is harmonic. Give the general solution.

Hint: The result has to be of the form: *1pt*

$$g(\theta) = (C_1 - C \cot\theta) \sin^2\theta, \qquad f(\varphi) = A \sin\varphi + B \cos\varphi.$$

e. Combine the results and show that a general Killing vector on S^2 has the following form: *1pt*

$$X = A(\sin\varphi \partial_\theta + \cos\varphi \cot\theta \partial_\varphi) + B(\cos\varphi \partial_\theta - \sin\varphi \cot\theta \partial_\varphi) + C_1 \partial_\varphi \ . \quad (3)$$

f. Identify the three basis vectors of the Killing vector in equation (3) with the angular momentum $L_i = \sum_{j,k} \epsilon_{ijk} x_j \partial_k$. Is S^2 a maximally symmetric space? *2pt*

Problem 3 Spectral Red Shift: The spherically symmetric solution to Einstein's equations for a massive object of mass m (here $1 = c$) is given by:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

with

$$A(r) = \left(1 - \frac{2GM}{r}\right) = \frac{1}{B(r)} \ .$$

Assume that an *emitter* sends a signal at a fixed point $(r_E, \vartheta_E, \varphi_E)$ and that the signal propagates along the null geodesic and is detected by the *receiver* at a fixed point $(r_R, \vartheta_R, \varphi_R)$. If t_E is the time coordinate of emission and t_R the time coordinate of reception, the signal propagates from event $(t_E, r_E, \vartheta_E, \varphi_E)$ to $(t_R, r_R, \vartheta_R, \varphi_R)$.

a. Draw the spacetime diagram and illustrate both events. *1pt*

b. Let λ be the affine parameterization of the null geodesic with $\lambda_{E/R}$ at the points of emission/reception. Show that the following holds $(i, j = 1, 2, 3)$:

$$\frac{dt}{d\lambda} = \left[\left(1 - \frac{2GM}{r}\right)^{-1} g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\right]^{1/2} \ .$$

2pt

c. Take the above even to argue that

$$\Delta t_E = \Delta t_R \qquad \textit{with} \qquad \Delta t = t^{(2)} - t^{(1)}$$

corresonds to the difference of the coordinate times of signals 1 and 2. *1pt*

d. A clock at the position of an observer measures the proper time τ instead of the coordinate time t. Find the relation among the two time parameterization and conclude:

$$\frac{\Delta\tau_R}{\Delta\tau_E} = \left[\frac{1 - 2MG/r_R}{1 - 2MG/r_e}\right]^{1/2} \ .$$

1pt

e. Now assume that the emitter is pulsed with frequency $\nu_E = \frac{n}{\Delta\tau_E}$, i.e. n pulses per proper time interval $\Delta\tau_E$ are emitted. Analogous relations hold for the receiver. Find the relation among the two frequencies ν_E/ν_R. Expand the relation for $r_E, r_R \gg 2GM$ and discuss what happens if the emitter (receiver) is closer to the massive object than the receiver (emitter). *1.5pt*