V QED PROCESSES IN LOWEST ORDER

1) CROSS SECTION

2) $e^- e^+ \to \mu^- \mu^+$ IN CoM SYSTEM

3) $e^- \pi^- \to e^- \pi^-$ IN LAB SYSTEM

4) EXTERNAL PHOTONS : PHOTON POLARIZATION SUM

1) CROSS SECTION

⇒ TRANSITION PROBABILITY PER UNIT TIME

↳ CONSIDER PROCESS

$$1 + 2 \longrightarrow 1' + 2' + \ldots + N'$$

e.g. $e^- + p \longrightarrow e^- + p \qquad (N' = 2)$

S_{fi}
$$|\tilde{i}\rangle = |\bar{p}_1, s_1 \,;\, \bar{p}_2, s_2\rangle$$
$$|\tilde{f}\rangle = |\bar{p}_1', s_1' \,;\, \ldots \,;\, \bar{p}_N' s_N'\rangle$$

$|S_{fi}|^2$: TRANSITION PROBABILITY

TO PROPERLY SPEAK ABOUT PROBABILITY WE NEED TO NORMALIZE $|i\rangle, |f\rangle$ SUCH THAT

$$\sum_f |S_{fi}|^2 = 1$$

$$S_{fi} = \langle f | S | i \rangle$$

WITH
$$|i\rangle = N\,|\tilde{i}\rangle$$
$$|f\rangle = N\,|\tilde{f}\rangle$$

WITH $\langle i | i' \rangle = \delta_{ii'}$

$\langle f | f' \rangle = \delta_{ff'}$

FOR 1 PARTICLE STATE

$|\tilde{i}\rangle = |\bar{p}_i, s_i\rangle$ HAS COVARIANT NORMALIZATION

$$\langle \tilde{i} | \tilde{i}' \rangle = \delta_{s_i s_i'} (2\pi)^3 (2E_i)\, \delta^3(\bar{p}_i - \bar{p}_i')$$

$\rightsquigarrow$ DISCRETIZE MOMENTUM (FINITE VOLUME V)

$$\sum_{\bar{p}} \cdots \iff \frac{V}{(2\pi)^3} \int d^3\bar{p} \cdots$$

$$\delta_{\bar{p}\bar{p}'} \iff \frac{(2\pi)^3}{V} \delta^3(\bar{p} - \bar{p}')$$

$\therefore$ $\langle i | i' \rangle = N^2 \langle \tilde{i} | \tilde{i}' \rangle$

$$\delta_{s_i s_i'} \delta_{\bar{p}_i \bar{p}_i'} = N^2 \delta_{s_i s_i'} (2E_i) (2\pi)^3 \delta^3(\bar{p}_i - \bar{p}_i')$$

$$\Downarrow$$

$$\underline{\underline{N = \frac{1}{(2E_i V)^{1/2}}}}$$

$\therefore$ WHEN CALCULATING FEYNMAN AMPLITUDE USING COVARIANT NORMALIZATION (AS BEFORE) WE NEED TO MULTIPLY ALL EXTERNAL LINES BY FACTOR $\frac{1}{(2EV)^{1/2}}$ TO CONVERT TO PROBABILITIES

$$S_{fi} = \delta_{fi} + (2\pi)^4 \delta^4 \left(\sum_{f=1}^{N} p_f' - p_1 - p_2 \right) \cdot \frac{1}{(2VE_i)^{1/2}} \cdot \frac{1}{(2VE_2)^{1/2}} \prod_f \left(\frac{1}{2VE_f'} \right)^{1/2} \cdot \mathcal{M}_{fi}$$

1° TERM δ_{fi} : NO SCATTERING IF $i = f$

2° TERM $\mathcal{M}_{fi}$: FEYNMAN AMPLITUDE USING COVARIANT NORMALIZATION WHERE ENERGY - MOMENTUM CONSERVING δ - FUNCTION IS TAKEN OUT

↳ TRANSITION PROBABILITY W (FOR $i \neq f$)

$$W = \left[(2\pi)^4 \delta^4 \left(\sum_{f=1}^{N} p_f' - p_1 - p_2 \right) \right]^2 \cdot \frac{1}{(2VE_1)} \cdot \frac{1}{(2VE_2)} \prod_f \left(\frac{1}{2VE_f'} \right) |\mathcal{M}|^2$$

↳ SQUARING δ - FUNCTION

- IN 1 DIM

$$(2\pi)\,\delta(E_f - E_i) = \int_{-\infty}^{+\infty} dt\; e^{+it\Delta E} \qquad \text{WITH } \Delta E \equiv E_f - E_i$$

$$= \lim_{T\to\infty} \int_{-T/2}^{T/2} dt\; e^{it\Delta E}$$

$$= \lim_{T\to\infty} \frac{2}{\Delta E} \sin\left(\frac{\Delta E\, T}{2}\right)$$

$$\left[(2\pi)\,\delta(E_f - E_i)\right]^2 = (2\pi)\,\delta(E_f - E_i) \lim_{T\to\infty} \lim_{\Delta E\to 0} \frac{2}{\Delta E} \sin\left(\frac{\Delta E\, T}{2}\right)$$

$$= (2\pi)\,\delta(E_f - E_i) \lim_{T\to\infty} T$$

- IN 4 DIM

$$\left[(2\pi)^4\,\delta^4(P_f - P_i)\right]^2 = \lim_{T,V\to\infty} (T\,V) \cdot (2\pi)^4\,\delta^4(P_f - P_i)$$

↳ TRANSITION PROBABILITY PER UNIT TIME

$$w \equiv \frac{W}{T}$$

$$= V\,(2\pi)^4\,\delta^4\Big(\sum_f p_f' - p_1 - p_2\Big)$$

$$\cdot \frac{1}{(2VE_1)} \cdot \frac{1}{(2VE_2)} \prod_f \left(\frac{1}{2VE_f'}\right) |\mathcal{M}|^2.$$

+ TAKE LIMIT $V \to \infty$

↳ THIS IS FOR ONE SPECIFIC FINAL STATE

TO OBTAIN TRANSITION PROBABILITY TO A GROUP OF FINAL STATES WITH MOMENTA BETWEEN $[\vec{p}_f', \vec{p}_f' + d\vec{p}_f']$ $f = 1 \dots N$

$$\prod_{f'} \left(\frac{V}{(2\pi)^3} \cdot d^3\vec{p}_f'\right)$$

$$w = V \prod_f \left(\frac{V}{(2\pi)^3}\, d^3\vec{p}_f'\right) \frac{1}{(2VE_1)} \cdot \frac{1}{(2VE_2)}$$

$$\cdot (2\pi)^4\delta^4\Big(\sum_f p_f' - p_1 - p_2\Big) \prod_f \left(\frac{1}{2VE_f'}\right) |\mathcal{M}|^2$$

$$w = \frac{1}{V} \cdot \frac{1}{(2E_1)(2E_2)} \prod_f \left(\frac{d^3\vec{p}_f'}{(2\pi)^3\, 2E_f'}\right) \cdot (2\pi)^4\delta^4\Big(\sum_f p_f' - p_1 - p_2\Big) \cdot |\mathcal{M}|^2$$

⇒ CROSS SECTION

↳ REACTION CROSS SECTION $d\sigma$ (FOR EACH TARGET PARTICLE)

$$\boxed{d\sigma = \frac{w}{\phi}}$$

w : TRANSITION PROB. PER UNIT TIME

ϕ : INCIDENT FLUX (PER BEAM PARTICLE)

$$\phi = \frac{1}{V} \cdot v_{rel}$$

DIMENSION $[\phi] = \frac{1}{T \cdot L^2} = \frac{1}{(\text{TIME})(\text{SURFACE})}$

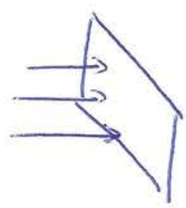
v_{rel} : RELATIVE VELOCITY BETWEEN BEAM & TARGET

$\frac{1}{V}$: DENSITY FOR 1 BEAM PARTICLE IN VOL. V

- BEAM ON FIXED TARGET (LAB SYSTEM)

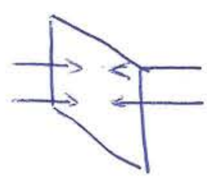
$\bar{P}_2 = 0$

$$\boxed{v_{rel} = \frac{|\bar{P}_1|}{E_1}}$$



- COLLIDER (CoM SYSTEM)

CENTRE-OF-MASS



$\bar{P}_2 = -\bar{P}_1$

$$\boxed{v_{rel} = \frac{|\bar{P}_1|}{E_1} + \frac{|\bar{P}_2|}{E_2} = \frac{(E_1 + E_2)}{E_1 E_2} |\bar{P}_1|}$$

SUM OF 2 FLUXES

↳

$$\boxed{d\sigma = \frac{1}{(2E_1)(2E_2)\, v_{rel}} \cdot \prod_f \left(\frac{d^3 \vec{p}_f^{\,\prime}}{(2\pi)^3 2E_f^{\prime}} \right) \cdot (2\pi)^4 \delta^4\left(\sum_f p_f^{\prime} - p_1 - p_2\right) \cdot |\mathcal{M}|^2}$$

DIMENSION $[d\sigma] = L^2$ SURFACE

FOR EACH INITIAL PARTICLE : FACTOR $\frac{1}{2E}$

FOR EACH FINAL PARTICLE : FACTOR $\left(\frac{d^3 \vec{p}}{(2\pi)^3 2E} \right)$

↑

LORENTZ INVARIANT PHASE SPACE

$|\mathcal{M}|^2$: SQUARED FEYNMAN AMPLITUDE (COVARIANT NORMALIZATION)

GLOBAL ENERGY - MOMENTUM CONSERVATION

<u>NOTE</u> : IN EXPRESSION OF $d\sigma$ THE VOLUME DEPENDENCE V DROPS OUT !

⇒ UNITS OF $d\sigma$

IN MASS DIMENSION (GeV)

$$[d\sigma] \text{ in } GeV^{-2}$$

- TO CONVERT TO SURFACE

MULTIPLY WITH $(\hbar c)^2$

$$d\sigma\,(\text{in } GeV^{-2}) \cdot \underbrace{(\hbar c)^2}_{(0.197)^2 \; GeV^2\, fm^2}$$

- CONVENTIONAL UNIT : barn (b)

$$1\,b = 100\, fm^2 = 10^{-28}\, m^2$$

- $d\sigma\,(\text{in } b) = d\sigma\,(\text{in } GeV^{-2}) \cdot (0.197)^2\, 10^{-2}$

- MILLI-BARN $\quad 1\, mb = 10^{-3}\, b$

MICRO-BARN $\quad 1\, \mu b = 10^{-6}\, b$

NANO-BARN $\quad 1\, nb = 10^{-9}\, b$

PICO-BARN $\quad 1\, pb = 10^{-12}\, b$

$$d\sigma\,(\text{in } mb) = d\sigma\,(\text{in } GeV^{-2}) \cdot (0.197)^2 \cdot 10$$
$$d\sigma\,(\text{in } \mu b) = d\sigma\,(\text{in } GeV^{-2}) \cdot (0.197)^2 \cdot 10^4$$
$$d\sigma\,(\text{in } nb) = d\sigma\,(\text{in } GeV^{-2}) \cdot (0.197)^2 \cdot 10^7$$
$$d\sigma\,(\text{in } pb) = d\sigma\,(\text{in } GeV^{-2}) \cdot (0.197)^2 \cdot 10^{10}$$

⇒ SPIN STATES

SPIN s_1 $\Rightarrow$ $(2s_1+1)$ STATES

- IF WE HAVE UNPOLARIZED INITIAL PARTICLES

  $\Rightarrow$ AVERAGE OVER INITIAL POLARIZATIONS

$$\frac{1}{(2s_1+1)(2s_2+1)} \sum_{s_1, s_2}$$

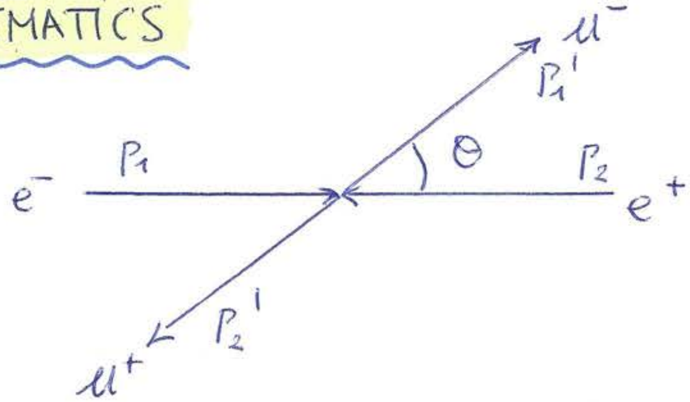
- IF WE DON'T OBSERVE POLARIZATION OF FINAL STATE PARTICLES

  $\Rightarrow$ SUM OVER FINAL POLARIZATIONS

$$\sum_{s'_1} \cdots \sum_{s'_N}$$

2) $e^- e^+ \to \mu^- \mu^+$ IN CoM FRAME

⇒ **KINEMATICS**



CoM $\parallel$ $\bar{P}_2 = -\bar{P}_1$

$$P_1 = (E, \bar{P}_1) \qquad E = \sqrt{\bar{P}_1^2 + m_e^2}$$

$$P_2 = (E, -\bar{P}_1) \qquad m_e : \text{MASS } e^- = \text{MASS } e^+$$

μ^- : MUON : SPIN 1/2 PARTICLE

MASS $m_\mu \simeq 105.7$ MeV

$\simeq 207\ m_e$

$$P_1' = (E', \bar{P}_1')$$

$$P_2' = (E', -\bar{P}_1')$$

ENERGY CONSERVATION REQUIRES

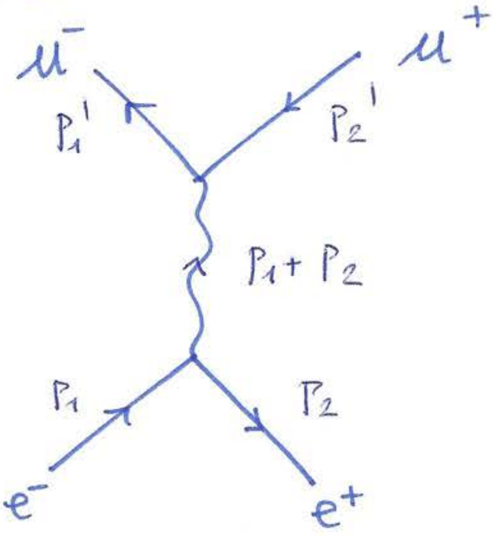
$$2E' = 2E \implies E = \sqrt{\bar{P}_1'^2 + m_\mu^2}$$

$$\gtrsim m_\mu$$

$$\Downarrow$$

$$|\bar{P}_1'|^2 = E^2 - m_\mu^2$$

⟹ INVARIANT AMPLITUDE $\mathcal{M}$



$$\mathcal{M} = \bar{u}(p_1', s_1')(ie\gamma^\mu)\, v(p_2', s_2') \cdot \frac{(-i g_{\mu\nu})}{(p_1+p_2)^2 + i\varepsilon} \cdot \bar{v}(p_2, s_2)(ie\gamma^\nu)\, u(p_1, s_1)$$

⟹ SPIN SUMS

↳ TO CALCULATE UNPOLARIZED CROSS SECTION WE NEED TO EVALUATE

$$\frac{1}{4} \sum_{s_1} \sum_{s_2} \sum_{s_1'} \sum_{s_2'} |\mathcal{M}|^2$$

<u>NOTE</u>: $$\left(\bar{u}(p_1', s_1')\, \gamma^\mu\, v(p_2', s_2') \right)^*$$

$$= \bar{v}(p_2', s_2')\, \gamma^\mu\, u(p_1', s_1')$$

$$\frac{1}{4}\sum_{s_1}\sum_{s_2}\sum_{s_1'}\sum_{s_2'}|\mathcal{M}|^2$$

$$= \frac{e^4}{\left[(p_1+p_2)^2\right]^2}\cdot\frac{1}{4}\sum_{s_1}\sum_{s_2}\sum_{s_1'}\sum_{s_2'} \bar{u}(p_1' s_1')\gamma^\alpha v(p_2' s_2')$$

$$\cdot\ \bar{v}(p_2, s_2)\gamma_\alpha u(p_1, s_1)$$

$$\cdot\ \bar{v}(p_2', s_2')\gamma^\beta u(p_1' s_1')$$

$$\cdot\ \bar{u}(p_1, s_1)\gamma_\beta v(p_2, s_2).$$

$$= \frac{e^4}{(4E^2)^2}\cdot\frac{1}{4}$$

$$\cdot\sum_{s_1'}\sum_{s_2'} \bar{u}(p_1' s_1')\gamma^\alpha v(p_2' s_2')\bar{v}(p_2' s_2')\gamma^\beta u(p_1' s_1')$$

$$\cdot\sum_{s_1}\sum_{s_2} \bar{v}(p_2, s_2)\gamma_\alpha u(p_1, s_1)\bar{u}(p_1, s_1)\gamma_\beta v(p_2, s_2)$$

$$\downarrow$$

USE SPIN SUMS

$$\sum_s u(p,s)\,\bar{u}(p,s) = \not{p} + m$$

$$\sum_s v(p,s)\,\bar{v}(p,s) = \not{p} - m$$

$$\frac{1}{4}\sum_{s_1}\sum_{s_2}\sum_{s_1'}\sum_{s_2'}|\mathcal{M}|^2$$

$$= \frac{e^4}{16E^4}\cdot\frac{1}{4}\,\mathrm{Tr}\left\{\gamma^\alpha(\not{p}_2'-m_\mu)\gamma^\beta(\not{p}_1'+m_\mu)\right\}$$

$$\cdot\,\mathrm{Tr}\left\{\gamma_\alpha(\not{p}_1+m_e)\gamma_\beta(\not{p}_2-m_e)\right\}$$

$\hookrightarrow$ $\mathrm{Tr}\left\{\gamma^\alpha(\not{p}_2'-m_\mu)\gamma^\beta(\not{p}_1'+m_\mu)\right\}$

ONLY Tr OF EVEN # γ-MATRICES IS NON-ZERO

$$\mathrm{Tr}\{\not{a}\,\not{b}\} = 4\,a.b$$

$$\mathrm{Tr}\{\not{a}\,\not{b}\,\not{c}\,\not{d}\} = 4\{(a.b)(c.d)-(a.c)(b.d) + (a.d)(b.c)\}$$

$$\therefore\ \mathrm{Tr}\left\{\gamma^\alpha(\not{p}_2'-m_\mu)\gamma^\beta(\not{p}_1'+m_\mu)\right\}$$

$$= \mathrm{Tr}\left\{\gamma^\alpha\not{p}_2'\gamma^\beta\not{p}_1'\right\} - m_\mu^2\,\mathrm{Tr}\left\{\gamma^\alpha\gamma^\beta\right\}$$

$$= 4\left\{p_2'^\alpha p_1'^\beta + p_1'^\alpha p_2'^\beta - g^{\alpha\beta}(m_\mu^2 + p_1'\cdot p_2')\right\}$$

$$\hookrightarrow \mathrm{Tr}\left\{ \gamma_\alpha (\not{p}_1 + m_e)\, \gamma_\beta\, (\not{p}_2 - m_e) \right\}$$

$$= 4 \left\{ p_{1\alpha}\, p_{2\beta} + p_{1\beta}\, p_{2\alpha} - g_{\alpha\beta} \left(m_e^2 + p_1 \cdot p_2 \right) \right\}$$

$$\hookrightarrow \frac{1}{4} \sum_{s_1} \sum_{s_2} \sum_{s_1'} \sum_{s_2'} |\mathcal{M}|^2$$

$$= \frac{e^4}{4E^4} \cdot \left\{ p_1'^{\alpha} p_2'^{\beta} + p_1'^{\beta} p_2'^{\alpha} - g^{\alpha\beta} \left(m_\mu^2 + p_1' \cdot p_2' \right) \right\}$$

$$\cdot \left\{ p_{1\alpha}\, p_{2\beta} + p_{1\beta}\, p_{2\alpha} - g_{\alpha\beta} \left(m_e^2 + p_1 \cdot p_2 \right) \right\}$$

$$= \frac{e^4}{2E^4} \left\{ (p_1 \cdot p_1')(p_2 \cdot p_2') + (p_1 \cdot p_2')(p_2 \cdot p_1') \right.$$

$$- (p_1' \cdot p_2')(m_e^2 + p_1 \cdot p_2)$$

$$- (p_1 \cdot p_2)(m_\mu^2 + p_1' \cdot p_2')$$

$$+ 2 \left(m_e^2 m_\mu^2 + m_\mu^2 (p_1 \cdot p_2) + m_e^2 (p_1' \cdot p_2') \right.$$

$$\left. \left. + (p_1 \cdot p_2)(p_1' \cdot p_2') \right) \right\}$$

$$= \frac{e^4}{2E^4} \left\{ (p_1 \cdot p_1')(p_2 \cdot p_2') + (p_1 \cdot p_2')(p_2 \cdot p_1') \right.$$

$$\left. + m_e^2 (p_1' \cdot p_2') + m_\mu^2 (p_1 \cdot p_2) + 2 m_e^2 m_\mu^2 \right\}$$

$\Rightarrow$ CROSS SECTION IN CoM FRAME

$$\hookrightarrow d\sigma = \frac{1}{(4E^2)v_{rel}} \cdot \frac{d^3\bar{p}_1'}{(2\pi)^3 2E_1'} \cdot \frac{d^3\bar{p}_2'}{(2\pi)^3 2E_2'}$$

$$\cdot (2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2')$$

$$\cdot \frac{1}{4} \sum_{s_1} \sum_{s_2} \sum_{s_1'} \sum_{s_2'} |\mathcal{M}|^2$$

WITH $v_{rel} = \frac{|\bar{p}_1|}{E_1} + \frac{|\bar{p}_2|}{E_2} = 2\frac{|\bar{p}_1|}{E}$ (IN CoM)

$$\hookrightarrow \int \frac{d^3\bar{p}_2'}{(2\pi)^3} (2\pi)^3 \delta^3(\bar{p}_1 + \bar{p}_2 - \bar{p}_1' - \bar{p}_2') \ \ldots$$

$$d\sigma = \frac{1}{8|\bar{p}_1|E} \cdot \frac{1}{(2\pi)^2} \frac{d^3\bar{p}_1'}{(2E')^2} \delta(2E - 2E')$$

$$\cdot \frac{1}{4} \sum_{s_1} \sum_{s_2} \sum_{s'} \sum_{s_2'} |\mathcal{M}|^2$$

$$\hookrightarrow \delta(2E - 2E')$$

$$= \frac{1}{2} \delta\left(E - \sqrt{\bar{p}_1'^2 + m_\mu^2}\right)$$

$$= \frac{1}{2} \frac{1}{\left|\frac{\partial E'}{\partial |\bar{p}_1'|}\right|} \delta\left(|\bar{p}_1'| - \sqrt{E^2 - m_\mu^2}\right)$$

$$\frac{\partial E'}{\partial |\bar{p}_1'|} = \frac{|\bar{p}_1'|}{E'}$$

$$\therefore \quad \delta(2E - 2E') = \frac{E'}{2|\bar{p}_1'|} \delta\left(|\bar{p}_1'| - \sqrt{E^2 - m_\mu^2}\right)$$

$$\hookrightarrow \frac{d^3\bar{p}_1'}{(2E')^2} = d\Omega \, d|\bar{p}_1'| \frac{|\bar{p}_1'|^2}{4E'^2}$$

$$\hookrightarrow \int \frac{d^3\bar{p}_1'}{(2E')^2} \delta(2E - 2E')$$

$$= d\Omega \int d|\bar{p}_1'| \frac{|\bar{p}_1'|^2}{4E'^2} \cdot \frac{E'}{2|\bar{p}_1'|} \delta\left(|\bar{p}_1'| - \sqrt{E^2 - m_\mu^2}\right)$$

SOLID ANGLE

$$= d\Omega \frac{|\bar{p}_1'|}{8E} \quad \text{WHERE } |\bar{p}_1'| = \sqrt{E^2 - m_\mu^2}$$

$\hookrightarrow \left(\frac{d\sigma}{d\Omega}\right)_{CoM}$ DIFFERENTIAL CROSS SECTION IN CoM FRAME PER UNIT OF SOLID ANGLE

$$d\Omega = d\phi \, d\cos\theta$$

UNIT $[d\Omega] = sr$

$$\left(\frac{d\sigma}{d\Omega}\right)_{CoM} = \frac{|\bar{p}_1'|}{64\,|\bar{p}_1|\,E^2}\cdot\frac{1}{(2\pi)^2}\cdot\frac{1}{4}\sum_{s_1}\sum_{s_2}\sum_{s_1'}\sum_{s_2'}|\mathcal{M}|^2$$

$$= \left(\frac{e^2}{4\pi}\right)^2\cdot\frac{|\bar{p}_1'|}{32\,|\bar{p}_1|\,E^6}\Big\{(p_1\cdot p_1')(p_2\cdot p_2') + (p_1\cdot p_2')(p_2\cdot p_1') + m_e^2(p_1'\cdot p_2') + m_\mu^2(p_1\cdot p_2) + 2m_e^2 m_\mu^2\Big\}$$

↳ AS $E > m_\mu \simeq 207\ m_e$

$E,\ |\bar{p}_1| \gg m_e \implies$ WE CAN NEGLECT ALL m_e TERMS

$|\bar{p}_1| = E$

$$\rightsquigarrow\ p_1\cdot p_1' = E\left(E - |\bar{p}_1'|\cos\theta\right) = p_2\cdot p_2'$$

$$\rightsquigarrow\ p_1\cdot p_2' = E\left(E + |\bar{p}_1'|\cos\theta\right) = p_2\cdot p_1'$$

$$\rightsquigarrow\ \{\dots\} \simeq E^2\left(E - |\bar{p}_1'|\cos\theta\right)^2 + E^2\left(E + |\bar{p}_1'|\cos\theta\right)^2 + m_\mu^2\, 2E^2$$

$$= 2E^2\left\{E^2 + |\bar{p}_1'|^2\cos^2\theta + m_\mu^2\right\}$$

$$\hookrightarrow \left(\frac{d\sigma}{d\Omega}\right)_{CoM} = \left(\frac{e^2}{4\pi}\right)^2 \frac{1}{16E^4} \cdot \left(\frac{|\vec{p}_1'|}{E}\right)$$

$$\cdot \left(E^2 + |\vec{p}_1'|^2 \cos^2\theta + m_\mu^2\right)$$

INTRODUCE $\alpha \equiv \frac{e^2}{4\pi} \simeq \frac{1}{137}$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{CoM} = \frac{\alpha^2}{16E^4} \cdot \left(\frac{|\vec{p}_1'|}{E}\right) \cdot \left(E^2 + |\vec{p}_1'|^2 \cos^2\theta + m_\mu^2\right)}$$

WITH $|\vec{p}_1'| = \sqrt{E^2 - m_\mu^2}$

$\Longrightarrow$ TOTAL CROSS SECTION

$$\sigma_{tot} \equiv \int d\Omega \left(\frac{d\sigma}{d\Omega}\right)$$

$$\downarrow \qquad \int d\Omega \cos^2\theta = 2\pi \int_{-1}^{1} dx \; x^2 = \frac{4\pi}{3}$$

$$\boxed{\sigma_{tot} = \frac{\pi\alpha^2}{4E^4} \left(\frac{|\vec{p}_1'|}{E}\right) \left(E^2 + \frac{1}{3}|\vec{p}_1'|^2 + m_\mu^2\right)}$$

⟹ ULTRA-RELATIVISTIC LIMIT

↳ FOR $E \gg m_\mu$

WE CAN ALSO NEGLECT ALL m_μ DEPENDENCE

$$|\vec{p}_1'| = E$$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{CoM} = \frac{\alpha^2}{16E^2}\left(1 + \cos^2\theta\right)}$$

$$\boxed{\sigma_{tot} = \frac{\pi\alpha^2}{3E^2}}$$

↳ NUMERICAL EXAMPLE:

FOR $E = 1$ GeV

$$\sigma_{tot} = \frac{\pi}{3}\alpha^2 \ \mathrm{GeV}^{-2}$$

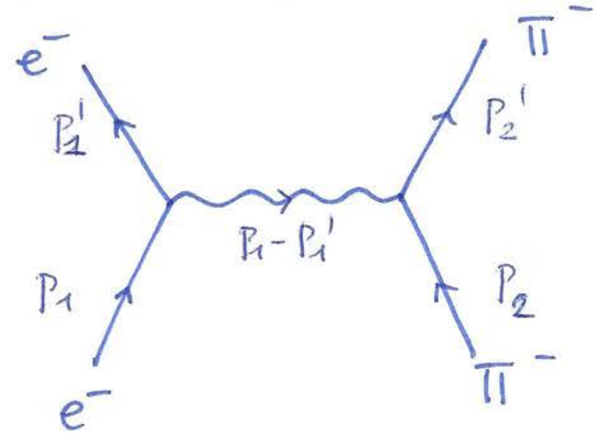
$$= (5.6)\ 10^{-5}\ \mathrm{GeV}^{-2}$$

$$= (5.6)\ 10^{-5} \cdot (0.1973)^2\ 10^{7}\ \mathrm{nb}$$

$$= 21.8\ \mathrm{nb}$$

3) $e^- \pi^- \to e^- \pi^-$ IN LAB FRAME

$\Rightarrow$ KINEMATICS



$\hookrightarrow$ π^- : SPIN-0 PARTICLE (WE WILL CONSIDER IT AS POINT PARTICLE)

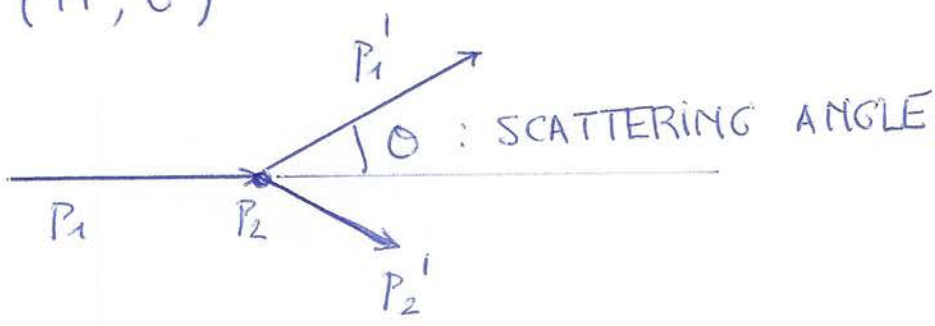
MASS $M \simeq 140$ MeV $\gg m_e$

$\downarrow$

WE WILL NEGLECT m_e

$\hookrightarrow$ LAB SYSTEM

$P_2 \, (M, 0)$



$P_1 (E, \bar{P}_1)$ WITH $|\bar{P}_1| = E$

$P_1' (E', \bar{P}_1')$ WITH $|\bar{P}_1'| = E'$

$$P_2'^2 = M^2 = (P_1 + P_2 - P_1')^2$$

$$= P_2^2 + 2P_2 \cdot (P_1 - P_1') + (P_1 - P_1')^2$$

$$= M^2 + 2M(E - E') - 2P_1 \cdot P_1'$$

MOMENTUM TRANSFER (CARRIED BY γ)

$$q \equiv P_1 - P_1'$$

$$q^2 = -2P_1 \cdot P_1'$$

$$= -2EE'(1 - \cos\theta)$$

$$= -4EE' \sin^2\theta/2$$

$< 0 \Rightarrow$ SPACE LIKE VIRTUAL γ EXCHANGED

$$\boxed{-q^2 \equiv Q^2 = 4EE' \sin^2\theta/2}$$

$\therefore \quad P_2'^2 = M^2$

$$\Downarrow$$

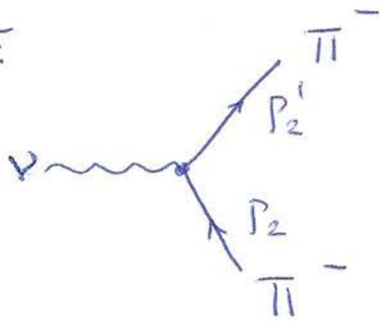
$$q^2 = 2M(E' - E)$$

$$\boxed{E' = \frac{E}{1 + \frac{2E}{M}\sin^2\frac{\theta}{2}}}$$

SCATTERED e^- ENERGY

$\Longrightarrow$ FEYNMAN AMPLITUDE

$\hookrightarrow$ FEYNMAN RULE



$$ie\,(p_2 + p_2')^\nu$$

$\hookrightarrow$ $\mathcal{M}(e^-\pi^- \to e^-\pi^-)$

$$= \bar{u}(p_1', s_1')\,(ie\gamma^\mu)\,u(p_1, s_1)\left(-\frac{i g_{\mu\nu}}{q^2}\right)$$

$$\cdot\left(ie\,(p_2 + p_2')^\nu\right)$$

$$= \frac{ie^2}{q^2}\,\bar{u}(p_1', s_1')\,(\not{p}_2 + \not{p}_2')\,u(p_1, s_1)$$

$$\downarrow \qquad p_2' = p_1 + p_2 - p_1'$$

$$= \frac{ie^2}{q^2}\,\bar{u}(p_1', s_1')\,(2\not{p}_2 + \not{p}_1 - \not{p}_1')\,u(p_1, s_1)$$

$$\downarrow \qquad \begin{aligned} \not{p}_1 u &= 0 \\ \bar{u}\not{p}_1' &= 0 \end{aligned}$$

$$= \frac{2ie^2}{q^2}\,\bar{u}(p_1', s_1')\,\not{p}_2\,u(p_1, s_1)$$

$\Longrightarrow$ SPIN SUM

$$\frac{1}{2}\sum_{s_1}\sum_{s_1'} |\mathcal{M}|^2$$

$$= \frac{4e^4}{Q^4}\frac{1}{2}\sum_{s_1}\sum_{s_2} \bar{u}(p_1', s_1')\, \not{p}_2\, u(p_1, s_1) \cdot \bar{u}(p_1, s_1)\, \not{p}_2\, u(p_1', s_1')$$

$$= \frac{2e^4}{Q^4} \mathrm{Tr}\left\{ \not{p}_2 \not{p}_1 \not{p}_2 \not{p}_1' \right\}$$

$$= \frac{8e^4}{Q^4}\left\{ 2(p_1 \cdot p_2)(p_1' \cdot p_2) - M^2\, p_1 \cdot p_1' \right\}$$

$$= e^4 \frac{8M^2}{Q^4}\left\{ 2EE' - EE'(1 - \cos\theta) \right\}$$

$$= e^4 \frac{8M^2}{Q^4} EE'(1 + \cos\theta)$$

$$= e^4 \frac{16 M^2 EE'}{Q^4} \cos^2\theta/2$$

$\Longrightarrow$ CROSS SECTION

$$\hookrightarrow \quad d\sigma = \frac{1}{(2M)(2E)\, v_{rel}} \frac{d^3\bar{p}_1'}{(2\pi)^3 2E_1'} \frac{d^3\bar{p}_2'}{(2\pi)^3 2E_2'}$$

$$\cdot \, (2\pi)^4 \, \delta^4(p_1 + p_2 - p_1' - p_2')$$

$$\cdot \, \frac{1}{2} \sum_{s_1} \sum_{s_1'} |\mathcal{M}|^2$$

$$v_{rel} = \frac{|\bar{p}_1|}{E_1} \simeq 1 \qquad (\mathrm{LAB})$$

$$\hookrightarrow \quad d\sigma = \frac{1}{(2M)(2E)} \frac{1}{(2\pi)^2} \frac{d^3\bar{p}_1'}{2E_1' \, 2E_2'}$$

$$\cdot \, \delta\left(E + M - E_1' - \sqrt{(\bar{p}_1 - \bar{p}_1')^2 + M^2}\right)$$

$$\cdot \, \frac{1}{2} \sum_{s_1} \sum_{s_1'} |\mathcal{M}|^2$$

$$\hookrightarrow \quad \delta\left(E + M - |\bar{p}_1'| - \sqrt{(\bar{p}_1 - \bar{p}_1')^2 + M^2}\right)$$

$$= \frac{1}{1 + \frac{1}{E_2'}\left(|\bar{p}_1'| - |\bar{p}_1| \cos\theta\right)} \, \delta\left(|\bar{p}_1'| - \cdots\right)$$

$$|\bar{p}_1'| + E_2' = M + E$$

$$= \frac{E_2'}{M + E(1 - \cos\theta)} \, \delta\left(|\bar{p}_1'| - \cdots\right)$$

$$\hookrightarrow \int \frac{d^3\bar{p}_1'}{2E_1' \, 2E_2'} \, \delta\left(E + M - |\bar{p}_1'| - \sqrt{(\bar{p}_1 - \bar{p}_1')^2 + M^2}\right)$$

$$= d\Omega \cdot \frac{E'}{4\left(M + 2E\sin^2\theta/2\right)}\Bigg|_{\text{WITH } E' = \frac{E}{1 + \frac{2E}{M}\sin^2\theta/2}}$$

$$= d\Omega \left(\frac{E'}{E}\right) \frac{1}{4M} \frac{E}{\left(1 + \frac{2E}{M}\sin^2\theta/2\right)}$$

$$= d\Omega \left(\frac{E'}{E}\right)\left(\frac{E'}{4M}\right)$$

$$\hookrightarrow \left(\frac{d\sigma}{d\Omega}\right)_{LAB} = \frac{1}{4ME} \frac{1}{(2\pi)^2} \cdot \left(\frac{E'}{E}\right) \cdot \frac{E'}{4M}$$

$$\cdot \frac{1}{2} \sum_{s_1} \sum_{s_1'} |\mathcal{M}|^2$$

$$= \left(\frac{e^2}{4\pi}\right)^2 \cdot \frac{E'^2}{4M^2E^2} \cdot \frac{16 M^2 EE'}{Q^4} \cos^2\theta/2$$

$$\downarrow \qquad Q^2 = 4EE'\sin^2\theta/2$$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{LAB} = \left(\frac{\alpha^2\cos^2\theta/2}{4E^2\sin^4\theta/2}\right) \cdot \left(\frac{E'}{E}\right)}$$

↳ NOTE 1 IN LIMIT $m_e \ll E \ll M$

SCATTERING OF RELATIVISTIC e^-
BY A COULOMB FIELD (STATIC TARGET)

$$E' \simeq E$$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{LAB} = \frac{\alpha^2 \cos^2 \theta/2}{4E^2 \sin^4 \theta/2}}$$

MOTT CROSS SECTION

NUM. EXAMPLE $E = 1\ \mathrm{GeV}$ $(M \gg 1\ \mathrm{GeV})$

$$\left(\frac{d\sigma}{d\Omega}\right)_{LAB}^{\theta = 90^\circ} = \frac{\alpha^2}{2E^2}$$

$$= \frac{\alpha^2}{2}\ \mathrm{GeV}^{-2}$$

$$= \frac{\alpha^2}{2} (0.1973)^2\ 10^7\ \mathrm{nb}$$

$$\simeq 10.4\ \mathrm{nb}$$

↳ NOTE 2 : IN NON-RELATIVISTIC LIMIT

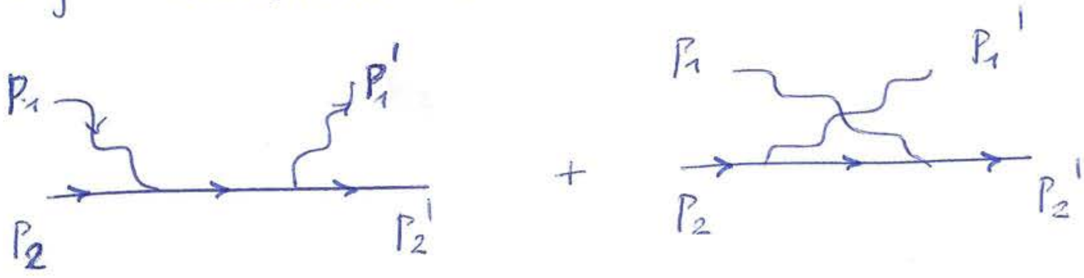
$$\left(\frac{d\sigma}{d\Omega}\right)_{LAB} = \frac{\alpha^2}{4m^2 v^4 \sin^4 \theta/2}$$

RUTHERFORD FORMULA

4) EXTERNAL PHOTONS : PHOTON POLARIZATION SUM

FOR PROCESSES WITH EXTERNAL PHOTONS

e.g. COMPTON SCATTERING



↳ FEYNMAN AMPLITUDE

$$\mathcal{M} = \varepsilon_\mu(p_1, \lambda_1)\, \varepsilon_\nu^*(p_1', \lambda_1')\, \mathcal{M}^{\mu\nu}$$

↳ UNPOLARIZED SQUARE AMPLITUDE

$$\frac{1}{2} \sum_{\lambda_1} \sum_{\lambda_2} |\mathcal{M}|^2$$

$$= \frac{1}{2} \sum_{\lambda_1} \varepsilon_\mu(p_1, \lambda_1)\, \varepsilon_{\mu'}^*(p_1, \lambda_1)$$

$$\cdot \sum_{\lambda_1'} \varepsilon_{\nu'}(p_1', \lambda_1')\, \varepsilon_\nu^*(p_1', \lambda_1')$$

$$\cdot\, \mathcal{M}^{\mu\nu} \left(\mathcal{M}^{\mu'\nu'}\right)^*$$

↳ GAUGE INVARIANCE

$$\varepsilon_\mu(p_1,\lambda_1) \to \varepsilon_\mu(p_1,\lambda_1) + a\,(p_1)_\mu$$

$$(p_1)_\mu\, \mathcal{M}^{\mu\nu} = 0$$

$$(p_1')_\nu\, \mathcal{M}^{\mu\nu} = 0$$

↳ $$\sum_{\lambda_1=\pm1} \varepsilon_\mu(p_1,\lambda_1)\, \varepsilon^*_{\mu'}(p_1,\lambda_1)$$

$$= -g_{\mu\mu'} + \underbrace{\text{TERMS IN } (p_1)_\mu \text{ OR } (p_1)_{\mu'}}$$

↓

GIVE 0 UPON CONTRACTION WITH $\mathcal{M}^{\mu\nu}(\mathcal{M}^{\mu'\nu'})^*$

↳ $$\frac{1}{2}\sum_{\lambda_1}\sum_{\lambda_2} |\mathcal{M}|^2$$

$$= \frac{1}{2}(-g_{\mu\mu'})(-g_{\nu\nu'})\, \mathcal{M}^{\mu\nu}(\mathcal{M}^{\mu'\nu'})^*$$

$$= \frac{1}{2}\, \mathcal{M}^{\mu\nu}(\mathcal{M}_{\mu\nu})^*$$