Problem Sheet 7

for the course

"Introduction to Lattice Gauge Theory"

Summer 2019

Lecturer: PD Dr. G. von Hippel

1. *Reading List*

   1. Smit, chapter 5 (p. 115-148)
   2. Les Houches Lecture Notes, sections 1.5.4 and 1.5.5 (p. 63-73)
   3. DeGrand/DeTar, chapter 12 (p. 235-244) and section 5.4 (p. 92-97)

2. *Area Law at Strong Coupling*

   (a) Rewrite the Wilson action as

   $$S = \sum_{\text{Plaq.}} \frac{\beta}{2N_c} \left(\mathrm{Tr}(\mathbb{1} - U_\Box) + \mathrm{Tr}(\mathbb{1} - U_\Box^\dagger) \right)$$

   and expand the path integral into a power series in β.

   (b) Show that the leading contribution to the expectation value of a rectangular Wilson loop with side lengths Ra and Ta is given by

   $$\langle \mathcal{W}_{RT} \rangle = \left(\frac{\beta}{2N_c} \right)^{RT} \left(\frac{1}{N_c} \right)^{2RT+R+T} N_c^{(R+1)(T+1)} .$$

   (c) Conclude that in the strong-coupling regime the Wilson loop satisfies an area law

   $$\langle \mathcal{W}_{RT} \rangle = \mathrm{e}^{-\sigma RT}$$

   and determnine the value of σ in this regime.

   (d) What can you conclude for the potential between two static colour sources?

3. *Mass Gap at Strong Coupling*

   (a) Derive a relationship between the mass of the lightest eigenstat of the Hamiltonian and the expectation value of the product of two spatial plaquettes,

   $$C(t) = \langle U_{kl}(t, \mathbf{0}) U_{kl}(0, \mathbf{0}) \rangle .$$

   (b) Determine the leading contribution to $C(t)$ in the strong-coupling regime, and use it to derive the mass of the lightest eigenstate in that regime.

   (c) Can you take the continuum limit for the results of the preceding problem?

4. *Grassmann Algebra and Grassmann Analysis*

In order to represent fermions in the path integral formalism, we have to introduce anticommuting variables, so-called Grassmann variables. These variables form an algebra with N generators subject to the anticommutation relations

$$\{\eta_i, \eta_j\} = 0.$$

(a) Show that any function of Grassmann variables can be written as

$$f(\eta) = \sum_{n=0}^{N} \sum_{i_1 < \ldots < i_n} a_{i_1 \cdots i_n} \eta_{i_1} \cdots \eta_{i_n}$$

with complex coefficients $a_{i_1 \cdots i_n}$.

(b) The derivative $\frac{\partial}{\partial \eta_i}$ is to be defined such that is satisfies

$$\frac{\partial}{\partial \eta_i} 1 = 0 \quad \text{and} \quad \frac{\partial}{\partial \eta_i} \eta_i = 1\,.$$

Show that consistency requires it to satisfy

$$\frac{\partial}{\partial \eta_i} \frac{\partial}{\partial \eta_j} = -\frac{\partial}{\partial \eta_j} \frac{\partial}{\partial \eta_i} \quad \text{and} \quad \frac{\partial}{\partial \eta_i} \eta_j = -\eta_j \frac{\partial}{\partial \eta_i}$$

for $i \neq j$.

(c) For the integration over Grassmann variables we demand $\mathbb{C}$-linearity and translation invariance, i.e.

$$\int \mathrm{d}\eta_i \, (z_1 f_1(\eta) + z_2 f_2(\eta)) = z_1 \int \mathrm{d}\eta_i \, f_1(\eta) + z_2 \int \mathrm{d}\eta_i \, f_2(\eta) \quad \text{and}$$

$$\int \mathrm{d}\eta_i \, f(\eta + \xi) = \int \mathrm{d}\eta_i \, f(\eta)$$

for $z_i \in \mathbb{C}$ and $\xi \neq \eta_i$. Show that this implies

$$\int \mathrm{d}\eta_i = 0 \qquad \int \mathrm{d}\eta_i \, \eta_i = 1 \qquad \mathrm{d}\eta_i \mathrm{d}\eta_j = -\mathrm{d}\eta_j \mathrm{d}\eta_i$$

if the integral is normalized appropriately.

(d) Show that for a change of variables $\eta' = M\eta$ the transformation formula

$$\mathrm{d}\eta_N \cdots \mathrm{d}\eta_1 = \det(M) \mathrm{d}\eta'_N \cdots \mathrm{d}\eta'_1$$

holds, and compare with the case of real variables.

(e) Show

$$\int \mathrm{d}\eta_N \mathrm{d}\bar{\eta}_N \cdots \mathrm{d}\eta_1 \mathrm{d}\bar{\eta}_1 \, \mathrm{e}^{\bar{\eta} M \eta} = \det(M)$$

for $M \in \mathbb{C}^{N \times N}$ with the Grassmann variables η_i, $\bar{\eta}_i$ taken to be independent.

5. *Perturbation Theory on the Lattice*

(a) Explain why in the limit of weak coupling $g_0 \to 0$ the link variables can be parameterized $U_{x,\mu} = \mathrm{e}^{ig_0 a A_{x,\mu}}$ with $A_{x,\mu} \in \mathfrak{su}(N_c)$.

(b) Let $G(U) = 0$ be an arbitrary gauge condition. Following the Fadeev-Popov procedure known from the continuum, derive the identities

$$\begin{aligned}\int \mathrm{D}U\, \mathrm{e}^{-S(U)} &= \int \mathrm{D}U\, \mathrm{e}^{-S(U)} \delta(G(U)) \det(M(U)) \\ &= C(\alpha) \int \mathrm{D}U \mathrm{D}\bar{c} \mathrm{D}c\, \mathrm{e}^{-S(U) - \bar{c}M(U)c - \frac{1}{2\alpha}|G(U)|^2} \\ &= C(\alpha) \int \mathrm{D}A \mathrm{D}\bar{c} \mathrm{D}c\, \mathrm{e}^{-S(U) - \bar{c}M(U)c - \frac{1}{2\alpha}|G(U)|^2 - \mu(A)}\end{aligned}$$

with α an arbitrarily chosen gauge parameter, and c and $\bar{c}$ a pair of Grassmann-valued scalar fields (Fadeev-Popov ghosts). Give suitable expressions for $M(U)$ and $\mu(A)$.

(c) Which steps remain to be done in order to state the Feynman rules for gauge theory with a lattice regulator? Look up the results (e.g. in DeGrand/DeTar chap. 12) and verify at least part thereof.

(d) Why is perturbation theory so much more complicated on the lattice than in the continuum?