Problem Sheet 6

for the course

„Introduction to Lattice Gauge Theory“

Summer 2019

Lecturer: PD Dr. G. von Hippel

1. *Reading List*

   1. Smit, chapters 4 and 5 (p. 83-148)
   2. Gattringer/Lang, chapters 2 and 3 (p. 25-71)
   3. Les Houches Lecture Notes, section 1.5 (p. 50-75)

2. *Naive Continuum Limit of the Wilson Action*

   The Wilson action for a $\mathrm{SU}(N_c)$ gauge theory on the lattice is given by the product of the link variables around each plaquette on the lattice:

   $$S_W(U) = \beta \sum_{x\in\Lambda} \sum_{\mu<\nu} \left(1 - \frac{1}{N_c} \mathrm{Re}\ \mathrm{tr}\ U_{x,\mu\nu}\right)$$

   with

   $$U_{x,\mu\nu} = U^\dagger_{x,\nu} U^\dagger_{x+\hat\nu,\mu} U_{x+\hat\mu,\nu} U_{x,\mu}.$$

   (a) Introduce the gauge potential $A_\mu(x)$ via $U_{x,\mu} = \mathrm{e}^{aA_\mu(x+\frac{1}{2}\hat\mu)}$ and use the Baker-Campbell-Hausdorff (BCH) formula

   $$\mathrm{e}^X \mathrm{e}^Y = \mathrm{e}^{X+Y+\frac{1}{2}[X,Y]+\dots}$$

   to write $U_{x,\mu\nu}$ as a single exponential, keeping terms up to order a^2 explicit.

   (b) Expand the gauge potentials in the exponent into a Taylor series around x to show $U_{x,\mu\nu} = \mathrm{e}^{a^2F_{\mu\nu}(x)+\mathrm{O}(a^3)}$.

   (c) Determine the value of β for which the Wilson action reproduces the Yang-Mills action in the naive continuum limit $a \to 0$.

   (d) In the case of electrodynamics, we could also discretize the Maxwell action directly in terms of $A_\mu(x)$. Why is this not possible for the Yang-Mills action for a non-abelian theory?

3. *The Haar Measure on $SU(N_c)$*

The Haar measure $\mathrm{d}\mu(U)$ on a compact semi-simple Lie group G is uniquely defined by the properties

$$\mathrm{d}\mu(VU) = \mathrm{d}\mu(UV) = \mathrm{d}\mu(U) \qquad \text{für alle } V \in G$$

and

$$\int_G \mathrm{d}\mu(U) = 1\,.$$

In this problem we will construct the Haar measure on $\mathrm{SU}(N_c)$.

(a) Show that

$$(X, Y) = \mathrm{tr}\left(X^\dagger Y\right)$$

defines a scalar product on the space $\mathbb{C}^{N_c \times N_c}$ of complex $N_c \times N_c$-matrices which is invariant under multiplication with a unitary matrix $U \in \mathrm{SU}(N_c)$.

(b) Conclude that an invariant metric on $\mathrm{SU}(N_c)$ is given by

$$\mathrm{d}s^2 = (\mathrm{d}U, \mathrm{d}U) = g_{mn}(\alpha)\mathrm{d}\alpha^m \mathrm{d}\alpha^n\,,$$

where $U(\alpha) = \mathrm{e}^{\alpha_k T^k}$ is a parameterization of $\mathrm{SU}(N_c)$ by antihermitian generators T^k that satisfy $[T^a, T^b] = f_c^{ab} T^c$.

(c) Show: The Haar measure on $\mathrm{SU}(N_c)$ is

$$\mathrm{d}\mu(U) = C\sqrt{|\det(g)|}\prod_k \mathrm{d}\alpha_k$$

with a suitable constant C. (Hint: Use the transformation properties of the metric tensor under a coordinate transformation $\alpha \mapsto \alpha'$).

(d) Use the identity $UU^\dagger = \mathbb{1}$ to derive an expression for the metric tensor $g_{mn}(\alpha)$.

(e) Derive the following integrals:

$$\begin{aligned} \int_{\mathrm{SU(N_c)}} \mathrm{d}\mu(U) U_{ij} &= 0 \\ \int_{\mathrm{SU(N_c)}} \mathrm{d}\mu(U) U_{ij} U_{kl}^* &= \frac{1}{N_c}\delta_{ik}\delta_{jl} \end{aligned}$$

4. *Transfer Matrix for Pure Gauge Theory with Wilson Action*

In this problem we will construct the transfer matrix for the pure gauge theory with fields $U_{x,\mu} \in \mathrm{SU}(N_c)$ and the Wilson action

$$S_{\mathrm{W}} = \beta \sum_x \sum_{\nu<\mu} \frac{1}{N_c} \mathrm{Re}\ \mathrm{Tr}(\mathbb{1} - U_{x,\mu\nu})$$

where $U_{x,\mu\nu} = U_{x,\mu} U_{x+\hat{\mu},\nu} U^\dagger_{x+\hat{\nu},\mu} U^\dagger_{x,\nu}$ is the smallest possible Wilson loop on the lattice (a plaquette).

As Hilbert space we consider the space $\mathcal{H}$ spanned by the eigenstates $|U\rangle$ of the spatial link operators $\hat{U}_{\mathbf{x},k}$ (with $\hat{U}_{\mathbf{x},k}|U\rangle = U_{\mathbf{x},k}|U\rangle$). Orthonormality is taken to be

$$\langle U'|U\rangle = \prod_{\mathbf{x},k} \delta(U'_{\mathbf{x},k} - U_{\mathbf{x},k})$$

with a Dirac distribution normalized according to

$$\int \mathrm{d}\mu(U)\, \delta(U - U') = 1\,.$$

(a) Confirm that with these definitions the scalar product between two arbitrary states is given by

$$\langle\phi|\psi\rangle = \int \prod_{\mathbf{x},k} \mathrm{d}\mu(U_{\mathbf{x},k})\, \phi^*(U)\psi(U)$$

with suitably defined and normalized wavefunctions ψ, ϕ.

(b) Under a gauge transformation Ω the spatial link variables transform according to $U^\Omega_{\mathbf{x},k} = \Omega_{\mathbf{x}} U_{\mathbf{x},k} \Omega^\dagger_{\mathbf{x}+\hat{\mathbf{k}}}$. Show that $(\hat{D}(\Omega)\psi)(U) = \psi(U^\Omega)$ defines a representation of the gauge group on the Hilbert space $\mathcal{H}$.

(c) Physical states must be gauge invariant. Show that

$$(\hat{P}_{\mathrm{phys}}\psi)(U) = \int \prod_{\mathbf{x}} \mathrm{d}\mu(\Omega_{\mathbf{x}})\, \psi(U^\Omega)$$

defines a projector on the subspace $\mathcal{H}_{\mathrm{phys}}$ of gauge-invariant states in $\mathcal{H}$.

(d) Decompose the Wilson action into a purely spatial and a spatio-temporal part and conclude that the it can be written in the form

$$S_{\mathrm{W}} = \sum_n \left(\frac{1}{2} V(U^{(n+1)}) + K(U^{(n+1)}, U_0^{(n)}, U^{(n)}) + U + \frac{1}{2} V(U^{(n)}) \right)$$

with purely spatial link variables $U^{(n)}$ on the timeslice $t = na$ and temporal link variables $U_0^{(n)}$ between the timeslices $t = na$ and $t = (n+1)a$. Determine explicit expressions for V and K.

(e) Conclude that the transfer matrix is given by $\hat{T} = \mathrm{e}^{\frac{1}{2}\hat{V}}\hat{T}_K\mathrm{e}^{\frac{1}{2}\hat{V}}$ with $\hat{V}|U\rangle = V(U)|U\rangle$ and $\hat{T}_K = \hat{T}_K^0\hat{P}_{\text{phys}}$ with

$$\langle U'|\hat{T}_K^0|U\rangle = \mathrm{e}^{\beta\sum_{\mathbf{x},k}\frac{1}{N_c}\mathrm{Re\ Tr}\left(\mathbb{1}-\mathrm{U}'_{\mathbf{x},\mathrm{k}}\mathrm{U}^\dagger_{\mathbf{x},\mathrm{k}}\right)}\,.$$

What is the role of the temporal gauge links U_0?

(f) Show that $[\hat{P}_{\text{phys}}, \hat{T}] = 0$, and conclude that the time evolution of the theory preserves $\mathcal{H}_{\text{phys}}$.

(g) Show that $\langle\psi|\hat{T}|\psi\rangle > 0$ for all physical states $|\psi\rangle \in \mathcal{H}_{\text{phys}}$, provided that $\langle\phi|\hat{T}_K^0|\phi\rangle > 0$ for arbitrary states $|\phi\rangle \in \mathcal{H}$. Why does the latter inequality hold?

5. *Wilson Loops and Static Potential*

We consider the theory of a static scalar field $\phi_x \in \mathbb{C}^{N_c}$ with action

$$S = S_{\mathrm{W}} + a^4\sum_x \left(\nabla_0\phi_x^\dagger\nabla_0\phi_x + m^2\phi^\dagger\phi\right)$$

where the covariant derivative is given by $\nabla_\mu\phi_x = a^{-1}\left(U_{x,\mu}\phi_{x+\hat{\mu}} - \phi_x\right)$.

(a) Show that the propagator of the scalar field for a given gauge field U is given by

$$\langle\phi_x\phi_y^\dagger\rangle = \frac{a}{2\sinh(a\omega)}\mathrm{e}^{-\omega|x_0-y_0|}\theta(x_0-y_0)\delta(\mathbf{x}-\mathbf{y})\prod_{n=0}^{a^{-1}|x_0-y_0|-1} U_{y+n\hat{0},0}\,.$$

(b) Consider the operator $O_R(t) = \phi^\dagger_{x+R\hat{k}}\left(\prod_{m=0}^{R-1}U_{x+m\hat{k}}\right)\phi_x$ with $x_0 = t$, $\mathbf{x} = \mathbf{0}$, and show that

$$\langle O_R^\dagger(T)O_R(0)\rangle = C\langle\mathcal{W}_{RT}\rangle\mathrm{e}^{-E_0T}$$

with constants C and E_0, where $\mathcal{W}_{RT}$ is a rectangular Wilson loop with sides R and T.

(c) Use the transfer matrix to explain why this implies that the potential $V(R)$ between two static sources obeys

$$V(R) \sim -\frac{1}{T}\log\langle\mathcal{W}_{RT}\rangle$$

for $0 \ll T \ll N_t$.

(d) Conclude that an *area law* for the Wilson loop implies a linear potential $V(R) \sim \sigma R$.