Problem Sheet 5

for the course
„Introduction to Lattice Gauge Theory"
Summer 2019

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1. *Reading List*
   1. Kogut, sections V-VI (p. 681-694)
   2. Smit, sections 4.1 and 4.5 (p. 83-85, 97-98)
   3. Les Houches Lecture Notes, section 1.5.1 (p. 50-54)

2. *The Ising Gauge Model*

   (a) The Ising model has a global $\mathbb{Z}_2$-symmetry $\sigma \mapsto -\sigma$. How must a link variable λ_{ij} transform in order for the Hamiltonian

   $$H = -J \sum_{(i,j)} \sigma_i \lambda_{ij} \sigma_j$$

   to be invariant under *local* $\mathbb{Z}_2$-transformations $\sigma_i \mapsto \omega_i \sigma_i$ with arbitrary $\omega_i \in \mathbb{Z}_2$?

   (b) Show that the product of link variables along an arbitrary closed path is invariant under the transformations found in the preceding problem (*gauge transformations*).

   (c) Consider now a theory of only link variables with the Hamiltonian

   $$H = -\sum_i \sum_{\mu<\nu} \lambda_{i,i+\hat{\mu}} \lambda_{i+\hat{\mu},i+\hat{\mu}+\hat{\nu}}, \lambda_{i+\hat{\nu},i+\hat{\mu}+\hat{\nu}} \lambda_{i,i+\hat{\nu}}$$

   (i.e. the product of the link variables around each plaquette of the lattice) and show that it is invariant under gauge transformations.

   (d) Show that in the case $D = 2$ the theory from the preceding problem (the Ising gauge model) can be written as a collection of Ising models that become uncoupled in the infinite-volume limit.

3. *Elitzur's Theorem*

   (a) The spontaneous magnetization is defined by $\lim_{h\to 0} \lim_{V\to\infty} \langle \lambda_{ij} \rangle_h$ (with the limits taken in that order), where

   $$\langle O \rangle_h = \frac{\sum_{\{\lambda\}} O(\lambda) \mathrm{e}^{-\beta H(\lambda) + h \sum_{(i,j)} \lambda_{ij}}}{\sum_{\{\lambda\}} \mathrm{e}^{-\beta H(\lambda) + h \sum_{(i,j)} \lambda_{ij}}}$$

is the expectation value in the presence of a homogeneous external field. Consider the gauge transformation

$$\omega_i = \begin{cases} -1, & i = n \\ +1, & \text{sonst} \end{cases}$$

and use it to show that

$$\langle \lambda_{n,n+\hat{\mu}} \rangle_h = -\langle \lambda_{n,n+\hat{\mu}} \mathrm{e}^{-2h \sum_\mu (\lambda_{n,n+\hat{\mu}} + \lambda_{n-\hat{\mu},n})} \rangle_h \,.$$

(b) Use the result of the preceding problem to derive the bound $2|\langle \lambda_{n,n+\hat{\mu}} \rangle_h| \leq |\langle \lambda_{n,n+\hat{\mu}} \rangle_h| \cdot |1 - \mathrm{e}^{4Dh}|$

(c) Conclude that the spontaneous magnetization has to vanish in the Ising gauge model.

4. *Wilson Loops in the Ising Gauge Model*

For a closed loop Γ on the lattice we defined P_Γ as its perimeter, A_Γ as the area of the minimal surface it encloses, and

$$W_\Gamma = \prod_{\ell \in \Gamma} \lambda_\ell$$

as the corresponding Wilson loop.

(a) Use the formula of problem 6 (a) from problem sheet 2 to write the expectation value of the Wilson loop as

$$\langle W_\Gamma \rangle = \frac{\sum_{\{\sigma\}} \prod_{\text{Plaq}} (1 + \lambda\lambda\lambda\lambda \tanh \beta) \prod_\ell \lambda_\ell}{\sum_{\{\sigma\}} \prod_{\text{Plaq}} (1 + \lambda\lambda\lambda\lambda \tanh \beta)} .$$

(b) For high temperatures we can expand numerator and denominator in powers of $\kappa = \tanh \beta$ (and accordingly in terms of plaquettes). Which geometric condition must be fulfilled in order to obtain a non-vanishing contribution? What is the leading contribution? Show that it yields an *area law*

$$\langle W_\Gamma \rangle = \mathrm{e}^{-\sigma A_\Gamma}$$

and determine the value of σ.

(c) For low temperatures (and $D > 2$) we expand around the ground state in which all plaquettes take the value take the value +1; one of many realizations of this state that are equivalent up to gauge transformations is $\lambda_\ell = +1$ for all links. Explain why the contribution of a state with n flipped links $\lambda_\ell = -1$ to the numerator of $\langle W_\Gamma \rangle$ is given by $\frac{(V - 2P_\Gamma)^n}{n!} \mathrm{e}^{-4n(D-1)\beta}$ up to exclusion effects (which become negligible in the infinite-volume limit and for long paths). Also explain why the corresponding contribution to the denominator is given by $\frac{V^n}{n!} \mathrm{e}^{-4n(D-1)\beta}$. Conclude that this results in a *perimeter law*

$$\langle W_\Gamma \rangle = \mathrm{e}^{-\alpha P_\Gamma}$$

and determine the value of α.

(d) How would you proceed in each case in order to derive higher-order corrections to the results of the preceding problems?

5. *Scalar U(1) Gauge Theory*

(a) Consider the theory of a complex scalar field ϕ coupled to Maxwell electrodynamics with Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} |D\phi|^2 - \frac{m^2}{2} |\phi|^2 \,,$$

where $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative, and $F_{\mu\nu} = -\frac{i}{e}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor. Show that the theory is invariant under gauge transformations

$$\phi \mapsto e^{ie\chi}\phi \qquad A_\mu \mapsto A_\mu + \partial_\mu \chi$$

with an arbitrary function $\chi \in \mathcal{C}^\infty(\mathbb{R}^{1,3})$.

(b) Formulate the naive discretization of this theory (continuing it to Euclidean metric first) and show that it is not invariant under the discrete version of the gauge transformation. Where does it fail, and why?

(c) Consider the parallel transporter along a path Γ (Wilson line)

$$U_\Gamma = \mathrm{e}^{ie \int_\Gamma dx^\mu A_\mu}$$

and determine its behavior under gauge transformations.

(d) Conclude that expressions of the form $\phi^*(y) U_\Gamma \phi(x)$ are gauge-invariant if Γ leads from $x$to y. What can you say about expressions of the form $f(U_\Gamma)$ if Γ is a closed curve?

(e) Discretize the action by introducing link variables $U_\mu(x)$ with the transformation behavior of a Wilson line from x to $x+ae_\mu$ and using the discrete covariant derivative $\Delta_\mu^+ \phi(x) = a^{-1} \left(\phi(x + a\hat{\mu}) - U_\mu(x)\phi(x) \right)$. Show that this discretization is invariant under gauge transformations and has the correct continuum limit. (Hint: identify $U_\mu(x) = \mathrm{e}^{ieaA_\mu(x)}$).

(f) Consider the alternative discretization of the Maxwell term given by

$$S = \beta \sum_x \sum_{\mu<\nu} \mathrm{Re}\; U_{\mu\nu}(x) \,,$$

where $U_{\mu\nu}(x) = U_\nu^*(x) U_\mu^*(x + a\hat{\nu}) U_\nu(x + a\hat{\mu}) U_\mu(x)$ is the directed product of the link variables around a plaquette in the (μ, ν) plane. Show that this discretization is invariant under gauge transformations and has the correct continuum limit for a suitably chosen value of the parameter β.