Problem Sheet 4

for the course

„Introduction to Lattice Gauge Theory“

Summer 2019

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1. *Reading List*
   1. Smit, section 3.9 (p. 60-67)
   2. Gattringer/Lang, chapter 4 (p. 73-79, 89-100)
   3. Les Houches Lecture Notes, sections 6.2.1-6.2.2 (p. 346-357)

2. *Markov Chains*

   Let X be a finite set of states, P a probability distribution on X, and $O_i : X \to \mathbb{R}$ a family of functions (observables) on X. In order to determine

$$\langle O_i \rangle \equiv \sum_{s \in X} O_i(s) P(s)\,,$$

   we seek a method to generate a sequence of N states $s_k \in X$ that are distributed according to P such that

$$\overline{O}_i \equiv \frac{1}{N} \sum_{k=1}^{N} O_i(s_k) = \langle O_i \rangle + \mathrm{O}(N^{-1/2})\,.$$

   One possibility to achieve this is by mans of a Markov chain.
   A Markov chain $(s_k)_{k \in \mathbb{N}}$ on X is a sequence of states $s_k \in X$, where s_{k+1} is generated from s_k by a stochastic process with constant transition probability for each pair of states (Markov process). We represent a Markov process by a matrix T, the elements $T_{s's}$ of which give the probability to transition from s to s'.
   Clearly, T must satisfy

$$T_{s's} \geq 0 \text{ für alle } s, s' \text{ und } \sum_{s'} T_{s's} = 1 \text{ für alle } s \tag{1}$$

   We also demand that the Markov process be aperiodic, i.e.

$$\sum_{s} T_{s's} P_s = P_{s'} \text{ für alle } s', \text{ und} \tag{2}$$

$$T_{ss} > 0 \text{ für alle } s. \tag{3}$$

   Finally, we demand the Markov process to be ergodic, i.e. for every non-empty subset $S \subset X$ there exists a pair of states s, s' with $s \in S$, $s' \notin S$, and $T_{s's} > 0$.

(a) Show that the transition probability

$$T^0_{ij} = \begin{cases} \frac{1}{3} \text{ für } i = j, i = j \pm 1 \bmod |X|\,, \\ 0 \text{ sonst}\,, \end{cases}$$

on $X = \{0, \ldots, |X|-1\}$ satisfies our demands w.r.t the probability distribution $P_0(s) = \frac{1}{|X|}$.

(b) Show that the transition probability

$$T_{ij} = T^0_{ij} A_{ij} + \delta_{ij} \sum_k T^0_{kj}(1 - A_{kj})$$

with

$$A_{ij} = \min\{1, P(i)/P(j)\}$$

on $X = \{0, \ldots, |X|-1\}$ satisfies our demand w.r.t the (arbitrary) probability distribution $P(s)$.

(c) Let T now be a Markov process satisfying our demand w.r.t. some probability distribution P. Let $\mathcal{H}$ be the space of all functions $f : X \to \mathbb{R}$ with the norm $\|f\|_1 = \sum_{s \in X} |f(s)|$. We define an operator $\hat{T}$ on $\mathcal{H}$ by

$$(\hat{T} f)(s') = \sum_{s \in X} T_{s's} f(s)\,.$$

Show: For all $f \in \mathcal{H}$ the inequality $\|\hat{T} f\|_1 \leq \|f\|_1$ holds. (Hint: Decompose $f = f_+ - f_-$ with $f_\pm \geq 0$.)

(d) Show: $P \in \mathcal{H}$ is the only eigenvector to the eigenvalue 1 of $\hat{T}$. (Hint: Consider the action of $\hat{T}$ on $f_\pm$.)

(e) Let us defined a scalar product on $\mathcal{H}$ by

$$(f, g) = \sum_{s \in X} \frac{f(s) g(s)}{P(s)}$$

and let $\|f\| = \sqrt{(f, f)}$ be the corresponding norm. Show: if T satisfies our demands, then so does the Markov process with associated operator $\hat{Q} = \hat{T}^\dagger \hat{T}$, where $\hat{T}^\dagger$ is defined w.r.t. the scalar product defined above.

(f) Let now $\mathcal{H}_0 = \{f \in \mathcal{H} | \sum_{s \in x} f(s) = 0\}$. Show: there exists a $\rho \in [0; 1)$ such that $\|\hat{T} f\| \leq \rho \|f\|$ for all $f \in \mathcal{H}_0$. (Hint: Consider the spectrum of $\hat{Q}$ on $\mathcal{H}_0$).

(g) Consider now a Markov chain with initial state s_1. Show: The probability $P_k(s)$ that $s_k = s$ tends to $P(s)$ for $k \to \infty$ as

$$P_k(s) = P(s) + \mathrm{O}(\mathrm{e}^{-k/\tau})$$

with the so-called exponential autocorrelation time $\tau = -1/\log \rho$.

(h) Conclude that the elements $(s_k)_{k \gg \tau}$ of a Markov chain constitute an appropriate sample in the sense of the original problem.