Problem Sheet 3

for the course

„Introduction to Lattice Gauge Theory“

Summer 2019

Lecturer: PD Dr. G. von Hippel

1. *Reading List*
   1. Smit, section 3.10 (p. 67-71)
   2. Parisi, chapter 7 (p. 112-136)

2. *Real-Space Renormalization Group and critical exponents*

   A powerful method to study the behavior of statistical systems in the neighborhood of a critical point is the use of renormalization group transformations that in the case of the Ising model can be written as a repeated average over blocks B,

   $$\sigma_i^{(k)} = \operatorname{sgn}\left(\frac{1}{|B|}\sum_{j\in B_i}\sigma_j\right) , \qquad \sigma_i^{(0)} = \sigma_i .$$

   For odd block sizes $|B|$, this corresponds to the majority rule. The block variables $\sigma_i^{(k)}$ can then be considered as fields on a coarser lattice.

   (a) Explain why the partition function

   $$Z = \sum_{\{\sigma\}} \mathrm{e}^{-\beta H(\sigma)}$$

   can be written as

   $$Z = \sum_{\{\sigma^{(k)}\}} \mathrm{e}^{-\beta H^{(k)}(\sigma^{(k)})}$$

   and give an expression for $H^{(k)}(\sigma^{(k)})$ in terms of $H^{(k-1)}(\sigma^{(k-1)})$.

   (b) Interpret this expression as a transformation acting in the space of possible Hamilton operators, $H^{(k)} = R(H^{(k-1)})$. Interpreting $H^{(k)}$ as the evaluation of a continuous operator-valued function $H^{(\lambda)}$ at $\lambda = k \in \mathbb{N}$, derive a differential equation which $H^{(\lambda)}$ must satisfy. Which necessary condition must H^* satisfy in order for $\lim_{\lambda\to\infty} H^{(\lambda)} = H^*$ to hold?

   (c) Consider now a finite-dimensional approximation $H^{(\lambda)}(\sigma) = \sum_{a=1}^{N_{\mathrm{op}}} c_a(\lambda) g_a(\sigma)$ and express the corresponding flow equation in the form $\frac{\mathrm{d}}{\mathrm{d}\lambda} c_a = b_a(c)$. Which necessary condition must c^* satisfy in order for $H^*(\sigma) = \sum_{a=1}^{N_{\mathrm{op}}} c_a^* g_a(\sigma)$ to hold?

(d) Parameterize small deviations from the fixed point by $c = c^* + \delta c$ and expand the flow equation to first order in δc to derive a linear differential equation for δc. Which quantities characterize the attractive or repulsive nature of a fixed point?

(e) Explain why the existence of two attractive fixed points necessitates the existence of a critical surface, i.e. a hypersurface $\mathcal{C}$ in $\mathbb{R}^{N_{\mathrm{op}}}$ such that trajectories starting on $\mathcal{C}$ remain in $\mathcal{C}$ for all values of λ.

(f) Conclude that a fixed point with a repulsive direction must exist on the critical surface.

(g) Explain by physical arguments why the high- and low-temperature regimes of the Ising model each constitute an attractive fixed point.

(h) Conclude that β parameterizes the direction normal to the critical surface (at $\beta = \beta_c$), and that the fixed point on the critical surface corresponds to a strongly-correlated, self-similar system.

(i) Conclude moreover that for $|\beta - \beta_c| \to 0$, $\lambda \to \infty$, the Hamiltonian can be parameterized as $H = H_c + (\beta - \beta_c)\mathrm{e}^{\lambda\omega_R} H_R$ with $\omega_R > 0$.

(j) Explain why for $\lambda \sim -\frac{1}{\omega_R} \log|\beta - \beta_c|$ a change of regime (strongly-correlated vs. uncorrelated) must occur, and determine the corresponding block size.

(k) Conclude that for $\beta \to \beta_c$ the correlation length ξ of the system diverges like $\xi \propto |\beta - \beta_c|^{-\frac{1}{\omega_R}}$, corresponding to a critical exponent $\nu = \frac{1}{\omega_R}$.

(l) Explain in physical terms why the continuum limit of a lattice-regularized quantum field theory corresponds to a critical point of the corresponding statistical model.