Problem Sheet 2

for the course
„Introduction to Lattice Gauge Theory“
Summer 2019

Lecturer: PD Dr. G. von Hippel

1. *Reading List*

   1. Smit, chapter 3 (p. 32–55)
   2. Parisi, chapters 3 and 4 (p. 22–66)

2. *The Hopping-Parameter Expansion*

   (a) Consider the theory of a free scalar field ϕ with mass m_0 and split the action into a local term and a term coupling neighboring lattice points. Perform a rescaling $\phi = \sqrt{2\kappa}\varphi$ and determine the value of κ as a function of m_0 for which the coefficient of the local term becomes equal to one.

   (b) In the following we assume that κ has the value determined in the preceding problem. Which critical value κ_c of κ corresponds to the massless case?

   (c) Consider now a correlation function of the form $\langle \varphi_x \varphi_y \rangle$ and expand the corresponding path integral into a power series in κ. Which terms give a non-vanishing contribution? Interpret the result graphically.

3. *From the ϕ^4-Theory to the Ising Model*

   (a) We now add an interaction $V_I(\phi) = \frac{\lambda_0}{4!} \sum_x \phi_x^4$ to the theory from the preceding problem. For which values of κ and λ as a function of m_0 and λ_0 does the local term take the form $\varphi_x^2 + \lambda(\varphi_x^2 - 1)^2$?

   (b) Consider now the limit $\lambda \to \infty$ and show that in this limit the theory can be written as an Ising model with partition function

   $$Z = \sum_{\{\sigma=\pm 1\}} \mathrm{e}^{-\beta H(\sigma)}$$

   and Hamiltonian

   $$H = -J \sum_{(i,j)} \sigma_i \sigma_j$$

   where the sum over (i, j) run over all pairs of neighboring lattice sites.

4. *Solution of the Ising model in one dimension and in infinitely many dimensions*

   (a) Write the partition function for the one-dimensional Ising model using the transfer matrix (here literally a 2×2 matrix). By diagonalizing the transfer matrix, derive the exact solution for the free energy per site $\frac{F}{L} = -\frac{1}{\beta L} \log Z$ in the thermodynamic limit $L \to \infty$.

   (b) In the opposite case of infinitely many dimensions, $D \to \infty$, explain why we can replace the sum of the neighboring spins by the expectation value of the spin: $\sum_{(i,j)} \sigma_j \to 2D\langle\sigma_i\rangle$. Use this ansatz to derive an equation which the magnetization $m = \langle\sigma_i\rangle$ has to satisfy.

   (c) If we use the infinite-dimensional ansatz also in a finite number D of dimensions, we obtain an approximation (the mean-field approximation). Under which conditions does the resulting equation have how many solutions? Interpret the result graphically and physically.

5. *Low-Temperature Expansion of the Ising Model*

   (a) Find the ground state of the Ising model in a volume $V = L^D$ with periodic boundary conditions, and determine its energy.

   (b) Considering now states that differ from the ground state by flipping one and two spins, respectively, determine how many such states there are in each case, and determine their energies.

   (c) Use the results of the preceding two questions to expand the partition function Z into a power series in a suitable variable λ (which is to satisfy $\lambda \to 0$ for $\beta \to \infty$).

   (d) Explain why the free energy per site, $\frac{F}{V}$, is independent of V.

6. *High-Temperature Expansion of the Ising Model*

   (a) Show the identity

   $$e^{\beta ss'} = \cosh\beta\,(1 + ss' \tanh\beta)$$

   for $s, s' = \pm 1$.

   (b) Use this identity to rewrite the partition function of the Ising model as

   $$Z = (\cosh(\beta J))^V \sum_{\{n_{ij}=0,1\}} \kappa^{\sum_{(i,j)} n_{ij}} \sum_{\{\sigma\}} \prod_{(i,j)} (\sigma_i \sigma_j)^{n_{ij}}$$

   with $\kappa = \tanh(\beta J)$.

   (c) Consider the contribution of an individual spin σ_l to the partition function and show that it contributes a factor 2 to $\sum_{\{\sigma\}} \prod_{(i,j)} (\sigma_i\sigma_j)^{n_{ij}}$ if $\sum_k n_{lk}$ is even, whereas the partition function vanishes otherwise.

   (d) Use the result of the preceding question to rewrite Z as a a sum over configurations of only the auxiliary variables n_{ij}, and interpret the result geometrically.