Cosmology and General Relativity: HW 3

Turn in solutions to “Wittig” (Hausaufgaben) mailbox in KPH by **noon, 23 May 2019**

Problem 1 More differential forms: For a manifold M, let T_pM be the tangent space at $p \in M$ with basis $\{e_{(\mu)}\}$ and T_p^*M be the corresponding cotangent space with basis $\{\theta^{(\mu)}\}$. A metric is a symmetric bilinear (0,2)-tensor

$$g = g_{\mu\nu}\theta^{(\mu)} \otimes \theta^{(\nu)} \in T_p^*M \otimes T_p^*M \ .$$

One can naturally define am isomorphism $\phi : T_pM \to T_p^*M$ such that

$$[\phi(x)]\,(y) = g(x, y) \ ,$$

for vectors $x, y \in T_pM$. This operation acts by “lowering” the indices of vectors. Let $g^* : T_p^*M \times T_p^*M \to \mathbb{R}$ be a (2,0)-tensor such that $g^*(\phi(x), \phi(y)) = g(x, y)$.

a. Write out $g^{*\mu\nu} \equiv g^*(e_{(\mu)}, e_{(\nu)})$ explicitly in terms of $g_{\mu\nu}$. Use this to prove that ϕ is bijective (i.e., invertible). *2pt*

On an n-dimensional manifold with metric g, using the **Hodge star *** one can associate A to the $(n-p)$-form $(*A)$ with components:

$$(*A)_{\mu_1\ldots\mu_{n-p}} = \frac{1}{p!}\sqrt{|\det g|}\epsilon_{\mu_1\ldots\mu_n} g^{\mu_{n-p+1}\nu_1}\ldots g^{\mu_n\nu_p} A_{\nu_1\ldots\nu_p} \ .$$

The combination $\sqrt{|\det g|}\epsilon_{\mu_1\ldots\mu_n}$ is needed for $(*A)$ to transform as a tensor.

b. Consider $M = \mathbb{R}^3$ together with a scalar function $\phi(x)$ and a vector $\vec{u}(x)$ and rewrite the known operators grad, div, rot in terms of the differential forms language. Rederive the known identities: *3pt*

   (i) rot grad $\phi = 0$,

   (ii) div rot $\vec{u} = 0$,

   (iii) Let $\vec{v}$ be another vector field. Express $\vec{u} \times \vec{v}$ also in differential form.

c. In the standard model, one defines gauge fields $\boldsymbol{A} : TM \to G$ which are 1-forms that take a tangent vector and return an element of the gauge group G. Yang-Mills theory defines a field-strength tensor as the $(0, 2)$-tensor

$$\boldsymbol{F} = d\boldsymbol{A} + \boldsymbol{A} \wedge \boldsymbol{A} \ .$$

Written out in components,

$$\boldsymbol{F}_{\mu\nu} = \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu + \frac{1}{2}[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu] \ ,$$

where $[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu] \equiv \boldsymbol{A}_\mu \boldsymbol{A}_\nu - \boldsymbol{A}_\nu \boldsymbol{A}_\mu$ is the commutator. For the abelian group $G = U(1)$, the commutator term vanishes and we recover the electromagnetic field-strength tensor. Naïvely, one may think that $\boldsymbol{A} \wedge \boldsymbol{A} = 0$ by antisymmetry. Why does this exterior product not vanish? *1pt*

Problem 2 Push-forwards and pull-backs: Let M, N be differentiable manifolds with charts (U, φ) and (V, ψ), respectively, and $f : M \to N$ a smooth map with $f(p) \in V$ for $p \in U$. The local coordinates are $x = \varphi(p)$ for (U, φ) and $y = \psi(f(p))$ for (V, ψ). *Hint:* For this problem, it may help to write out a diagram to keep track of the different spaces, elements, and functions.

a. Using $V = V^\mu \frac{\partial}{\partial x^\mu} \in T_pM$ and *push-forward* $f_*V = W^\alpha \frac{\partial}{\partial y^\alpha}$, show that $W^\alpha = V^\mu \frac{\partial y^\alpha}{\partial x^\mu}$. *2pt*

b. Show that for $\omega = \omega_\alpha dy^\alpha \in T^*_{f(p)}N$ the induced 1-form $f^*\omega = \xi_\mu dx^\mu \in T^*_pM$ (the *pull-back*) has the following components: $\xi_\mu = \omega_\alpha \frac{\partial y^\alpha}{\partial x^\mu}$. *2pt*

c. Let M be a Riemannian manifold. Consider the curve $\gamma : [a, b] \to M$, $\lambda \mapsto p(\lambda)$ and for simplicity assume that $\mathrm{Im}(\gamma) \subset M$ can be covered by a single chart. Compute the pull-back of the metric g on this curve. What is the geometric interpretation of this expression? *2pt*

Problem 3 Geometric interpretation of the torsion tensor: Let $p \in M$ with coordinates p^μ. $X = \epsilon^\mu \partial_\mu$ and $Y = \delta^\mu \partial_\mu$ are taken as two infinitesimal vectors in T_pM, which can be viewed as small displacements; i.e., they define two points q and s near p with coordinates $p^\mu + \epsilon^\mu$ and $p^\mu + \delta^\mu$, respectively. By parallel transporting s in the X direction, one obtains a new point r_1. Likewise, transporting q in the Y direction, one obtains a new point r_2. Let $(\nabla_\mu X)^\nu = \left(\frac{\partial}{\partial x^\mu} X\right)^\nu + C^\nu_{\mu\sigma} X^\sigma$, with *no symmetry conditions* on the indices of $C^\nu_{\mu\sigma}$.

a. Show that the vectors $\overline{qr_1}$ and $\overline{sr_2}$ are given by *2pt*

$$\epsilon^\mu - C^\mu_{\alpha\beta} \delta^\alpha \delta^\beta \quad \text{and} \quad \delta^\mu - C^\mu_{\beta\alpha} \delta^\alpha \epsilon^\beta.$$

b. The components of the torsion tensor are given by:

$$T^\lambda_{\mu\nu} = C^\lambda_{\mu\nu} - C^\lambda_{\nu\mu}.$$

Argue that the torsion tensor measures by how much the parallelogram does not close, which arises by small displacements of vectors and their parallel transports. We often demand that the connection be torsion-free, so $C^\nu_{\mu\sigma}$ is the Cristoffel symbol $\Gamma^\nu_{\mu\sigma} = \Gamma^\nu_{\sigma\mu}$. *1pt*