Cosmology and General Relativity: HW 2

Turn in solutions to "Wittig" (Hausaufgaben) mailbox in KPH by **noon, 9 May 2019**

Problem 1 Noether's current for Lorentz transformation: Recall that the Lorentz transformation ${\Lambda^\mu}_\nu$ is one that leaves the metric $g^{\mu\nu} = (+, -, -, -)$ invariant; $g^{\mu\nu}{\Lambda^\sigma}_\mu{\Lambda^\rho}_\nu = g^{\rho\sigma}$. One demands the action be invariant under global Lorentz transformations $x^\mu \to {\Lambda^\mu}_\nu x^\nu$.

a. Consider an infinitesimal Lorentz transformation ${\Lambda^\mu}_\nu = \delta^\mu_\nu + {\omega^\mu}_\nu$ where ω^2 is negligible. What are the properties of ${\omega^\mu}_\nu$? Define a basis for ω, ${({\omega^\rho}_\sigma)^\mu}_\nu$ — use the "natural" choice in which μ and ν have the same symmetry as ρ and σ.

   Hint: consider the transformation of the metric tensor. *2pt*

b. Denote the field $\phi(x)$ which transforms $\phi(x) \mapsto \phi(\Lambda x)$. The action is given by $\mathcal{S} = \int d^4x\ \mathcal{L}(\phi, \partial_\mu\phi)$. Define the local infinitesimal transformation by assigning a transformation to each point in space ${\omega^\mu}_\nu \to {\omega^\mu}_\nu(x)$. Derive the Noether currents for each of the basis tensors, ${({\omega^\rho}_\sigma)^\mu}_\nu$, in terms of the stress-energy tensor, *3pt*

$$ {T^\mu}_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - {\delta^\mu}_\nu\mathcal{L}\ . $$

c. Write down the conserved charges associated with the Lorentz transformation. Show that *1pt*

$$ \frac{d}{dt}\int d^3\vec{x}\ x^i T^{00} = \text{const.} $$

Problem 2 Charts of S^2: A 2-sphere S^2 can be embedded in a 3-dimensional Euclidean space via the defining equation

$$ x^2 + y^2 + z^2 = 1. $$

a. Use the defining equation to construct *open charts* for S^2. Give the transition maps and verify that they are smooth. *2pt*

Another example of a manifold is $\mathbb{CP}^1$, defined as the space of all lines passing through the origin of $\mathbb{C}^2$. Take into account that we are considering complex lines here, i.e. copies of $\mathbb{C}$. An element of $\mathbb{CP}^1$ is given by

$$ [z_1 : z_2] = \{(z_1, z_2) \neq (0,0) | z_1, z_2 \in \mathbb{C}\}/\sim, $$

where the equivalence relation is given by ($\mathbb{C}^* = \mathbb{C}\backslash\{0\}$)

$$ (z_1, z_2) \sim (w_1, w_2) \quad \Leftrightarrow \quad \exists\lambda \in \mathbb{C}^* \text{ such that } (z_1, z_2) = \lambda(w_1, w_2). $$

b. Show that each element of $\mathbb{CP}^1$ can either be represented by $[1 : a]$ or $[b : 1]$. *1pt*

c. Now restrict to $\mathbb{RP}^1$. Consider $\mathbb{R}^2$ and draw the lines $x = 1$ and $y = 1$ (notice that these lines are not elements of the projective space!). Which objects correspond to the representatives of problem part b in this picture? How many lines (i.e., elements of $\mathbb{RP}^1$) exist that do not intersect with $x = 1$? From this picture, what is the topology of $\mathbb{RP}^1$? *3pt*

d. Argue that $\mathbb{CP}^1$ can be endowed with two charts, which are each isomorphic to $\mathbb{C}$. Show that the transition maps are given by

$$\phi : \mathbb{C}^* \to \mathbb{C}^*, \quad z_1 \mapsto z_2 = z_1^{-1},$$

where z_i denote the coordinates in each copy of $\mathbb{C}$. *1pt*

Problem 3 Tensors and differential forms: The *contravariant* components x^μ belong to a four-vector x, which is an element of the tangential space $T_p(M)$ of the manifold M at point $p \in M$.
Consider a n-dimensional manifold, i.e., $T_p(M)$ is a n-dimensional vector space with basis $\{\hat{e}_{(0)}, \ldots, \hat{e}_{(n-1)}\}$. Each vector $v \in T_p(M)$ can be expressed as $v = v^\mu \hat{e}_{(\mu)}$. The *dual vector space* $V^* = T_p^*(M)$ is the space of linear maps $T_p(M) \to \mathbb{R}$ with basis $\{\hat{e}^{(0)}, \ldots, \hat{e}^{(n-1)}\}$ defined by:

$$\hat{e}^{(\mu)}\left(\hat{e}_{(\nu)}\right) = \delta^\mu_\nu.$$

The definition of a co- or contravariant vector can be generalized to (p, q)-tensors via the multilinear map:

$$T : \underbrace{V^* \times \ldots \times V^*}_{p \text{ times}} \times \underbrace{V \times \ldots \times V}_{q \text{ times}} \to \mathbb{R}$$

with components

$$T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q} = T\left(\hat{e}^{(\mu_1)}, \ldots, \hat{e}^{(\mu_p)}, \hat{e}_{(\nu_1)}, \hat{e}_{(\nu_q)}\right).$$

One can show that there exists a natural isomorphism between the vector space $L^{p+q}(V^* \times \ldots \times V^*, V \times \ldots \times v; \mathbb{R})$ of (p, q)-tensors and the space of tensor products:

$$\underbrace{V^* \otimes \ldots \otimes V^*}_{p \text{ times}} \otimes \underbrace{V \otimes \ldots \otimes V}_{q \text{ times}} \to \mathbb{R},$$

where **tensor products** of vector spaces are defined as follows: let V, W be vector spaces of dimensions d_V, d_W with bases $\{v_{(\alpha)}\}, \{u_{(i)}\}$. The basis of $V \otimes W$ of dimension $d_{V\otimes W} = d_V \cdot d_W$ is then given by $\{v_{(\alpha)} \otimes u_{(i)}\}$, i.e.

$$V \otimes W = \{a^{\alpha i} v_{(\alpha)} \otimes u_{(i)} | a^{\alpha i} \in \mathbb{R}\}.$$

The elements of $V \otimes W$ can be represented by a matrix $(a^{\alpha i})_{\alpha \in \{1 \dots d_V\}, i \in \{1 \dots d_W\}}$. Note that the basis elements of the tensor product space are products of basis elements of the individual factors. In general, an element of $V \otimes W$ can, however, not be written as a single tensor product of an element of V with an element of W. Elements, where this is possible, are called *pure tensors*. Each tensor is a linear combination of pure tensors.

a. Consider V be a two dimensional vector space with basis $\{v_1, v_2\}$ and W be a three dimensional vector space with basis $\{w_1, w_2, w_3\}$. Write down a pure basis of $V \otimes W$. Also write down an example of an element that is not pure. *1pt*

Let now $V = T_p(M)$ be the tangential space to M in point p and $V^* = T_p^*(M)$ the corresponding cotangential or dual space. A **differential form** A_r of order r (or an r-form) is the totally antisymmetric $(0, r)$-tensor,

$$A_r = \frac{1}{r!} A_{\mu_1 \dots \mu_r} \hat{e}^{(\mu_1)} \wedge \dots \wedge \hat{e}^{(\mu_r)} ,$$

where the **wedge product** is defined by:

$$\hat{e}^{(\mu_1)} \wedge \dots \wedge \hat{e}^{(\mu_r)} = \sum_{\sigma \in \mathrm{Sym}(r)} \mathrm{sgn}(\sigma) \hat{e}^{(\mu_{\sigma(1)})} \otimes \dots \otimes \hat{e}^{(\mu_{\sigma(r)})} ,$$

where the sum is over permutations of r elements, and $\mathrm{sgn}(\sigma)$ is the sign of the permutation. The wedge product can be generalized to arbitrary forms:

$$\begin{aligned} A_p \wedge B_q &= \frac{1}{p!} \frac{1}{q!} A_{\mu_1 \dots \mu_p} B_{\nu_1 \dots \nu_q} \hat{e}^{(\mu_1)} \wedge \dots \wedge \hat{e}^{(\mu_p)} \wedge \hat{e}^{(\nu_1)} \wedge \dots \wedge \hat{e}^{(\nu_q)} \\ &\equiv \frac{1}{(p+q)!} (A_p \wedge B_q)_{\mu_1 \dots \mu_{p+q}} \hat{e}^{(\mu_1)} \wedge \dots \wedge \hat{e}^{(\mu_{p+q})} . \end{aligned}$$

The set of p-forms on a vector space V forms a vector space $A^p(V)$.

b. Write $(A_p \wedge B_q)_{\mu_1 \dots \mu_{p+q}}$ explicitly. What does a $(n+1)$-form on a n-dimensional vector space look like? *1.5pt*

c. How many independent components has a p-form on an n-dimensional vector space? Compare with the number of independent components of a *symmetric* (k, l)-tensor with $k + l = p$. *1pt*

The **external derivatives** $d : A^p(V) \to A^{p+1}(V)$ constitute an important class of forms. The action of a p-form A_p is given by:

$$dA_p = d \left(\frac{1}{p!} A_{\mu_1 \dots \mu_p} \hat{e}^{(\mu_1)} \wedge \dots \wedge \hat{e}^{(\mu_p)} \right) = \frac{1}{p!} \partial_\rho A_{\mu_1 \dots \mu_p} \hat{e}^{(\rho)} \wedge \hat{e}^{(\mu_1)} \wedge \dots \wedge \hat{e}^{(\mu_p)} .$$

d. Write out (dA_p) in components and verify that the exterior derivative transforms as a Lorentz tensor. Show that $d^2 = 0$ and that the Leibniz rule holds: *3.5pt*

$$d(A_p \wedge B_q) = (dA_p) \wedge B_q + (-1)^p A_p \wedge (dB_q) .$$