Chapter 2

Lagrangian Formalism

Symmetries are important in the construction of EFTs and therefore we revisit the Lagrangian formalism in this chapter. The Lagrangian formalism is particular useful in incorporating symmetries in a theory.
Let us treat the fields describing particles as dynamical variables, i.e.

$$\phi(x) = \phi(t, \mathbf{x}) \tag{2.1}$$

$$\partial_\mu \phi(x) = \frac{\partial \phi}{\partial x^\mu} = \left(\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial \mathbf{x}}\right) \tag{2.2}$$

I'll be using the metric tensor

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.3}$$

2.1 Classical Equation of Motion

The Lagrangian is given via the Lagrangian density

$$L(t) = \int\limits_{\mathbb{R}^3} d^3x \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \tag{2.4}$$

Applying the least action principle we obtain the classical equations of motion

$$S(\phi) = \int\limits_{\mathbb{R}^4} d^4x \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \tag{2.5}$$

where S is the action of the theory. Let us define an infinitesimal change of the fields

$$\phi_\varepsilon(x) = \phi(x) + \varepsilon h(x) \tag{2.6}$$

with

$$h(x) = 0 \qquad \forall x \in \partial\mathbb{R}^4 \tag{2.7}$$

The change in the action reads

$$F(\varepsilon) = \int d^4x \mathcal{L}(\phi(x) + \varepsilon h(x), \partial_\mu(\phi(x) + \varepsilon h(x))). \tag{2.8}$$

Expanding in ε and keeping terms to first order only

$$F(\varepsilon) = \int d^4x \mathcal{L}(\phi(x), \partial_\mu\phi(x)) + \varepsilon \int d^4x \Big[h\frac{\partial\mathcal{L}}{\partial\phi} + h'(x)\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\Big] + \mathcal{O}(\varepsilon^2) \tag{2.9}$$

The variation in the action should vanish, i.e. $F'(0) = \delta S = 0$. This means

$$0 = \int d^4x \Big[h\frac{\partial\mathcal{L}}{\partial\phi} + h'(x)\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\Big] \tag{2.10}$$

Note that the total derivative for a function depending on $x, \phi(x)$ and $\partial_\nu\phi(x)$ reads

$$D_\mu = \partial_\mu + \partial_\mu\phi\frac{\partial}{\partial\phi} + \partial_\mu\partial_\nu\phi\frac{\partial}{\partial\partial_\nu\phi} \tag{2.11}$$

then

$$D_\mu\Big(h\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\Big) = \Big(D_\mu h\Big)\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} + hD_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} \tag{2.12}$$

The idea is to shift the derivative from $h(x)$ to the remainder.

$$0 = \int_{\mathbb{R}^4} d^4x \Big[h\frac{\partial\mathcal{L}}{\partial\phi} + \Big\{\underbrace{D_\mu\Big(h\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\Big)}_{0} - hD_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\Big\}\Big] \tag{2.13}$$

The second term is zero because the integral over the total derivative is just the argument of the derivative evaluated at the surface $\partial\mathbb{R}^4$, i.e.

$$h\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = 0 \qquad \forall x \in \partial\mathbb{R}^4 \tag{2.14}$$

which is zero because of Eq (2.7). Now the extremum is given by

$$0 = \int_{\mathbb{R}^4} d^4xh\Big[\frac{\partial\mathcal{L}}{\partial\phi} - D_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\Big] \tag{2.15}$$

Now we can apply the fundamental lemma of variational calculus, which states that

$$\int_{x_1}^{x_2} dxh(x)g(x) = 0 \qquad \text{for arbitrary } h(x) \tag{2.16}$$

assuming $h(x)$ has continuous derivatives and

$$h(x_1) = h(x_2) = 0,$$

then

$$g(x) = 0 \quad \text{in } [x_1, x_2]. \tag{2.17}$$

Applying this lemma to Eq. (2.15)

$$\frac{\partial\mathcal{L}}{\partial\phi} - D_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = 0 \tag{2.18}$$

we arrived at the Euler-Lagrange equations of motion (EOM). The generalization to n fields is straightforward

- Perform n independent variations of ϕ_i

$$\phi_{i,\varepsilon_i} = \phi_i + \varepsilon_i h_i \qquad , i = 1, \ldots, n \tag{2.19}$$

$$\Rightarrow F(\underbrace{\varepsilon_1, \ldots, \varepsilon_n}_{\boldsymbol{\varepsilon}}) = S[\phi_{i,\varepsilon_i}] \tag{2.20}$$

- For each field find extremum

$$\frac{\partial F}{\partial\varepsilon_i}|_{\boldsymbol{\varepsilon}=0} = 0. \tag{2.21}$$

  Leading to n EOM for ϕ_i.

Let us look at an example of a free scalar particle

$$\mathcal{L} = \frac{1}{2}\Big(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\Big). \tag{2.22}$$

The EOM can be constructed via

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{2}{2} m^2 \phi \tag{2.23}$$

$$\frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi} = \frac{\partial}{\partial \partial_\alpha \phi} \Big(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi \Big) \tag{2.24}$$

$$= \frac{\partial}{\partial \partial_\alpha \phi} \Big(\frac{1}{2} \partial_\mu \phi \partial_\nu \phi \; g^{\nu\mu} \Big) \tag{2.25}$$

$$= \Big(\frac{1}{2} g_\mu^\alpha \partial_\nu \phi \; g^{\nu\mu} + \frac{1}{2} \partial_\mu \phi g_\nu^\alpha \; g^{\nu\mu} \Big) \tag{2.26}$$

$$= \partial^\alpha \phi \tag{2.27}$$

Putting it all together[1]

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi} = 0 \tag{2.28}$$

$$-m^2 \phi - \partial_\alpha \partial^\alpha \phi = 0 \tag{2.29}$$

$$(m^2 + \Box)\phi = 0 \tag{2.30}$$

This is the EOM of a free scalar particle, i.e. the Klein-Gordon-Equation. The solution of the differential equations reads

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega(\mathbf{k})} \Big[a(\mathbf{k}) e^{-ikx} + a^*(\mathbf{k}) e^{ikx} \Big] \tag{2.31}$$

This leads to the on-shell condition

$$k^2 - m^2 = 0. \tag{2.32}$$

What happens when higher derivatives are present?

$$\mathcal{L}(\phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi, \dots)$$

We proceed in the same way, i.e.

$$\phi_\varepsilon(x) = \phi(x) + \varepsilon h(x), \tag{2.33}$$

where

$$h(x) = 0, \quad \partial_\mu h(x) = 0 \qquad \forall x \in \partial R \tag{2.34}$$

[1] Note that we replaced the total derivative with the partial one, however we mean that ∂_μ acts on explicit and implicit dependence of x.

Again expand the action in ε

$$F(\varepsilon) = \int\limits_{\mathbb{R}^4} d^4x \mathcal{L}(\phi + \varepsilon h, \partial_\mu(\phi + \varepsilon h), \partial_\mu\partial_\nu(\phi + \varepsilon h), \dots) \tag{2.35}$$

$$= F(0) + \varepsilon \int\limits_{\mathbb{R}^4} d^4x \Big[\frac{\partial \mathcal{L}}{\partial \phi} h + \frac{\mathcal{L}}{\partial\partial_\mu\phi}\partial_\mu h + \frac{\partial \mathcal{L}}{\partial\partial_\mu\partial_\nu\phi}\partial_\mu\partial_\nu h + \dots\Big] + \mathcal{O}(\varepsilon^2) \tag{2.36}$$

The goal is to shift the derivatives from h to the remainder such that we can apply the fundamental lemma. Let us examine the third term

$$\begin{aligned}
\int\limits_{\mathbb{R}^4} d^4x \frac{\partial \mathcal{L}}{\partial\partial_\mu\partial_\nu\phi}\partial_\mu\partial_\nu h &= \int\limits_{\mathbb{R}^4} d^4x \underbrace{D_\mu\left[\frac{\partial \mathcal{L}}{\partial\partial_\mu\partial_\nu\phi}\partial_\nu h\right]}_{0} - \int\limits_{\mathbb{R}^4} d^4x D_\mu \frac{\partial \mathcal{L}}{\partial\partial_\mu\partial_\nu\phi}\partial_\nu h \\
&= - \int\limits_{\mathbb{R}^4} d^4x D_\mu \frac{\partial \mathcal{L}}{\partial\partial_\mu\partial_\nu\phi}\partial_\nu h \\
&= - \int\limits_{\mathbb{R}^4} d^4x D_\nu \left[D_\mu \frac{\partial \mathcal{L}}{\partial\partial_\mu\partial_\nu\phi} h\right] + \int\limits_{\mathbb{R}^4} d^4x D_\nu D_\mu \frac{\partial \mathcal{L}}{\partial\partial_\mu\partial_\nu\phi} h \\
&= + \int\limits_{\mathbb{R}^4} d^4x D_\nu D_\mu \frac{\partial \mathcal{L}}{\partial\partial_\mu\partial_\nu\phi} h
\end{aligned} \tag{2.37}$$

where the surface terms in the first and third line vanish because of Eq. (2.34). It can be seen that we defined h such that we can shift the derivative one by one to the remainder, where each time we incur a sign shift. This leads to the following for Eq. (2.36)

$$0 = \int\limits_{\mathbb{R}^4} d^4x h(x) \left[\frac{\partial \mathcal{L}}{\partial \phi} - D_\mu \frac{\partial \mathcal{L}}{\partial\partial_\mu\phi} + D_\nu D_\mu \frac{\partial \mathcal{L}}{\partial\partial_\mu\partial_\nu\phi} + \dots\right], \tag{2.38}$$

where the alternating sign come from even/odd numbers of integration by parts.

Example toy model:

$$\mathcal{L} = \frac{1}{2}\Big(\partial_\mu\phi\partial^\mu\phi - m_1^2\phi^2\Big) + \frac{1}{2}\Big(\partial_\mu\varphi\partial^\mu\varphi - m_2^2\varphi^2\Big) - g\big(\Box\phi\big)^2\varphi^2 \tag{2.39}$$

The corresponding EOM read

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi} = \frac{\partial}{\partial \partial_\mu \partial_\nu \phi} g\big(\partial_\alpha \partial_\beta \phi g^{\alpha\beta}\big)^2 \varphi^2 \tag{2.40}$$
$$= 2g\big(\Box\phi\big) g^\mu_\alpha g^\nu_\beta g^{\alpha\beta} \varphi^2 \tag{2.41}$$
$$= 2g\big(\Box\phi\big) \varphi^2 g^{\mu\nu} \tag{2.42}$$

and therefore

$$-m_1^2\phi - \Box\phi - g\partial_\mu\partial_\nu 2g\big(\Box\phi\big)\varphi^2 g^{\mu\nu} = 0 \tag{2.43}$$
$$-m_1^2\phi - \Box\phi - 2g^2\Box\big(\Box\phi\big)\varphi^2 = 0 \tag{2.44}$$

2.2 Noether-Theorem

A continuous global symmetry that leaves the Lagrangian invariant leads to a conserved current. Let $\mathcal{L}$ depend on n fields ϕ_i with only one derivative acting on it, i.e.

$$\mathcal{L}(\phi_i, \partial_\mu \phi_i).$$

This gives n EOMs

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} = 0, \qquad i = 1, \ldots, n \tag{2.45}$$

Consider n infinitesimal transformations depending on r real parameters

$$\begin{aligned} \phi_i(x) \to \phi_i'(x) &= \phi_i(x) + \delta\phi_i(x) \\ &= \phi_i(x) - i\varepsilon_a(x) F_i^a(\phi_i(x)), \qquad a = 1, \ldots, r \end{aligned} \tag{2.46}$$

What is the change in the Lagrangian $\mathcal{L}$ under this transformation?

$$\delta\mathcal{L} = \mathcal{L}(\phi_i', \partial_\mu \phi_i') - \mathcal{L}(\phi_i, \partial_\mu \phi_i) \tag{2.47}$$
$$= \frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \partial\mu\delta\phi_i \tag{2.48}$$

The derivative of the change in the field after transformations is

$$\partial_\mu \delta\phi_i = -i\big(\partial_\mu \varepsilon_a(x)\big) F_i^a(\phi_j(x)) - i\varepsilon_a(x)\partial_\mu F_i^a(\phi_j). \tag{2.49}$$

Inserting this derivative into the variation of the Lagrangian results in

$$\delta\mathcal{L} = -i\varepsilon_a(x)\Big\{\frac{\partial \mathcal{L}}{\partial \phi_i} F_i^a + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\partial_\mu F_i^a\Big\} - i\partial_\mu \varepsilon_a(x) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} F_i^a. \tag{2.50}$$

Let us rewrite the change

$$\begin{aligned} \delta\mathcal{L} &= \varepsilon_a \partial_\mu J^{\mu,a} + \partial_\mu \varepsilon_a J^{\mu,a} \\ \Rightarrow J^{\mu,a} &= -i \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} F_i^a. \end{aligned} \tag{2.51}$$

Why is this correct?

$$\partial_\mu J^{\mu,a} = -i \underbrace{\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\right)}_{\text{EOM}\rightarrow \frac{\partial \mathcal{L}}{\partial \phi_i}} F_i^a - i \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \partial_\mu F_i^a \tag{2.52}$$

$$= -i \frac{\partial \mathcal{L}}{\partial \phi_i} F_i^a - i \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \partial_\mu F_i^a, \tag{2.53}$$

which reproduces the first term in Eq. (2.50). We can now derive the current and its divergence from the variation of the Lagrangian

$$J^{\mu,a} = \frac{\partial \delta \mathcal{L}}{\partial \partial_\mu \varepsilon_a}, \quad \partial_\mu J^{\mu,a} = \frac{\partial \delta \mathcal{L}}{\partial \varepsilon_a}. \tag{2.54}$$

If we assume that the Lagrangian is invariant under **global** transformations, i.e.

$$\delta\mathcal{L} = 0 \quad \text{and} \quad (\partial_\mu \varepsilon_a(x)) = 0 \tag{2.55}$$

it follows that the current is conserved

$$\partial_\mu J^{\mu,a} = 0. \tag{2.56}$$

From the current we can derive the conserved charges

$$Q^a(t) = \int d^3x J^{0,a}(\mathbf{x}, t) \tag{2.57}$$

i.e. the charge is conserved over time

$$\frac{dQ^a(t)}{dt} = \int d^3x \frac{\partial}{\partial t} J^{0,a}(\mathbf{x}, t) \tag{2.58}$$

$$\stackrel{?}{=} 0 \tag{2.59}$$

where the last equality is not very obvious. The way to show this is via Gauß-Law

$$\int d^3x \boldsymbol{\nabla} \boldsymbol{x} = \int d\boldsymbol{F} \boldsymbol{J^a} = \lim_{R\to\infty} R^2 \int d\Omega \boldsymbol{J^a} \tag{2.60}$$

$$= 0 \tag{2.61}$$

The last equality only holds if the density $\boldsymbol{J^a}(t, \boldsymbol{x})$ falls of faster then $1/r^2$ as $|r| \to \infty$. This is not necessarily true, e.g. for massless charges particles. Let us assume for now that the current density has the desired property, then we write

$$\begin{aligned}
\frac{dQ^a(t)}{dt} &= \int d^3x \frac{\partial}{\partial t} J^{0,a}(\mathbf{x}, t) &(2.62)\\
&= \int d^3x \Big(\frac{\partial}{\partial t} J^{0,a}(\mathbf{x}, t) + \boldsymbol{\nabla} \boldsymbol{J^a} \Big) &(2.63)\\
&= \int d^3x \partial_\mu J^{\mu,a}(x) &(2.64)\\
&= 0 \qquad , \text{for } \delta\mathcal{L} = 0 &(2.65)
\end{aligned}$$

2.3 Quantization

Canonical quantization usually proceeds in the Hamilton formalism. We define the conjugate variables

$$p = \frac{L}{\partial \dot{q}} \to \Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}, \tag{2.66}$$

where the arrow indicates the change from classical formulation to the one using fields. The Hamiltonian density is given by

$$\mathcal{H} = \Pi\dot{\phi} - \mathcal{L}, \tag{2.67}$$

the relation to the Hamiltonian is

$$H = \int d^3x \mathcal{H} \tag{2.68}$$

Let us look at a free scalar field

$$\begin{aligned}
\Pi &= \dot{\phi} &(2.69)\\
\mathcal{H} &= \dot{\phi}^2 - \frac{1}{2}\Big(\dot{\phi}^2 - (\boldsymbol{\nabla\phi})^2 - m^2\phi^2 \Big) &(2.70)\\
&= \frac{1}{2}\Big(\Pi^2 + (\boldsymbol{\nabla\phi})^2 + m^2\phi^2 \Big). &(2.71)
\end{aligned}$$

For a real scalar field one sees that the Hamiltonian is positive, i.e.

$$H = \int d^3x \mathcal{H} \geq 0 \tag{2.72}$$

there are no negative energy solutions. The canonical quantization proceeds analogously to the case of quantum mechanics, where one uses the commutation relations

$$[q_i, p_j] = \delta_{ij}, \tag{2.73}$$

$$[q_{H_i}(t), q_{H_j}(t)] = 0 \qquad [p_{H_i}(t), p_{H_j}(t)] = 0. \tag{2.74}$$

Here the index H denotes time dependent operators in the Heisenberg picture, which relate to the operators in the Schroedinger picture as

$$\mathcal{O}_H(t) = e^{iHt}\mathcal{O}_S e^{-iHt} \tag{2.75}$$

Applied to fields this leads to the equal time commutation relations (ETCR)

$$\Big[\phi(t, \boldsymbol{x}), \Pi(t, \boldsymbol{y})\Big] = \delta^3(\boldsymbol{x} - \boldsymbol{y}) \tag{2.76}$$

$$\Big[\phi(t, \boldsymbol{x}), \phi(t, \boldsymbol{y})\Big] = \Big[\Pi(t, \boldsymbol{x}), \Pi(t, \boldsymbol{y})\Big] = 0 \tag{2.77}$$

Now ϕ must satisfy ETCR and EOM at the same time, and the solution has the form

$$\phi(t, \boldsymbol{x}) = \int \underbrace{\frac{d^3k}{(2\pi)^3 2\omega}}_{d^3\tilde{k}} \Big[a(\boldsymbol{k})e^{-ikx} + a^\dagger(\boldsymbol{k})e^{ikx} = \phi^\dagger(t, \boldsymbol{x})\Big] \tag{2.78}$$

where

$$k_0 = \omega = \sqrt{\boldsymbol{k} + m^2}, \tag{2.79}$$

and the $a, a^\dagger$ are interpreted as operators. What is the interpretation of these operators? To that end let us look at the effect of acting with $a, a^\dagger$ on eigentstates of the Hamiltonian. First we write the Hamiltonian in terms of $a, a^\dagger$

$$H = \int d^3x \mathcal{H} = \int d^3x \frac{1}{2}\Big(\dot{\phi}(x)^2 + (\boldsymbol{\nabla}\boldsymbol{\phi})^2 + m^2\phi^2\Big) \tag{2.80}$$

$$= \frac{1}{2}\int d^3\tilde{k} k_0 \Big(a^\dagger(\boldsymbol{k})a(\boldsymbol{k}) + a(\boldsymbol{k})a^\dagger(\boldsymbol{k})\Big). \tag{2.81}$$

Let $|E\rangle$ be an eigenstate of H, i.e. $H|E\rangle = E|E\rangle$. Since H is hermitian E is real

$$Ha(\boldsymbol{k})|E\rangle = \Big(a(\boldsymbol{k})H + \underbrace{[H, a(\boldsymbol{k})]}_{-k_0 a(\boldsymbol{k})}\Big)|E\rangle \tag{2.82}$$

$$= (E - k_0)\underbrace{a(\boldsymbol{k})|E\rangle}_{|E'\rangle} \tag{2.83}$$

therefore a acting on $|E\rangle$ gives an eigenstate of H, namely $|E'\rangle$ with an energy reduced by k_0, i.e. a destroys an energy quantum of k_0. Repeat the procedure for the momentum operator, i.e.

$$\boldsymbol{P} = \int d^3\tilde{k}\boldsymbol{k}a^\dagger(\boldsymbol{k})a(\boldsymbol{k}). \tag{2.84}$$

We are lead to the conclusion that the operators $a(\boldsymbol{k})$ and $a^\dagger(\boldsymbol{k})$ destroy and create particles of energy $\omega(\boldsymbol{k})$ and momentum $\boldsymbol{k}$.
Remark on the ground state energy:
The action of an annihilation operator acting on the vacuum is

$$a(\boldsymbol{k})|0\rangle = 0, \langle 0|a^\dagger(\boldsymbol{k}) = 0 \tag{2.85}$$

one can therefore rewrite the Hamiltonian

$$\langle 0|a^\dagger a + aa^\dagger|0\rangle = \langle 0|2a^\dagger a + \underbrace{[a, a^\dagger]}_{\delta^{(3)}(\mathbf{0})}|0\rangle \tag{2.86}$$

where the first term is 0 and the second results in a δ-function. One can interpret the δ-function to give of contribution of size of the Volume, i.e.

$$\delta^{(3)}(\mathbf{0}) = \int d^3x 1 = V. \tag{2.87}$$

This means that the vacuum expectation value of the Hamiltonian from Eq. (2.81) is divergent. This is an unpleasant property of the theory, however this is a first hint that the theory is not valid for very high energies. We could remedy this problem by introducing a cutoff for the momenta we allow to be probed in our theory. One can also note that the Hamiltonian is not unique, i.e. we could introduce an arbitrary constant potential, that offsets this divergence. One may also note that the commutator vanishes for *classical* fields. An easy way out is that we introduce a normal ordering for operators, i.e. annihilation operators are shifted to the right of creation operators.

2.4 Lehmann-Symanzik-Zimmermann Reduction Formula

Now that we know how to quantize a classical field theory we need to understand how we can calculate observables from this theory. We will be concerned with matrix elements of the form $\langle f|i\rangle$ given the transition amplitudes of an initial state $|i\rangle$ of particles to a final state $|f\rangle$. First let us define initial and final states in the free theory, i.e.

$$|k\rangle = a^\dagger(\boldsymbol{k})|0\rangle \tag{2.88}$$

$$a(\boldsymbol{k})|k\rangle = 0 \tag{2.89}$$

where the states are normalized

$$\langle 0|0\rangle = 1, \qquad \langle k|k'\rangle = (2\pi)^3 2\omega\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k'}) \tag{2.90}$$

with

$$\omega^2 = \boldsymbol{k}^2 + m^2. \tag{2.91}$$

We will create a particle near the origin in position space localized near $\boldsymbol{k_1}$ in momentum space using a gaussian shaped form factor

$$a_1^]\dagger = \int d^3k f_1(\boldsymbol{k}) a^\dagger(\boldsymbol{k}) \tag{2.92}$$

$$= \int d^3k e^{-\frac{(\boldsymbol{k}-\boldsymbol{k_1})^2}{4\sigma^2}} a^\dagger(\boldsymbol{k}). \tag{2.93}$$

Two states with different momenta created at the same position will propagate in time such that they will be widely separated, i.e.

$$a_1^\dagger a_2^\dagger|0\rangle \quad \boldsymbol{k_1} \neq \boldsymbol{k_2} \tag{2.94}$$

for times $t \to \pm\infty$ represents a state of two non interacting particles. We suppose this feature is still true in the interacting theory (which is not a given). Let us define initial and final (two particle) states

$$|i\rangle = \lim_{t\to-\infty} a_1^\dagger a_2^\dagger|0\rangle \tag{2.95}$$

$$|f\rangle = \lim_{t\to\infty} a_{1'}^\dagger a_{2'}^\dagger|0\rangle \tag{2.96}$$

Given the initial and final states the scattering of particles is simply

$$\langle f|i\rangle. \tag{2.97}$$

One can show that

$$a_1^\dagger(\infty) - a_1^\dagger(-\infty) = \int\limits_{-\infty}^{\infty} \partial_0 a_1^\dagger(t) \tag{2.98}$$

$$= -i\int d^3k f_1(\boldsymbol{k}) \int d^4x e^{ikx}(\Box + m^2)\phi(x). \tag{2.99}$$

For the free theory using the EOM this is zero, i.e. in the free theory the one-particle state at $t = \pm\infty$ is the same. Note that this not true in the interacting theory. Let us write the relation for the annihilation operators

$$a_1(\infty) - a_1(-\infty) = \int_{-\infty}^{\infty} \partial_0 a_1(t) \tag{2.100}$$

$$= -i \int d^3k f_1(\boldsymbol{k}) \int d^4x e^{-ikx}(\Box + m^2)\phi(x). \tag{2.101}$$

Using the definition of initial and final state we write the scattering

$$\langle f|i\rangle = \langle 0|a_{1'}(\infty)a_{2'}(\infty)a_1(-\infty)a_2(-\infty)|0\rangle \tag{2.102}$$

$$=\langle 0|T\Big[a_{1'}(\infty)a_{2'}(\infty)a_1(-\infty)a_2(-\infty)\Big]|0\rangle. \tag{2.103}$$

Replacing annihilation (creation) operators at ∞ $(-\infty)$ using the relations eqs. (2.99) and (2.101) we obtain

$$\begin{aligned}\langle f|i\rangle = (i)^4 \int d^4x_1 \int d^4x_2 e^{ik_1x_1}(\Box + m^2) e^{ik_2x_2}(\Box + m^2)\\ \int d^4x_1' \int d^4x_2' e^{-ik_1'x_1'}(\Box + m^2) e^{-ik_2'x_2'}(\Box + m^2)\\ \langle 0|T\Big[\phi(x_1)\phi(x_2)\phi(x_1')\phi(x_2')\Big]|0\rangle\end{aligned} \tag{2.104}$$

where we have used the time ordering to eliminate the terms with creation and annihilation operators, and we have performed the integration over the momenta $\boldsymbol{k}$ in the limit $\sigma \to 0$, which generates $\delta(\boldsymbol{k} - \boldsymbol{k_1}$ functions and ensures momentum conservation. More generally for n (n') particles in the initial (final) states we obtain

$$\begin{aligned}\langle f|i\rangle =&(i)^{n+n'} \int d^4x_1 e^{ik_1x_1}(\Box + m^2)\cdots \int d^4x_1' \int d^4x_2' e^{-ik_1'x_1'}(\Box + m^2)\ldots\\ &\langle 0|T\Big[\phi(x_1)\phi(x_2)\ldots\phi(x_1')\phi(x_2')\ldots\Big]|0\rangle\end{aligned} \tag{2.105}$$

this is the LSZ reduction formula. It is valid provided that

- We only create one particle states

  $$\langle 0|\phi(x)|0\rangle = 0.$$

  Can be achieved by constant shift to ϕ.

- We want correctly normalized one particle states in the free theory

$$\langle \boldsymbol{k}|\phi(x)|0\rangle = e^{-ikx} \tag{2.106}$$

Can be achieved by multiplicative redefinition of the field ϕ.

We have related the scattering of particles to the calculation of VEV of time ordered products of field operators. We can calculate these using the path integral formalism, which for the case of quantum mechanics is given by

$$\langle q'', t''|T\Big[Q(t_1)Q(t_2)\Big]|q', t'\rangle = \int \mathcal{D}p\mathcal{D}q q(t_1)q(t_2)e^{iS} \tag{2.107}$$

The integrals is over all possible paths and the action is given by

$$S = \int dt\Big[p\dot{q} - H(p,q)\Big] = \int dtL. \tag{2.108}$$

In order to obtain the time ordered products we will introduce external sources in the action and will make use of functional derivatives

$$\frac{\delta}{\delta f(t_1)} f(t_2) = \delta(t_1 - t_2) \tag{2.109}$$

which is a generalization of

$$\partial_i x_j = \delta_{ij}. \tag{2.110}$$

Let us modify the Hamiltonian

$$H(p,q) \to H(p,q) - f(t)q(t) - h(t)p(t). \tag{2.111}$$

The matrix element then reads

$$Z = \langle q'', t''|q', t'\rangle_{f,h} = \int \mathcal{D}p\mathcal{D}q e^{iS(f,h)} \tag{2.112}$$

$$= \int \mathcal{D}p\mathcal{D}q \exp\Big\{i\int_{t'}^{t''} dt\Big[p\dot{q} - H(p,q) + f(t)q(t) + h(t)p(t)\Big]\Big\}$$

$$\frac{1}{i}\frac{\delta}{\delta f(t_1)}\langle\ldots\rangle|_{f=0} = \int \mathcal{D}p\mathcal{D}q q(t_1) \exp\Big\{i\int_{t'}^{t''} dt\Big[p\dot{q} - H(p,q) + f(t)q(t) + h(t)p(t)\Big]\Big\}$$

$$\frac{1}{i}\frac{\delta}{\delta h(t_1)}\langle\ldots\rangle|_{h=0} = \int \mathcal{D}p\mathcal{D}q p(t_1) \exp\Big\{i\int_{t'}^{t''} dt\Big[p\dot{q} - H(p,q) + f(t)q(t) + h(t)p(t)\Big]\Big\}$$

Using the functional derivative and external sources we can write every matrix element as functional derivative of Z. Z is called the generating functional

$$\langle q'', t''|T\Big[Q(t_1)Q(t_2)\dots P(t_{n+1})\dots\Big]|q', t'\rangle =$$
$$\frac{1}{i}\frac{\delta}{\delta f(t_1)}\cdots\frac{1}{i}\frac{\delta}{\delta f(t_n)}\cdots\frac{1}{i}\frac{\delta}{\delta h(t_{n+1})}\langle q'', t''|q', t'\rangle|_{f=h=0} \tag{2.113}$$

We are usually interested in ground state matrix elements, i.e.

$$\langle 0|0\rangle_{f,h} = \lim_{t'\to-\infty}\lim_{t''\to\infty}\int dq''\int dq'\Psi_0(q'')\langle q'', t''|q', t'\rangle\underbrace{\Psi_0(q')}_{\langle q'|0\rangle} \tag{2.114}$$

This expression is very inconvenient. Let us write the time dependence of the state explicitly

$$|q', t'\rangle = e^{iHt'}|q'\rangle \tag{2.115}$$
$$= \sum_n e^{iHt'}|n\rangle\langle n|q'\rangle \tag{2.116}$$
$$= \sum_n e^{iE_nt'}|n\rangle\Psi_n^*(q'). \tag{2.117}$$

We will use a trick and give thee Hamiltonian as small imaginary part, i.e.

$$H \to (1 - i\varepsilon)H \tag{2.118}$$

which results in damped exponentials as $t \to -\infty$. Therefore only the zero energy state survives.

$$|q', t'\rangle \overset{t'\to-\infty}{=} \Psi_0^*(q')|0\rangle \tag{2.119}$$

One can apply the same analysis to $\langle q'', t''|$ which leads to

$$\int dq''\Psi_0(q'')\langle q'', t''| \approx \langle 0|\text{const} \tag{2.120}$$

Once we add a small imaginary part to H we cab write

$$\langle 0|0\rangle_{f,h} = \int \mathcal{D}p\mathcal{D}q\exp\Big\{i\int_{t'}^{t''}dt\Big[p\dot{q} - (1 - i\varepsilon)H(p, q) + f(t)q(t) + h(t)p(t)\Big]\Big\}$$

and consequently all VEV of time ordered products as functional derivatives of that generating functional.
Example:

$$\mathcal{L} = \mathcal{L}_0(\phi) + \underbrace{j(x)\phi(x)}_{\mathcal{L}_{\text{ext}}} \tag{2.121}$$

Expand the external source part in powers of j

$$\begin{aligned} e^{i\int j(x)\phi(x)} &\approx 1 + i\int d^4x j(x)\phi(x) \\ &+ \frac{(i)^2}{2!}\int d^4x \int d^4y j(x)j(y)\phi(x)\phi(y) + \ldots \end{aligned} \tag{2.122}$$

It is clear that only the last term survives in

$$\langle 0|T\Big[\phi(x_1)\phi(x_2)\Big]|0\rangle = \frac{1}{i^2}\frac{\delta^2 e^{iW(j)}}{\delta j(x_1)\delta j(x_2)}|_{j=0} \tag{2.123}$$

$$= \langle 0|\frac{1}{2}\frac{\delta^2}{\delta j(x_1)j(x_2)}\int d^4x \int d^4y j(x)j(y)\phi(x)\phi(y)||0\rangle \tag{2.124}$$