# Exercise sheet 5
# Theoretical Physics 6a (QFT): SS 2019

13.05.2019

## Exercise 1 (30 points) : Dirac Field

The Free Dirac Lagrangian is given by:

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi,$$

where the normal mode expansion for the fields are:

$$\psi(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \sum_s \left\{ a(\vec{k},s)u(\vec{k},s)e^{-ikx} + b^\dagger(\vec{k},s)v(\vec{k},s)e^{ikx} \right\}$$
$$\bar{\psi}(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \sum_s \left\{ b(\vec{k},s)\bar{v}(\vec{k},s)e^{-ikx} + a^\dagger(\vec{k},s)\bar{u}(\vec{k},s)e^{ikx} \right\}.$$

**(a)(10 points)** Show, that the momentum operator

$$\vec{P} = \int d^3\vec{x}\ \psi^\dagger(x)(-i\vec{\nabla})\psi(x)$$

can be expressed as:

$$\vec{P} = \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_s \vec{k} \left\{ a^\dagger(\vec{k},s)a(\vec{k},s) + b^\dagger(\vec{k},s)b(\vec{k},s) \right\}.$$

**(b)(10 points)** Show, that the conserved charge

$$Q = \int d^3\vec{x}\ \bar{\psi}(x)\gamma_0\psi(x)$$

can be expressed as:

$$Q = \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_s \left\{ a^\dagger(\vec{k},s)a(\vec{k},s) - b^\dagger(\vec{k},s)b(\vec{k},s) \right\},$$

and explain the meaning of the relative minus sign in the expression above.

**(c)(10 points)** Show, that if the Dirac field were quantized according to the Bose-Einstein statistics, that is, through commutators as for the Klein-Gordon field, one would get the following Hamiltonian:

$$H = \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_s E_{\vec{k}} \left\{ a^\dagger(\vec{k},s)a(\vec{k},s) - b^\dagger(\vec{k},s)b(\vec{k},s) \right\}$$

and explain why this would lead to unphysical results for the energy spectrum.

## Exercise 2 (40 points) : The angular momentum operator

**(a)(20 points)** Starting from the transformation law for the classical Dirac field under Lorentz transformations, show that the generators of these transformations are given by:

$$M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \frac{1}{2}\sigma_{\mu\nu}.$$

**(b)(20 points)** The angular momentum of the Dirac field is given by:

$$M_{\mu\nu} = \int d^3x\, \psi^\dagger(x) \left[ i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \frac{1}{2}\sigma_{\mu\nu} \right] \psi(x).$$

Prove that

$$[M_{\mu\nu}, \psi(x)] = -i(x_\mu\partial_\nu - x_\nu\partial_\mu)\psi(x) - \frac{1}{2}\sigma_{\mu\nu}\psi(x).$$

## Exercise 3 (30 points) : Axial current

For a Dirac field, the transformations

$$\psi(x) \to \psi'(x) = e^{i\alpha\gamma_5}\psi(x), \qquad \psi^\dagger(x) \to \psi^{\dagger'}(x) = \psi^\dagger(x)e^{-i\alpha\gamma_5},$$

where $\alpha$ is an arbitrary real parameter, are called chiral phase transformations.

**(a)(15 points)** Show that the Dirac Lagrangian density $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ is invariant under chiral phase transformations in the zero-mass limit $m = 0$ only, and that the corresponding conserved current in this limit is the axial vector current $J_A^\mu \equiv \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$.

**(b)(15 points)** Deduce the equations of motion for the fields

$$\psi_L(x) \equiv \frac{1}{2}(\mathbb{1} - \gamma_5)\psi(x), \qquad \psi_R(x) \equiv \frac{1}{2}(\mathbb{1} + \gamma_5)\psi(x),$$

for non-vanishing mass, and show that they decouple in the limit $m = 0$.

Hence, the Lagrangian density $\mathcal{L} = i\bar{\psi}_L\partial\!\!\!/\psi_L$ describes massless fermions with negative helicity and massless anti-fermions with positive helicity only. This field is called the Weyl field and can be used to describe the neutrinos as far as the latter can be considered as massless.