Exercise sheet 4
Theoretical Physics 6a (QFT): SS 2019

06.05.2019

Exercice 1. (20 points) Dirac representation

In the standard Dirac representation, the Dirac matrices have the form

$$\gamma_s^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \vec{\gamma}_s = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix},$$

where $\vec{\sigma}$ is the vector of the 3 Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Using this representation of the gamma matrices, show the following identities:

$$[\gamma^\mu, \gamma^\nu]_+ \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}\mathbb{1},$$
$$\gamma^{\mu\dagger} = \gamma_0 \gamma^\mu \gamma_0.$$

Exercise 2. (30 points) : Weyl representation

In the so-called *Weyl representation*, the Dirac matrices have the form

$$\gamma_W^\mu \quad = \quad \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

where $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\vec{\sigma})$.

(a)(10 points) Write down a unitary matrix S connecting both representations $\gamma_s^\mu = S\gamma_W^\mu S^{-1}$.

(b)(10 points) Write the γ_5 matrix in both representations.

(c)(10 points) In the Weyl representation with $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, show that the bispinors ψ_L and ψ_R are independent for massless particles, and write down the eigenstates of the chirality operator γ_5 with their corresponding eigenvalues.

Exercise 3. (50 points) : Dirac matrix calculus

Without using an explicit representation of the gamma matrices, show the following identities:

$$\begin{aligned}
\gamma_\mu \gamma^\mu &= 4, \\
\mathrm{Tr}[\gamma^\mu \gamma^\nu] &= 4 g^{\mu\nu}, \\
\mathrm{Tr}\left[\not{a}\not{b}\not{c}\not{d}\right] &= 4(a \cdot b\ c \cdot d - a \cdot c\ b \cdot d + a \cdot d\ b \cdot c), \\
\gamma_5 &= \frac{i}{4!} \epsilon_{\mu\nu\alpha\beta} \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta, \\
\gamma_5^2 &= 1, \\
[\gamma_5, \gamma^\mu]_+ &= 0, \\
\mathrm{Tr}[\gamma_5] &= 0, \\
\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma_5] &= 0, \\
\mathrm{Tr}\left[\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta \gamma_5\right] &= 4 i \epsilon^{\mu\nu\alpha\beta}, \\
\mathrm{Tr}[\gamma^{\mu_1} ... \gamma^{\mu_n}] &= 0, \text{ if } n \text{ is odd},
\end{aligned}$$

where $\not{a} \equiv a^\mu \gamma_\mu$, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, and $\epsilon_{0123} = 1$