II. DIRAC FIELD

1) DIRAC EQUATION

2) QUANTIZATION OF DIRAC FIELD

3) FEYNMAN PROPAGATOR OF DIRAC FIELD

4) SYMMETRIES OF DIRAC THEORY

1) DIRAC EQUATION

⇒ DIRAC EQUATION AND SOLUTIONS

↳ DIRAC EQ. : DESCRIBES RELATIVISTIC SPIN 1/2 PARTICLE

$$\boxed{(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0}$$

↳ $\psi(x)$: 4×1 DIRAC SPINOR

↳ γ - MATRICES : (4×4 MATRICES)

$$\gamma^\mu\,(\gamma^0, \vec{\gamma})$$

IN DIRAC REPRESENTATION

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}$$

$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

$\vec{\sigma}\,(\sigma^1, \sigma^2, \sigma^3)$

↳ PAULI MATRICES (2×2)

γ - MATRICES SATISFY ANTI-COMMUTATION RELATIONS

$$\boxed{[\gamma^\mu, \gamma^\nu]_+ \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,\mathbb{1}_{4\times4}}$$

↑ ANTI-COMMUTATOR

↳ HERMITICITY CONDITION

$$\gamma^{0\dagger} = \gamma^0$$

$$\gamma^{i\dagger} = -\gamma^i \qquad (i = 1, 2, 3)$$

CAN BE COMBINED AS

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$$

↳ γ_5 MATRIX

IN FOLLOWING, WE WILL ALSO ENCOUNTER

A 5-th GAMMA :

$$\gamma_5 \equiv \frac{i}{4!} \varepsilon_{\mu\nu\alpha\beta} \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta$$

WITH $\varepsilon_{\mu\nu\alpha\beta}$ TOTAL ANTISYMMETRIC LEVI-CIVITA TENSOR IN 4-DIM

CONVENTION USED:

$$\varepsilon_{0123} = +1$$

NOTE: THIS CONVENTION IS EQUIVALENT WITH

$$\varepsilon^{0123} = -1$$

$$\gamma_5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3$$

$$(\gamma_5)^2 = \mathbb{1}_{4\times 4}$$

IN DIRAC REPRESENTATION

$$\gamma_5 = \begin{pmatrix} 0 & \mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times} & 0 \end{pmatrix}$$

CHANGES UPPER & LOWER COMPONENTS IN DIRAC SPINOR.

↳ ADJOINT DIRAC FIELD $\bar{\psi}$

$$\bar{\psi}(x) \equiv \psi^\dagger(x)\,\gamma^0$$

TAKE HERMITIAN CONJUGATE OF DIRAC EQ.

$$\left(-i(\partial_\mu \psi^\dagger)(\gamma^\mu)^\dagger - m\,\psi^\dagger\right) = 0$$

MULTIPLY ON RIGHT BY γ^0 & USE $(\gamma^0)^2 = \mathbb{1}$

$$\left(-i(\partial_\mu \psi^\dagger)\gamma^0\gamma^\mu(\gamma^0)^2 - m\,\psi^\dagger\gamma^0\right) = 0$$

$$\boxed{\left(i(\partial_\mu \bar{\psi})\gamma^\mu + m\,\bar{\psi}\right) = 0}$$

↳ PLANE-WAVE SOLUTIONS

CHARACTERIZED BY MOMENTUM $\bar{p}$, ENERGY $E_{\bar{p}} = \sqrt{\bar{p}^2 + m^2}$

& SPIN PROJECTION $s = \pm \frac{1}{2}$

(ALONG A CHOSEN QUANTIZATION AXIS)

$$\psi_{\bar{p},s+}(x) = e^{-ip.x}\, u(\bar{p},s)$$ POSITIVE ENERGY SOLUTION (PARTICLE)

$$\psi_{\bar{p},s-}(x) = e^{+ip.x}\, v(\bar{p},s)$$ NEGATIVE ENERGY SOLUTION (ANTI-PARTICLE)

↓ PLUG INTO DIRAC EQ.

$$(\not{p} - m)\, u(\bar{p},s) = 0$$

$$(\not{p} + m)\, v(\bar{p},s) = 0$$

WITH $\not{p} \equiv \gamma_\mu p^\mu$

EQUIVALENTLY FOR ADJOINT SPINORS

$$\bar{u}(\bar{p},s) \equiv u^{+}(\bar{p},s)\,\gamma^0 \Rightarrow \bar{u}(\bar{p},s)(\not{p} - m) = 0$$

$$\bar{v}(\bar{p},s) \equiv v^{+}(\bar{p},s)\,\gamma^0 \Rightarrow \bar{v}(\bar{p},s)(\not{p} + m) = 0$$

↳ FOR SPIN QUANTIZED ALONG AXIS $\hat{n}$

$$\bar{S} = \frac{\bar{\sigma}}{2} \quad \curvearrowright \quad \underline{\underline{\left(\frac{\bar{\sigma}}{2} \cdot \hat{n}\right) \chi_s = s \, \chi_s}}$$

$$s = \pm \frac{1}{2}$$

χ_s : 2×1 PAULI - SPINOR

- $\underline{\hat{n} = \bar{e}_z}$ SUPERSCRIPT ON χ INDICATES QUATIZATIO AXIS

$$\chi^{(z)}_{+\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi^{(z)}_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

SPIN UP SPIN DOWN

- $\underline{\hat{n} = \bar{e}_x}$

$$\sigma_x \, \chi^{(x)}_s = (2s) \, \chi^{(x)}_s .$$

e.g. $s = +\frac{1}{2}$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow a = b$$

$$\chi^{(x)}_{+\frac{1}{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \chi^{(x)}_{-\frac{1}{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

↑

NORMALIZATION $\chi^{+}_{s} \chi_{s'} = \delta_{ss'}$

↳ IN DIRAC REPRESENTATION : SOLUTION u

$$\left(\gamma^0 E_{\bar{p}} - \bar{\gamma}.\bar{p} - m \right) u(\bar{p}, s) = 0$$

$$\downarrow \qquad u(\bar{p}, s) = N \begin{pmatrix} \chi_s \\ \varphi_s \end{pmatrix}$$

$\uparrow$ NORMALIZATION

$$\begin{pmatrix} E_{\bar{p}} - m & -\bar{\sigma}.\bar{p} \\ \bar{\sigma}.\bar{p} & E_{\bar{p}} + m \end{pmatrix} \begin{pmatrix} \chi_s \\ \varphi_s \end{pmatrix} = 0$$

$$\hookrightarrow \quad (E_{\bar{p}} + m)\, \varphi_s = (\bar{\sigma}.\bar{p})\, \chi_s$$

$$\therefore \quad \boxed{ u(\bar{p}, s) = N \begin{pmatrix} \chi_s \\ \dfrac{\bar{\sigma}.\bar{p}}{E_{\bar{p}} + m}\, \chi_s \end{pmatrix} }$$

WHERE χ_s IS A PAULI-SPINOR $\quad s = \pm \frac{1}{2}$

NORMALIZATION (COVARIANT NORMALIZATION)

$$\bar{u}(\bar{p}, s)\, u(\bar{p}, s') = (2m)\, \delta_{ss'}$$

$$u^+(\bar{p}, s)\, u(\bar{p}, s') = (2E_{\bar{p}})\, \delta_{ss'}$$

$$\bar{U}(\bar{p},s)\,U(\bar{p},s') = N^2 \left(\chi_s^+ \;\; -\chi_s^+ \frac{\bar{\sigma}.\bar{p}}{E_{\bar{p}}+m} \right) \begin{pmatrix} \chi_{s'} \\ \frac{\bar{\sigma}.\bar{p}}{E_{\bar{p}}+m}\chi_{s'} \end{pmatrix}$$

$$= N^2 \left\{ \chi_s^+ \chi_{s'} - \chi_s^+ \frac{(\bar{\sigma}.\bar{p})(\bar{\sigma}.\bar{p})}{(E_{\bar{p}}+m)^2} \chi_s \right\}$$

$$\downarrow \quad (\bar{\sigma}.\bar{p})(\bar{\sigma}.\bar{p}) = \bar{p}^2 = E_{\bar{p}}^2 - m^2$$

$$= N^2 \chi_s^+ \chi_{s'} \left\{ 1 - \frac{E_{\bar{p}}^2 - m^2}{(E_{\bar{p}}+m)^2} \right\}$$

$$\downarrow$$

$$= N^2 \quad \delta_{ss'} \quad \frac{(2m)}{E_{\bar{p}}+m}$$

BY REQUIRING $\quad \bar{U}(\bar{p},s)\,U(\bar{p},s') = (2m)\,\delta_{ss'}$

$$\underline{\underline{N = \sqrt{E_{\bar{p}}+m}}}$$

↳ WITH THIS CONVENTION ONE SHOWS THAT

$$U^+(\bar{p},s)\;U(\bar{p},s') = 2E_{\bar{p}}\,\delta_{ss'}$$

↳ IN DIRAC REPRESENTATION : SOLUTION v

$$\left(\gamma^0 E_{\bar{p}} - \bar{\gamma}.\bar{p} + m \right) v(\bar{p}, s) = 0.$$

$$\downarrow \qquad v(\bar{p}, s) = N \begin{pmatrix} \varphi'_s \\ \chi'_s \end{pmatrix}$$

$$\begin{pmatrix} E_{\bar{p}} + m & -\bar{\sigma}.\bar{p} \\ \bar{\sigma}.\bar{p} & -(E_{\bar{p}} - m) \end{pmatrix} \begin{pmatrix} \varphi'_s \\ \chi'_s \end{pmatrix} = 0$$

$$(E_{\bar{p}} + m)\, \varphi'_s - \bar{\sigma}\bar{p}\, \chi'_s = 0$$

$$\therefore \quad \boxed{v(\bar{p}, s) = N \begin{pmatrix} \frac{\bar{\sigma}.\bar{p}}{E_{\bar{p}} + m} \chi'_s \\ \chi'_s \end{pmatrix}}$$

WITH
$$\chi'_{+\frac{1}{2}} = \chi_{-\frac{1}{2}}$$
$$\chi'_{-\frac{1}{2}} = \chi_{+\frac{1}{2}}$$

DIRAC : ABSENCE OF NEG. ENERGY SOLUTION OF SPIN PROJ $-1/2$ INTERPRETED AS ANTI-PARTICLE WITH SPIN PROJ $+1/2$

NORMALIZATION

$$\bar{v}(\bar{p},s)\, v(\bar{p},s')$$

$$= N^2 \left(\chi_s'^{+} \frac{\bar{\sigma}\cdot\bar{p}}{E_{\bar{p}}+m} \quad - \chi_s'^{+} \right) \begin{pmatrix} \frac{\bar{\sigma}\cdot\bar{p}}{E_{\bar{p}}+m}\, \chi'_{s'} \\ \chi'_{s'} \end{pmatrix}$$

$$= N^2 \left\{ \chi_s'^{+} \frac{(\bar{\sigma}\cdot\bar{p})(\bar{\sigma}\cdot\bar{p})}{(E_{\bar{p}}+m)^2}\, \chi'_{s'} - \chi_s'^{+}\chi'_{s'} \right\}$$

$$= N^2\, \chi_s'^{+}\chi'_{s'} \left\{ \frac{E_{\bar{p}}^2 - m^2}{(E_{\bar{p}}+m)^2} - 1 \right\}$$

$$\downarrow$$

$$= N^2\, \delta_{ss'} \frac{(-2m)}{E_{\bar{p}}+m}$$

$\downarrow$ WITH CHOICE $N = \sqrt{E_{\bar{p}}+m}$

$$= -(2m)\, \delta_{ss'}$$

$$\therefore \quad \bar{v}(\bar{p},s)\, v(\bar{p},s') = -(2m)\, \delta_{ss'}$$

$$v^{+}(\bar{p},s)\, v(\bar{p},s') = (2E_{\bar{p}})\, \delta_{ss'}$$

↳ WE CAN ALSO EASILY CHECK THE ORTHOGONALITY OF PARTICLE (u) AND ANTI-PARTICLE (v) SPINORS:

$$\bar{u}(\bar{p}, s)\, v(\bar{p}, s') = 0$$

$$\bar{v}(\bar{p}, s)\, u(\bar{p}, s') = 0.$$

NOTE: BECAUSE WE WANT TO WORK WITH LORENTZ INVARIANT NORMALIZATION WE NORMALIZE

$\bar{u}(\bar{p}, s)\, u(\bar{p}, s')$ TO A CONSTANT $(2m)\,\delta_{ss'}$

IN CONTRAST:

$$u^+(\bar{p}, s)\, u(\bar{p}, s') = 2E_{\bar{p}}\, \delta_{ss'}$$

IS NOT LORENTZ INVARIANT AS IT INVOLVES THE ENERGY $E_{\bar{p}}$

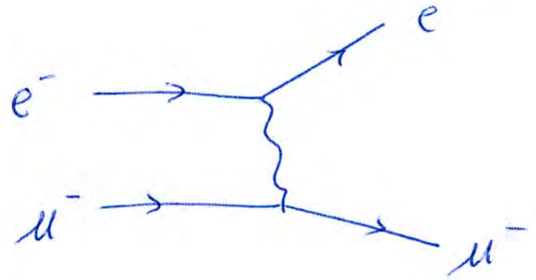
∴ IN QFT FOR DIRAC PARTICLES: WE WILL WORK WITH $\psi(x)$ AND $\bar{\psi}(x)$ ($\bar{\psi}$ INSTEAD OF ψ^+)

⇒ SPIN SUMS

IN FOLLOWING WE WILL FREQUENTLY ENCOUNTER PROCESSES IN WHICH THE POLARIZATION OF INITIAL SPIN 1/2 BEAM PARTICLE IS RANDOMLY ORIENTED,

AND POLARIZATION OF FINAL SPIN 1/2 PARTICLE IS NOT OBSERVED

e.g.



INITIAL e^- : SPIN s_e $\Rightarrow$ $\frac{1}{2}\sum_{s_e}$

FINAL e^- : SPIN s_e' $\Rightarrow$ $\sum_{s_e'}$

SUCH UNPOLARIZED PROCESSES WILL LEAD TO SPIN SUMS OF FORM

$$\sum_{s=\pm\frac{1}{2}} u(\bar{p},s)\,\bar{u}(\bar{p},s)$$

BY PLUGGING IN A SOLUTION

$$(E_{\bar{p}}+m)\sum_{s}\begin{pmatrix}\chi_s\\ \frac{\bar{\sigma}\cdot\bar{p}}{E_{\bar{p}}+m}\chi_s\end{pmatrix}\left(\chi_s^+ \quad -\chi_s^+\frac{\bar{\sigma}\cdot\bar{p}}{E_{\bar{p}}+m}\right)$$

$$= \left(E_{\bar p} + m\right) \cdot \begin{pmatrix} \sum_s \chi_s \chi_s^+ & - \sum_s \chi_s \chi_s^+ \frac{\bar\sigma . \bar p}{E_{\bar p} + m} \\ \frac{\bar\sigma . \bar p}{E_p + m} \sum_s \chi_s \chi_s^+ & - \frac{\bar\sigma . \bar p}{E_{\bar p} + m} \left(\sum_s \chi_s \chi_s^+ \right) \frac{\bar\sigma . \bar p}{E_p + m} \end{pmatrix}$$

$\downarrow$ USING COMPLETENESS FOR PAULI SPINORS

$$\sum_{s = \pm \frac{1}{2}} \chi_s \chi_s^+ = \mathbb{1}_{2\times 2}$$

$$= \left(E_{\bar p} + m \right) \begin{pmatrix} \mathbb{1} & - \frac{\bar\sigma . \bar p}{E_{\bar p} + m} \\ \frac{\bar\sigma . \bar p}{E_{\bar p} + m} & - \frac{\bar p^2}{(E_{\bar p} + m)^2} \end{pmatrix}$$

$$= \begin{pmatrix} E_{\bar p} + m & - \bar\sigma . \bar p \\ \bar\sigma . \bar p & - (E_{\bar p} - m) \end{pmatrix}$$

$$\therefore \quad \boxed{\sum_{s = \pm \frac{1}{2}} u(\bar p, s)\, \bar u(\bar p, s) = \not p + m.}$$

AND ANALOGOUSLY

$$\boxed{\sum_{s = \pm \frac{1}{2}} v(\bar p, s)\, \bar v(\bar p, s) = \not p - m.}$$

⇒ DIRAC LAGRANGIAN

↳ INTERPRETING $\psi(x)$ AND $\bar{\psi}(x)$

AS INDEPENDENT FIELDS SATISFYING

$$(i\gamma^\mu \partial_\mu - m)\psi = 0.$$

$$\left(i(\partial_\mu \bar{\psi})\gamma^\mu + m\bar{\psi}\right) = 0.$$

WE CAN DERIVE DIRAC EQ.

FROM THE DIRAC LAGRANGIAN:

$$\boxed{\mathcal{L}_{DIRAC} = \bar{\psi}(x)\,(i\gamma^\mu \partial_\mu - m)\,\psi(x)}$$

↳ EULER LAGRANGE EQS.

$$\left.\begin{aligned} &\cdot\ \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = (i\gamma^\mu \partial_\mu - m)\psi \\ &\cdot\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0 \end{aligned}\right\} \Rightarrow (i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$\left.\begin{aligned} &\cdot\ \frac{\partial \mathcal{L}}{\partial \psi} = \bar{\psi}(-m) \\ &\cdot\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \bar{\psi}\, i\gamma^\mu \end{aligned}\right\} \Rightarrow i(\partial_\mu \bar{\psi})\gamma^\mu + m\bar{\psi} = 0$$

⟹ DIRAC HAMILTONIAN

↳ CONJUGATE MOMENTA

$$\psi \longrightarrow \Pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i \bar{\psi} \gamma^0 = i \psi^+$$

$$\bar{\psi} \longrightarrow \bar{\Pi} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\bar{\psi}}} = 0$$

↳ HAMILTONIAN

$$\mathcal{H} = \Pi \dot{\psi} + \underbrace{\cancel{\dot{\bar{\psi}} \bar{\Pi}}}_{0} - \mathcal{L}$$

$$= i \bar{\psi} \gamma^0 \partial_0 \psi - \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi$$

$$= \bar{\psi} (-i \bar{\gamma} . \bar{\nabla} + m) \psi$$

WITH $\bar{\nabla}^i = \frac{\partial}{\partial x^i} = \partial_i$

$$\boxed{H = \int d^3\bar{x} \; \bar{\psi}(x) \left(-i \bar{\gamma} . \bar{\nabla} + m \right) \psi(x)}$$

$\Longrightarrow$ ENERGY - MOMENTUM TENSOR

$$\hookrightarrow \quad T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} (\partial^\nu \psi) - g^{\mu\nu} \mathcal{L}$$

$$\partial_\mu T^{\mu\nu} = 0$$

$$T^{\mu\nu}_{DIRAC} = \bar{\psi} i \gamma^\mu \partial^\nu \psi - g^{\mu\nu} \bar{\psi} (i \gamma^\alpha \partial_\alpha - m) \psi$$

$\hookrightarrow$ CONSERVED QUANTITY

4 - MOMENTUM P^ν

$$P^\nu = \int d^3\bar{x} \; T^{0\nu}$$

$$= \int d^3\bar{x} \left(\Pi \, \partial^\nu \psi - g^{0\nu} \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi \right)$$

$$P^0 = H$$

$$P^i = \int d^3\bar{x} \; \Pi (\partial^i \psi)$$

$(i = 1, 2, 3)$

$$= \int d^3\bar{x} \; \psi^\dagger (-i \bar{\nabla}^i) \psi \quad \longrightarrow \text{ 3 - MOMENTUM OF DIRAC FIELD}$$

2) QUANTIZATION OF DIRAC FIELD

⇒ **FERMI-DIRAC STATISTICS : ANTI-COMMUTATION RELATIONS**

↳ NORMAL MODE EXPANSION FOR ψ

$$\boxed{\psi(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3\bar{p}}{(2\pi)^3 \sqrt{2E_{\bar{p}}}} \left\{ a(\bar{p},s)\, u(\bar{p},s)\, e^{-ip.x} + b^{+}(\bar{p},s)\, v(\bar{p},s)\, e^{+ip.x} \right\}}$$

$$(i\gamma^\mu \partial_\mu - m)\, u(\bar{p},s)\, e^{-ip.x} = (\not{p} - m)\, u(\bar{p},s)\, e^{-ip.x} = 0$$

$$(i\gamma^\mu \partial_\mu - m)\, v(\bar{p},s)\, e^{+ip.x} = -(\not{p} + m)\, v(\bar{p},s)\, e^{ip.x} = 0$$

↳ NORMAL MODE EXPANSION FOR ADJOINT FIELD $\bar{\psi}$

$$\boxed{\bar{\psi}(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3\bar{p}}{(2\pi)^3 \sqrt{2E_{\bar{p}}}} \left\{ a^{+}(\bar{p},s)\, \bar{u}(\bar{p},s)\, e^{+ip.x} + b(\bar{p},s)\, \bar{v}(\bar{p},s)\, e^{-ip.x} \right\}}$$

↳ SINGLE- PARTICLE STATE (e^- : ELECTRON OF MOMENTUM $\bar{p}$, SPIN PROJ s)

$$|\bar{p}, s\rangle_{e^-} \equiv \sqrt{2E_{\bar{p}}}\, a^+(\bar{p}, s)\, |0\rangle$$

↳ SINGLE ANTI-PARTICLE STATE (e^+ : POSITRON OF MOMENTUM $\bar{p}$, SPIN PROJ s)

$$|\bar{p}, s\rangle_{e^+} \equiv \sqrt{2E_{\bar{p}}}\, b^+(\bar{p}, s)\, |0\rangle$$

↳ VACUUM

$$a(\bar{p}, s)|0\rangle = 0$$

$$b(\bar{p}, s)|0\rangle = 0$$

↳ FERMI- DIRAC STATISTICS.

e^- AND e^+ ARE BOTH SPIN 1/2 FERMIONS.

↳ PAULI PRINCIPLE : 2 FERMIONS CANNOT OCCUPY THE SAME STATE, STATE IS ANTI-SYMMETRIC UNDER EXCHANGE OF 2 FERMIONS

$$a^+(\bar{p}, s)\, a^+(\bar{p}, s)\, |0\rangle = 0$$

$$b^+(\bar{p}, s)\, b^+(\bar{p}, s)\, |0\rangle = 0$$

$$a^+(\bar{p}, s)\, a^+(\bar{p}', s')\, |0\rangle = -\, a^+(\bar{p}', s')\, a^+(\bar{p}, s)\, |0\rangle$$

$$b^+(\bar{p}, s)\, b^+(\bar{p}', s')\, |0\rangle = -\, b^+(\bar{p}', s')\, b^+(\bar{p}, s)\, |0\rangle$$

CAN BE SATISFIED BY
IMPOSING ANTI-COMMUTATION RELATIONS :

$$\left[a(\bar{p},s),\ a(\bar{p}',s')\right]_+ = 0$$

$$\left[a^+(\bar{p},s),\ a^+(\bar{p}',s')\right]_+ = 0$$

$$\left[a(\bar{p},s),\ a^+(\bar{p}',s')\right]_+ = (2\pi)^3\,\delta^3(\bar{p}-\bar{p}')\,\delta_{ss'}$$

$$\left[b(\bar{p},s),\ b(\bar{p}',s')\right]_+ = 0$$

$$\left[b^+(\bar{p},s),\ b^+(\bar{p}',s')\right]_+ = 0$$

$$\left[b(\bar{p},s),\ b^+(\bar{p}',s')\right]_+ = (2\pi)^3\,\delta^3(\bar{p}-\bar{p}')\,\delta_{ss'}$$

& ALL ANTI-COMMUTATORS BETWEEN PARTICLES & ANTI-PARTICLES VANISH

$$\left[a(\bar{p},s),\ b(\bar{p}',s')\right]_+ = \left[a(\bar{p},s),\ b^+(\bar{p}',s')\right]_+ = 0$$

$$\left[a^+(\bar{p},s),\ b(\bar{p}',s')\right]_+ = \left[a^+(\bar{p},s),\ b^+(\bar{p}',s')\right]_+ = 0.$$

WITH ANTI-COMMUTATOR

$$\left[A,B\right]_+ \equiv AB + BA$$

OFTEN ONE ALSO USES SYMBOL $\{A,B\}$ FOR ANTI-COMMUTATOR.

↳ WITH ANTI-COMMUTATION RELATIONS:
SINGLE-PARTICLE STATES ARE NORMALIZED AS:

$${}_{e^-}\langle \bar{p}', s' | \bar{p}, s \rangle_{e^-} = \sqrt{2E_{\bar{p}'}}\sqrt{2E_{\bar{p}}}\ \langle 0 | a(\bar{p}', s')\, a^{+}(\bar{p}, s) | 0 \rangle$$

$$= \sqrt{2E_{\bar{p}'}}\sqrt{2E_{\bar{p}}}\ \langle 0 | \left[a(\bar{p}', s'), a^{+}(\bar{p}, s) \right]_{+} | 0 \rangle$$

$$\underline{\underline{{}_{e^-}\langle \bar{p}', s' | \bar{p}, s \rangle_{e^-} = (2\pi)^3\, 2E_{\bar{p}}\ \delta^3(\bar{p} - \bar{p}')\, \delta_{ss'}}}$$

AND ANALOGOUSLY

$${}_{e^+}\langle \bar{p}', s' | \bar{p}, s \rangle_{e^+} = (2\pi)^3\, 2E_{\bar{p}}\ \delta^3(\bar{p} - \bar{p}')\, \delta_{ss'}$$

⟹ ANTI-COMMUTATION RELATIONS FOR FIELDS

↳ $\psi(\bar{x}, t) \quad \longrightarrow \quad \Pi_{\psi}(\bar{x}, t) = i\, \psi^{+}(\bar{x}, t).$

CONJUGATE MOMENTUM FIELD

EQUAL TIME ANTI-COMMUTATOR

$$\left[\psi(\bar{x}, t),\ \psi^{+}(\bar{x}', t) \right]_{+}$$

$$= \sum_{s} \sum_{s'} \int \frac{d^3 \bar{p}}{(2\pi)^3 \sqrt{2E_{\bar{p}}}} \int \frac{d^3 \bar{p}'}{(2\pi)^3 \sqrt{2E_{\bar{p}'}}}$$

$$\Big[a(\bar{p}, s)\, u(\bar{p}, s)\, e^{-iE_{\bar{p}}t}\, e^{i\bar{p}.\bar{x}} + b^{+}(\bar{p}, s)\, v(\bar{p}, s)\, e^{+iE_{\bar{p}}t}\, e^{-i\bar{p}.\bar{x}},$$

$$a^{+}(\bar{p}', s')\, u^{+}(\bar{p}', s')\, e^{+iE_{\bar{p}'}t}\, e^{-i\bar{p}'.\bar{x}'} + b(\bar{p}', s')\, v^{+}(\bar{p}', s')\, e^{-iE_{\bar{p}'}t}\, e^{+i\bar{p}'.\bar{x}'} \Big]_{+}$$

$$\Big\downarrow \quad \left[a(\bar p, s),\ a^\dagger(\bar p', s') \right]_+$$

$$= \left[b(\bar p, s),\ b^\dagger(\bar p', s') \right]_+$$

$$= (2\pi)^3\, \delta^3(\bar p - \bar p')\, \delta_{ss'}$$

$$\left[\psi(\bar x, t),\ \psi^\dagger(\bar x', t) \right]_+$$

$$= \sum_s \int \frac{d^3 \bar p}{(2\pi)^3\, 2E_{\bar p}} \left\{ u(\bar p, s)\, u^\dagger(\bar p, s)\, e^{i\bar p.(\bar x - \bar x')} + v(\bar p, s)\, v^\dagger(\bar p, s)\, e^{-i\bar p.(\bar x - \bar x')} \right\}$$

$$\Big\downarrow \quad \sum_s u(\bar p, s)\, u^\dagger(\bar p, s) = (\not p + m)\, \gamma^0$$

$$\sum_s v(\bar p, s)\, v^\dagger(\bar p, s) = (\not p - m)\, \gamma^0$$

$$= \int \frac{d^3 \bar p}{(2\pi)^3\, 2E_{\bar p}} \left\{ (\gamma^0 E_{\bar p} - \bar\gamma.\bar p + m)\, \gamma^0\, e^{i\bar p.(\bar x - \bar x')} + (\gamma^0 E_{\bar p} - \bar\gamma.\bar p - m)\, \gamma^0\, e^{-i\bar p.(\bar x - \bar x')} \right\}$$

$\Big\downarrow$ IN 2° TERM $\bar p \to -\bar p$

$$= \int \frac{d^3 \bar p}{(2\pi)^3\, 2E_{\bar p}} \left\{ 2E_{\bar p}\, e^{i\bar p.(\bar x - \bar x')} \right\} \mathbb{1}_{4\times 4}$$

↳ 4×4 MATRIX IN DIRAC SPACE

$$= \delta^3(\bar x - \bar x')\, \mathbb{1}_{4\times 4}$$

$$\therefore \quad \left[\psi(\bar{x},t),\ \psi^{+}(\bar{x}',t)\right]_{+} = (2\pi)^3\,\delta^3(\bar{x}-\bar{x}')\,\mathbb{1}_{4\times4}$$

$$\left[\psi(\bar{x},t),\ \psi(\bar{x}',t)\right]_{+} = 0$$

$$\left[\psi^{+}(\bar{x},t),\ \psi^{+}(\bar{x}',t)\right]_{+} = 0.$$

$\Rightarrow$ **HAMILTONIAN IN NORMAL MODE EXPANSION**

$$\hookrightarrow \quad H = \int d^3\bar{x}\ \bar{\psi}(x)\left(-i\,\bar{\gamma}.\bar{\nabla} + m\right)\psi(x)$$

$\downarrow$ INSERT NORMAL MODE EXPANSIONS FOR $\psi(x)$ & $\bar{\psi}(x)$

$$H = \int d^3\bar{x} \sum_{s}\sum_{s'} \int \frac{d^3\bar{p}}{(2\pi)^3\sqrt{2E_{\bar{p}}}} \int \frac{d^3\bar{p}'}{(2\pi)^3\sqrt{2E_{\bar{p}'}}}$$

$$\cdot \left(a^{+}(\bar{p},s)\,\bar{u}(\bar{p},s)\,e^{+ip.x} + b(\bar{p},s)\,\bar{v}(\bar{p},s)\,e^{-ip.x}\right)$$

$$\cdot \left(-i\,\bar{\gamma}.\bar{\nabla} + m\right)\left(a(\bar{p}',s')\,u(\bar{p}',s')\,e^{-ip'.x} + b^{+}(\bar{p}',s')\,v(\bar{p}',s')\,e^{+ip'.x}\right)$$

$$H = \int d^3\bar{x} \sum_{s}\sum_{s'} \int \frac{d^3\bar{p}}{(2\pi)^3\sqrt{2E_{\bar{p}}}} \int \frac{d^3\bar{p}'}{(2\pi)^3\sqrt{2E_{\bar{p}'}}}$$

$$\cdot \left(a^{+}(\bar{p},s)\,\bar{u}(\bar{p},s)\,e^{+ip.x} + b(\bar{p},s)\,\bar{v}(\bar{p},s)\,e^{-ip.x}\right)$$

$$\cdot \left(a(\bar{p}',s')\,(\bar{\gamma}.\bar{p}'+m)\,u(\bar{p}',s')\,e^{-ip'.x} + b^{+}(\bar{p}',s')\,(-\bar{\gamma}.\bar{p}'+m)\,v(\bar{p}',s') \cdot e^{+ip'.x}\right)$$

USING:
$$(\not{p}' - m)\, u(\bar{p}', s') = 0$$
$$\Downarrow$$
$$(\gamma^0 E_{\bar{p}'} - \bar{\gamma}.\bar{p} - m)\, u(\bar{p}', s') = 0.$$

$$(\not{p}' + m)\, v(\bar{p}', s') = 0$$
$$\Downarrow$$
$$(\gamma^0 E_{\bar{p}'} - \bar{\gamma}.\bar{p} + m)\, v(\bar{p}', s') = 0$$

$$H = \sum_s \sum_{s'} \int \frac{d^3\bar{p}}{(2\pi)^3 \sqrt{2E_{\bar{p}}}} \int \frac{d^3\bar{p}'}{(2\pi)^3 \sqrt{2E_{\bar{p}'}}}$$
$$\cdot \int d^3\bar{x} \Big\{ e^{i(E_{\bar{p}} - E_{\bar{p}'})t}\, e^{-i(\bar{p}-\bar{p}').\bar{x}}\, a^+(\bar{p}, s)\, a(\bar{p}', s')$$
$$\cdot\, \bar{u}(\bar{p}, s)\, \gamma^0 E_{\bar{p}'}\, u(\bar{p}', s')$$
$$-\, e^{i(E_{\bar{p}} + E_{\bar{p}'})t}\, e^{-i(\bar{p}+\bar{p}').\bar{x}}\, a^+(\bar{p}, s)\, b^+(\bar{p}', s')$$
$$\cdot\, \bar{u}(\bar{p}, s)\, \gamma^0 E_{\bar{p}'}\, v(\bar{p}', s')$$
$$+\, e^{-i(E_{\bar{p}} + E_{\bar{p}'})t}\, e^{i(\bar{p}+\bar{p}').\bar{x}}\, b(\bar{p}, s)\, a(\bar{p}', s')$$
$$\cdot\, \bar{v}(\bar{p}, s)\, \gamma^0 E_{\bar{p}'}\, u(\bar{p}', s')$$
$$-\, e^{-i(E_{\bar{p}} - E_{\bar{p}'})t}\, e^{i(\bar{p}-\bar{p}').\bar{x}}\, b(\bar{p}, s)\, b^+(\bar{p}', s')$$
$$\cdot\, \bar{v}(\bar{p}, s)\, \gamma^0 E_{\bar{p}'}\, v(\bar{p}', s') \Big\}$$

$$\int d^3\bar{x}\; e^{-i(\bar{p}-\bar{p}').\bar{x}} = (2\pi)^3\,\delta^3(\bar{p}-\bar{p}')$$

$$\downarrow$$

$$H = \sum_s \sum_{s'} \int \frac{d^3\bar{p}}{(2\pi)^3\, 2E_{\bar{p}}} \cdot E_{\bar{p}}$$

$$\cdot \Big\{ a^+(\bar{p},s)\, a(\bar{p},s')\quad u^+(\bar{p},s)\, u(\bar{p},s')$$

$$- e^{2iE_{\bar{p}}t}\, a^+(\bar{p},s)\, b^+(-\bar{p},s')\quad u^+(\bar{p},s)\, v(-\bar{p},s')$$

$$+ e^{-2iE_{\bar{p}}t}\, b(\bar{p},s)\, a(-\bar{p},s')\quad v^+(\bar{p},s)\, u(-\bar{p},s')$$

$$- b(\bar{p},s)\, b^+(\bar{p},s')\quad v^+(\bar{p},s)\, v(\bar{p},s') \Big\}$$

$$\downarrow$$

ORTHOGONALITY

$$u^+(\bar{p},s)\, u(\bar{p},s') = \delta_{ss'}\,(2E_{\bar{p}})$$

$$v^+(\bar{p},s)\, v(\bar{p},s') = \delta_{ss'}\,(2E_{\bar{p}})$$

$$u^+(\bar{p},s)\, v(-\bar{p},s') = 0$$

$$v^+(\bar{p},s)\, u(-\bar{p},s') = 0$$

$$H = \sum_s \int \frac{d^3\bar{p}}{(2\pi)^3}\, E_{\bar{p}} \Big\{ a^+(\bar{p},s)\, a(\bar{p},s) - b(\bar{p},s)\, b^+(\bar{p},s) \Big\}$$

USING $\left[b(\bar{p}, s), b^+(\bar{p}, s) \right]_+ = (2\pi)^3 \delta^3(0)$

$$H = \sum_s \int \frac{d^3\bar{p}}{(2\pi)^3} E_{\bar{p}} \left\{ a^+(\bar{p}, s)\, a(\bar{p}, s) + b^+(\bar{p}, s)\, b(\bar{p}, s) + \text{CONSTANT} \right\}$$

↳ CONSTANT TERM CAN BE DROPPED IF WE DEFINE VACUUM STATE AS STATE OF ZERO ENERGY
(i.e. EXPRESS ENERGY RELATIVE TO VACUUM STATE)

$$\langle 0 | H | 0 \rangle = \sum_s \int \frac{d^3\bar{p}}{(2\pi)^3} E_{\bar{p}} \cdot \{\text{CONSTANT}\}$$

BECAUSE
$$\langle 0 | a^+(\bar{p}, s)\, a(\bar{p}, s) | 0 \rangle = 0$$
$$\langle 0 | b^+(\bar{p}, s)\, b(\bar{p}, s) | 0 \rangle = 0$$

↳ NORMAL ORDERING

PRODUCT OF OPERATORS WHERE CREATION OPERATOR STANDS ON LEFT OF ANNIHILATION OPERATOR

$$N\left(a^{+}(\bar{p},s)\, a(\bar{p}',s')\right) = a^{+}(\bar{p},s)\, a(\bar{p}',s')$$

$$N\left(a(\bar{p},s)\, a^{+}(\bar{p}',s')\right) = -\, a^{+}(\bar{p}',s')\, a(\bar{p},s)$$

AND ANALOGOUS FOR b

WHEN INTERCHANGING FERMION OPERATORS:
FOR EVERY INTERCHANGE $\Rightarrow (-1)$

⇝ BY DEFINITION $\langle 0 | N(A.B) | 0 \rangle$

$\therefore$ $H - \langle 0 | H | 0 \rangle$

IS EQUIVALENT TO EXPRESSING H THROUGH NORMAL ORDERED PRODUCT

$$H = \int d^3\bar{x} \;\; \underset{\Uparrow}{N} \;\; \bar{\psi}(x)\left(-i\,\bar{\gamma}.\bar{\nabla} + m\right)\psi(x).$$

IN FOLLOWING: WE WILL EXPRESS ALL QUANTITIES (ENERGY, MOMENTUM, SPIN, CHARGE, ...) RELATIVE TO VACUUM

$\Downarrow$

NORMAL ORDERING IS UNDERSTOOD IN EXPRESSIONS

↳ ENERGY

$$\boxed{H = \sum_s \int \frac{d^3\bar{p}}{(2\pi)^3} E_{\bar{p}} \left\{ a^+(\bar{p},s)\, a(\bar{p},s) + b^+(\bar{p},s)\, b(\bar{p},s) \right\}}$$

↑ # PARTICLES OF $(\bar{p},s)$

↑ # ANTI-PARTICLE OF $(\bar{p},s)$

FOR FERMIONS: # PARTICLES $= 0, 1$

↳ MOMENTUM

ANALOGOUSLY, WE CAN OBTAIN FROM

$$\bar{P} = \int d^3\bar{x}\; N\; \psi^+(x) \left(-i\bar{\nabla}\right) \psi(x)$$

↓

$$\boxed{\bar{P} = \sum_s \int \frac{d^3\bar{p}}{(2\pi)^3}\, \bar{p} \left\{ a^+(\bar{p},s)\, a(\bar{p},s) + b^+(\bar{p},s)\, b(\bar{p},s) \right\}}$$

↳ NOTE: IF WE WOULD HAVE QUANTIZED DIRAC FIELD ACCORDING TO BOSE-EINSTEIN STATISTICS (i.e. THROUGH COMMUTATORS AS FOR KLEIN-GORDON FIELD), WE WOULD HAVE OBTAINED

$$H = \sum_s \int \frac{d^3\bar{p}}{(2\pi)^3} E_{\bar{p}} \left\{ a^+(\bar{p},s)\, a(\bar{p},s) - b^+(\bar{p},s)\, b(\bar{p},s) \right\}$$

DUE TO MINUS SIGN: H WOULD NOT HAVE A LOWER BOUND ⇒ NO STABLE GROUND STATE !

⟹ CHARGE

↳ $\mathcal{L}_{DIRAC} = \bar{\psi}(x)\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x)$

CONSIDER GLOBAL PHASE TRANSFORMATION

$\psi(x) \longrightarrow e^{i\alpha}\,\psi(x)$, INFINITESIMAL $\delta\psi = i\alpha\psi$

LEAVES $\mathcal{L}$ INVARIANT : $\mathcal{L} \to \mathcal{L}$

∴ SYMMETRY TF.

⇓

↳ CONSERVED CURRENT $\partial_{\mu} J^{\mu} = 0$

$$J^{\mu} \sim \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\psi)} \cdot \delta\psi$$

$$\boxed{J^{\mu} = \bar{\psi}(x)\,\gamma^{\mu}\psi(x)}$$

↳ CONSERVED CHARGE

$$Q \equiv \int d^3\bar{x}\;\bar{\psi}(x)\,\gamma^{0}\psi(x)$$

$$= \int d^3\bar{x}\;\psi^{+}(x)\,\psi(x)$$

↓ BY INSERTING NORMAL MODE EXPANSIONS.

AND CONSIDERING NORMAL ORDERING (EXERCISE!)

$$\boxed{Q = \sum_s \int \frac{d^3\bar{p}}{(2\pi)^3} \left\{ a^+(\bar{p},s)\, a(\bar{p},s) - b^+(\bar{p},s)\, b(\bar{p},s) \right\}}$$

PARTICLE & ANTI - PARTICLE HAVE OPPOSITE CHARGES !

↳ ELECTROMAGNETIC CURRENT & CHARGE

BY MULTIPLYING ABOVE J^μ WITH ELECTRIC CHARGE q, WE OBTAIN ELECTROMAGNETIC CURRENT J^μ_{EM}

$$\boxed{J^\mu_{EM} = q\, \bar{\psi}(x)\, \gamma^\mu\, \psi(x)}$$

CONSERVED CHARGE : ELECTRIC CHARGE

$$Q_{EM} = q \int d^3\bar{x}\, \psi^+(x)\, \psi(x)$$

e.g. FOR THEORY OF e^- AND e^+

FOR e^- : $q = -e$ $\quad \left(e > 0 ,\ \frac{e^2}{4\pi} \simeq \frac{1}{137} \right)$

$$\boxed{Q_{EM} = -e \sum_s \int \frac{d^3\bar{p}}{(2\pi)^3} \left\{ a^+(\bar{p},s)\, a(\bar{p},s) - b^+(\bar{p},s)\, b(\bar{p},s) \right\}}$$

RELATIVE MINUS SIGN : e^- AND e^+ HAVE OPPOSITE ELECTRIC CHARGES : $q_{e^-} = -e$, $q_{e^+} = +e$

3) FEYNMAN PROPAGATOR OF DIRAC FIELD

⇒ 2-POINT CORRELATION FUNCTIONS.

↳ $$\langle 0 | \psi_\alpha(x) \bar{\psi}_\beta(y) | 0 \rangle \equiv i S^+_{\alpha\beta}(x-y)$$

x, y : 2 DIFFERENT SPACE-TIME POINTS

$\alpha, \beta = 1,2,3,4$: SPINOR INDEX OF DIRAC FIELD

$S_{\alpha\beta}$: 4 x 4 MATRIX IN DIRAC SPACE

↳ INSERTING NORMAL MODE EXPANSIONS

$$i S^+(x-y)$$

$$= \sum_s \sum_{s'} \int \frac{d^3 \bar{p}}{(2\pi)^3 \sqrt{2E_{\bar{p}}}} \int \frac{d^3 \bar{p}'}{(2\pi)^3 \sqrt{2E_{\bar{p}'}}}$$

AS $b(\bar{p}, s)|0\rangle = 0$

$$\langle 0 | \left(a(\bar{p},s)\, u(\bar{p},s)\, e^{-ip.x} + \cancel{b^+}(\bar{p},s)\, v(\bar{p},s)\, e^{+ip.x} \right)$$

$$\cdot \left(a^+(\bar{p}',s')\, \bar{u}(\bar{p}',s')\, e^{+ip'.y} + \cancel{b(\bar{p}',s')}\, \bar{v}(\bar{p}',s')\, e^{-ip'.y} \right) | 0 \rangle$$

$$= \sum_s \sum_{s'} \int \frac{d^3 \bar{p}}{(2\pi)^3 \sqrt{2E_{\bar{p}}}} \int \frac{d^3 \bar{p}'}{(2\pi)^3 \sqrt{2E_{\bar{p}'}}} e^{-ip.x} e^{ip'.y} u(\bar{p},s)\bar{u}(p',s')$$

$$\cdot \langle 0 | a(\bar{p},s)\, a^+(\bar{p}',s') | 0 \rangle$$

$$\langle 0 | \, a(\bar{p}, s) \, a^{+}(\bar{p}', s') \, | 0 \rangle \ldots$$

$$= \langle 0 | \left[a(\bar{p}, s) , a^{+}(\bar{p}', s') \right]_{+} | 0 \rangle$$

$$= (2\pi)^3 \delta^3(\bar{p} - \bar{p}') \, \delta_{ss'}$$

$$i S^{+}(x-y) = \int \frac{d^3 \bar{p}}{(2\pi)^3 2E_{\bar{p}}} \, e^{-ip.(x-y)} \Big|_{p^0 = E_{\bar{p}}}$$

$$\cdot \underbrace{\sum_s U(\bar{p}, s) \, \bar{U}(\bar{p}, s)}_{(\not{p} + m)} \quad \text{SPIN SUM}$$

$$\therefore \boxed{ i S^{+}(x-y) = \int \frac{d^3 \bar{p}}{(2\pi)^3 2E_{\bar{p}}} \, e^{-ip.(x-y)} \, (\not{p} + m) \Big|_{p^0 = E_{\bar{p}}} }$$

$$\hookrightarrow \quad i S^{+}(x) = (i \gamma^{\mu} \partial_{\mu} + m) \underbrace{\int \frac{d^3 \bar{p}}{(2\pi)^3 2E_{\bar{p}}} \, e^{-ip.x} \Big|_{p^0 = E_{\bar{p}}}}_{i\Delta(x)}$$

SCALAR CORRELATOR FROM KLEIN-GORDON THEORY $i\Delta(x)$

$$\boxed{ S^{+}(x) = (i \gamma^{\mu} \partial_{\mu} + m) \, \Delta(x) }$$

↳ ANALOGOUSLY :

$$\langle 0 | \bar{\psi}_\beta(y)\, \psi_\alpha(x) | 0 \rangle \equiv i\, S^-_{\beta\alpha}(x-y)$$

$$i\, S^-_{\beta\alpha}(x-y)$$

$$= \sum_s \sum_{s'} \int \frac{d^3\bar{p}}{(2\pi)^3\sqrt{2E_{\bar{p}}}} \int \frac{d^3\bar{p}'}{(2\pi)^3\sqrt{2E_{\bar{p}'}}}\, e^{ip.x}\, e^{-ip'.y}$$

$$\cdot\ \bar{v}_\beta(\bar{p}', s')\, v_\alpha(\bar{p}, s)$$

$$\cdot \langle 0 | b(\bar{p}', s')\, b^+(\bar{p}, s) | 0 \rangle$$

$$= \int \frac{d^3\bar{p}}{(2\pi)^3\, 2E_{\bar{p}}}\, e^{ip.(x-y)} \underbrace{\sum_s v_\alpha(\bar{p}, s)\, \bar{v}_\beta(\bar{p}, s)}_{(\not{p}-m)_{\alpha\beta}} \Bigg|_{p^0 = E_{\bar{p}}}$$

$$\boxed{i\, S^-_{\beta\alpha}(x-y) = \int \frac{d^3\bar{p}}{(2\pi)^3\, 2E_{\bar{p}}}\, e^{ip.(x-y)}\, (\not{p}-m)_{\alpha\beta} \Bigg|_{p^0 = E_{\bar{p}}}}$$

$$i\, S^-_{\beta\alpha}(x) = -(i\gamma^\mu \partial_\mu + m)_{\alpha\beta} \underbrace{\int \frac{d^3\bar{p}}{(2\pi)^3\, 2E_{\bar{p}}}\, e^{ip.x}}_{i\Delta(-x)}$$

$$\boxed{S^-_{\beta\alpha}(x) = -(i\gamma^\mu \partial_\mu + m)_{\alpha\beta}\, \Delta(-x)}$$

⇒ TIME - ORDERED PRODUCT FOR FERMION FIELDS

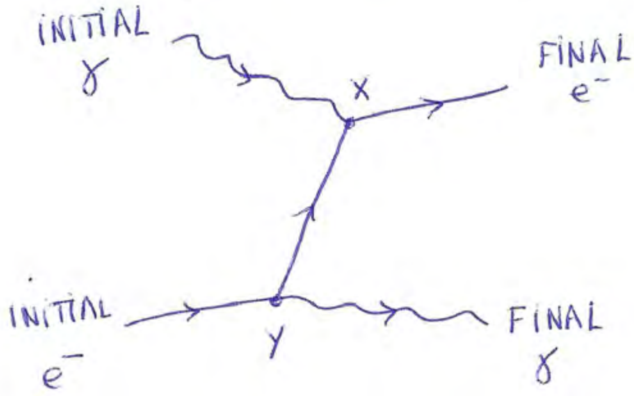
$$T\,\psi(x)\bar{\psi}(y) \equiv \psi(x)\bar{\psi}(y)\;\theta(x^0-y^0) - \bar{\psi}(y)\psi(x)\;\theta(y^0-x^0)$$

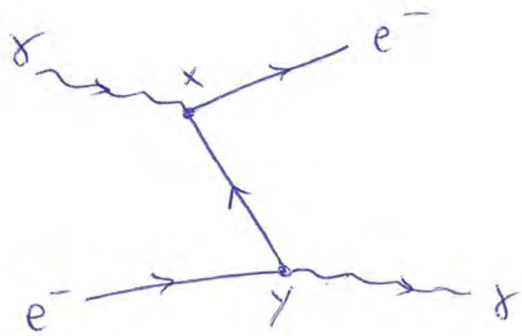
NOTE : − SIGN BECAUSE OF ANTI-COMMUTATION OF FERMION FIELDS

⇒ FEYNMAN PROPAGATOR FOR SPIN 1/2

↳ ENTERS PHYSICAL AMPLITUDES

e.g. COMPTON SCATTERING ON e^- : 2 CONTRIBUTIONS





⟶ TIME ⟶ TIME

$x^0 > y^0$

PROPAGATION OF PARTICLE FROM $y \to x$

e^- PROPAGATES FORWARD IN TIME FROM $y \to x$

$y^0 > x^0$

PROPAGATION OF ANTI-PARTICLE $x \to y$

- AT x : $e^- e^+$ PAIR IS CREATED
- e^+ PROPAGATES FORWARD IN TIME FROM $x \to y$
- AT y : e^+ ANNIHILATES WITH INITIAL e^-

↳ DEFINITION :

$$\boxed{\langle 0 | T\, \psi(x)\, \bar{\psi}(y) | 0 \rangle \equiv i S_F(x-y)}$$

$$\big(i S_F(x-y)\big)_{\alpha\beta} = \theta(x^0 - y^0)\, \langle 0 | \psi_\alpha(x)\, \bar{\psi}_\beta(y) | 0 \rangle$$

$$- \theta(y^0 - x^0)\, \langle 0 | \bar{\psi}_\beta(y)\, \psi_\alpha(x) | 0 \rangle$$

$$= \theta(x^0 - y^0)\; i S^+_{\alpha\beta}(x-y)$$

$$- \theta(y^0 - x^0)\; i S^-_{\beta\alpha}(x-y)$$

$$\big(S_F(x)\big)_{\alpha\beta} = \theta(x^0) \cdot (i\gamma^\mu \partial_\mu + m)_{\alpha\beta}\; \Delta(x)$$

$$- \theta(-x^0) \cdot (i\gamma^\mu \partial_\mu + m)_{\alpha\beta}\, (-1)\; \Delta(-x)$$

$$S_F(x) = (i\gamma^\mu \partial_\mu + m)\; \underbrace{\big(\theta(x^0)\, \Delta(x) + \theta(-x^0)\, \Delta(-x)\big)}_{\Delta_F(x)}$$

$$\boxed{S_F(x) = (i\gamma^\mu \partial_\mu + m)\; \Delta_F(x)}$$

WITH Δ_F : SPIN-0 (KLEIN-GORDON) FEYNMAN PROPAGATOR

↳ FEYNMAN PROPAGATOR IN MOMENTUM SPACE

$$\boxed{S_F(x) \equiv \int \frac{d^4p}{(2\pi)^4}\, e^{-ip.x}\, \tilde{S}_F(p)}$$

WHERE $\tilde{S}_F(p)$ IS FEYNMAN PROPAGATOR FOR SPIN 1/2 PARTICLE IN MOMENTUM SPACE

(NOTE: IN FOLLOWING WE WILL FOR SIMPLICITY DROP $\sim$ ON $\tilde{S}_F(p)$ WHEN IT IS UNDERSTOOD THAT WE WORK IN MOMENTUM SPACE)

↳ EXPLICIT FORM

$$S_F(x) = \left(i\gamma^\mu \partial_\mu + m\right) \Delta_F(x).$$

$$= \left(i\gamma^\mu \partial_\mu + m\right) \int \frac{d^4p}{(2\pi)^4}\, e^{-ip.x}\, \frac{1}{p^2 - m^2 + i\varepsilon}$$

$$= \int \frac{d^4p}{(2\pi)^4}\, e^{-ip.x}\, \frac{(\not{p} + m)}{p^2 - m^2 + i\varepsilon}$$

$$\therefore \quad \boxed{\tilde{S}_F(p) = \frac{(\not{p} + m)}{p^2 - m^2 + i\varepsilon}}$$

4) SYMMETRIES OF THE DIRAC THEORY

⇒ LORENTZ INVARIANCE

↳ EQUATIONS OF MOTION SHOULD BE SAME IN ANY INERTIAL FRAME

LORENTZ TF : $x^\mu \to x'^\mu = a^\mu{}_\nu \, x^\nu$

WITH $\underline{\underline{a^\mu{}_\nu \, a_\mu{}^\lambda = g^\lambda_\nu}}$

↓

ENSURES THAT $x'^\mu x'_\mu = x^\mu x_\mu$

SPACE-TIME INTERVAL IS SAME FOR ALL INERTIAL OBSERVERS

$$\det(a^\mu{}_\nu) = \pm 1$$

↗ $\det(a^\mu{}_\nu) = +1$ PROPER LORENTZ TF (3 ROTATIONS, 3 BOOSTS)

↘ $\det(a^\mu{}_\nu) = -1$ IMPROPER LORENTZ TF

↓

COMBINATIONS OF PROPER LORENTZ TF WITH

SPACE INVERSION OR TIME REVERSAL

$$a^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad a^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

↳ TF. FOR SPINOR FIELD

$$x^\mu \xrightarrow{a} x'^\mu = a^\mu{}_\nu x^\nu$$

$$\underline{\underline{\psi'(x') = S(a)\,\psi(x)}}$$ LINEAR TF.

IN MATRIX NOTATION $x' = a x$

$x = a^{-1} x'$

$$\underline{\underline{\psi(x) \xrightarrow{a} \psi'(x) = S(a)\,\psi(a^{-1}x)}}$$

INVERSE TF $\psi(x) = S^{-1}(a)\,\psi'(ax)$

$= S(a^{-1})\,\psi'(ax)$

$$\underline{\underline{S^{-1}(a) = S(a^{-1})}}$$

DIRAC EQ. IS SAME FOR BOTH INERTIAL OBSERVERS (COVARIANCE)

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi(x) = 0$$

↓

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right) S^{-1}(a)\,\psi'(x') = 0$$

↓ MULTIPLY ON LEFT BY $S(a)$

$$\left(i\,S(a)\gamma^\mu S^{-1}(a) \frac{\partial}{\partial x^\mu} - m\right)\psi'(x') = 0$$

↓

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} = a^\nu{}_\mu \frac{\partial}{\partial x'^\nu}$$

$$\left(i\, S(a)\, \gamma^\mu S^{-1}(a)\, a^\nu{}_\mu \frac{\partial}{\partial x'^\nu} - m \right) \psi'(x') = 0$$

LORENTZ COVARIANCE REQUIRES

$$\left(i\, \gamma^\nu \frac{\partial}{\partial x'^\nu} - m \right) \psi'(x') = 0$$

$$\Updownarrow$$

$$S(a)\, \gamma^\mu S^{-1}(a)\, a^\nu{}_\mu = \gamma^\nu$$

$$\Updownarrow$$

$$\boxed{S^{-1}(a)\, \gamma^\nu\, S(a) = a^\nu{}_\mu\, \gamma^\mu.}$$

↳ EXPLICIT CONSTRUCTION OF $S(a)$ FOR PROPER LORENTZ TF.

- CONSIDER INFINITESIMAL PROPER LORENTZ TF.

$$a^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu$$

$$(a^{-1})^\mu{}_\nu = g^\mu{}_\nu - \omega^\mu{}_\nu$$

FOR LORENTZ TF $\quad a^\mu{}_\nu\, g_\mu{}^\lambda = g_\nu^\lambda$

$$\left(g^\mu{}_\nu + \omega^\mu{}_\nu \right) \left(g_\mu{}^\lambda + \omega_\mu{}^\lambda \right) = g_\nu^\lambda$$

$$\omega^\lambda{}_\nu + \omega_\nu{}^\lambda = 0 \implies \underline{\underline{\omega_{\mu\nu} = -\,\omega_{\nu\mu}}}$$

ANTI - SYMMETRIC

- FOR LINEAR TF : $S(a)$ CAN BE CONSTRUCTED AS LINEAR IN $\omega^{\mu\nu}$

$$S(a) = \mathbb{1} - \frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu}$$

WITH $\sigma_{\mu\nu}$: 4×4 ANTI-SYMM MATRICES

$$\sigma_{\mu\nu} = - \sigma_{\nu\mu}$$

6 INDEPENDENT : ↗ 3 ROTATIONS, ↘ 3 BOOSTS.

- $\sigma_{\mu\nu}$ DETERMINED FROM

$$S^{-1}(a)\, \gamma^{\mu}\, S(a) = a^{\mu}{}_{\nu}\, \gamma^{\nu}$$

$$\left[\mathbb{1} + \frac{i}{4} \sigma_{\alpha\beta}\, \omega^{\alpha\beta}\right] \gamma^{\mu} \left[\mathbb{1} - \frac{i}{4} \sigma_{\alpha\beta}\, \omega^{\alpha\beta}\right]$$

$$= \gamma^{\mu} + \omega^{\mu}{}_{\nu}\, \gamma^{\nu}$$

$\Downarrow$ KEEPING TERMS LINEAR IN ω

$$\frac{i}{4} \left[\sigma_{\alpha\beta}, \gamma^{\mu}\right] \omega^{\alpha\beta} = \omega^{\mu\beta}\, \gamma_{\beta}$$

$$= \frac{1}{2} \omega^{\alpha\beta} \left(g^{\mu}_{\alpha}\, \gamma_{\beta} - g^{\mu}_{\beta}\, \gamma_{\alpha}\right)$$

$\uparrow$

BECAUSE OF ANTI-SYMM OF $\omega^{\alpha\beta}$

IN ORDER FOR THIS TO HOLD $\forall\ \omega^{\alpha\beta}$:

$$\left[\sigma_{\alpha\beta}\, ,\, \gamma^{\mu}\right] = -2i\left(g^{\mu}_{\alpha}\,\gamma_{\beta} - g^{\mu}_{\beta}\,\gamma_{\alpha}\right)$$

- THIS EQUATION IS SOLVED BY

$$\sigma_{\alpha\beta} = \frac{i}{2}\left[\gamma_{\alpha}, \gamma_{\beta}\right] = \frac{i}{2}\left(\gamma_{\alpha}\gamma_{\beta} - \gamma_{\beta}\gamma_{\alpha}\right)$$

PROOF :

$$\left[\sigma_{\alpha\beta}\, ,\, \gamma^{\mu}\right]$$

$$= \sigma_{\alpha\beta}\,\gamma^{\mu} - \gamma^{\mu}\,\sigma_{\alpha\beta}$$

$$= \frac{i}{2}\left\{\left(\gamma_{\alpha}\gamma_{\beta} - \gamma_{\beta}\gamma_{\alpha}\right)\gamma^{\mu} - \gamma^{\mu}\left(\gamma_{\alpha}\gamma_{\beta} - \gamma_{\beta}\gamma_{\alpha}\right)\right\}$$

$$= \frac{i}{2}\left\{\left(\gamma_{\alpha}\gamma_{\beta} - \gamma_{\beta}\gamma_{\alpha}\right)\gamma^{\mu} - \left(2g^{\mu}_{\alpha} - \gamma_{\alpha}\gamma^{\mu}\right)\gamma_{\beta} + \left(2g^{\mu}_{\beta} - \gamma_{\beta}\gamma^{\mu}\right)\gamma_{\alpha}\right\}$$

$$= \frac{i}{2}\left\{\left(\cancel{\gamma_{\alpha}\gamma_{\beta}} - \cancel{\gamma_{\beta}\gamma_{\alpha}}\right)\gamma^{\mu} - 2g^{\mu}_{\alpha}\gamma_{\beta} + \gamma_{\alpha}\left(2g^{\mu}_{\beta} - \cancel{\gamma_{\beta}\gamma^{\mu}}\right) + 2g^{\mu}_{\beta}\gamma_{\alpha} - \gamma_{\beta}\left(2g^{\mu}_{\alpha} - \cancel{\gamma_{\alpha}\gamma^{\mu}}\right)\right\}$$

$$\stackrel{!}{=} 2i\left(g^{\mu}_{\beta}\,\gamma_{\alpha} - g^{\mu}_{\alpha}\,\gamma_{\beta}\right)$$

$\therefore$ UNDER PROPER LORENTZ TF

$$x \xrightarrow{a} x' = a\, x$$

$$\psi(x) \xrightarrow{a} \psi'(x) = S(a)\, \psi(a^{-1} x)$$

↳ INFINITESIMAL $\quad a^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$

$$\boxed{\begin{array}{l} S(a) = \mathbb{1} - \frac{i}{4}\, \sigma_{\mu\nu}\, \omega^{\mu\nu} \\[1em] \text{WITH } \quad \sigma_{\mu\nu} = \frac{i}{2} \left[\gamma_\mu , \gamma_\nu \right] \end{array}}$$

↳ FINITE PROPER LORENTZ TF

$$\boxed{S(a) = \exp\left\{ -\frac{i}{4}\, \sigma_{\mu\nu}\, \omega^{\mu\nu} \right\}}$$

WITH $\omega^{\mu\nu}$ FINITE

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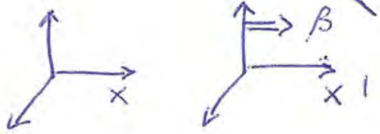
↳ LORENTZ BOOST ALONG X-AXIS



- $$a^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\left.\begin{aligned} \gamma &= \cosh \omega \\ \beta\gamma &= \sinh \omega \end{aligned}\right\} \quad \beta = \tanh \omega$$

- INFINITESIMAL TF

$$\gamma \simeq 1 \quad , \quad \Delta\beta \ll 1$$

$$\omega^{\mu}{}_{\nu} \simeq (\Delta\beta) \begin{pmatrix} 0 & -1 & & \\ -1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \equiv (\Delta\beta)\left(I_1\right)^{\mu}{}_{\nu}$$

$$I_1^{\,2} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

$$I_1^{\,2n+1} = I_1 \qquad n = 1, 2, 3 \ldots$$

$$I_1^{\,2n} = I_1^{\,2} \qquad n = 2, 3, 4, \ldots$$

$$\left(I_1\right)^{01} = 1$$

$$\left(I_1\right)^{10} = -1$$

- FINITE LORENTZ BOOST CAN BE OBTAINED FROM SUCCESSION OF INFINITESIMAL BOOSTS $\Delta\beta = \frac{\omega}{N}$

$$a^{\mu}{}_{\nu} = \left(\lim_{N \to \infty} \left(\mathbb{1} + \frac{\omega}{N} I_1 \right)^N \right)^{\mu}{}_{\nu}$$

$$= \left(\exp \left\{ \omega I_1 \right\} \right)^{\mu}{}_{\nu}$$

$$\exp\left\{\omega I_1\right\} = \left[\mathbb{1} + \frac{\omega^2}{2!} I_1^2 + \frac{\omega^4}{4!} I_1^4 + \cdots \right]$$

$$+ \left[\omega I_1 + \frac{\omega^3}{3!} I_1^3 + \cdots \right]$$

$$= \mathbb{1} - I_1^2 + (\cosh\omega)\, I_1^2 + (\sinh\omega)\, I_1$$

$$= \begin{pmatrix} \cosh\omega & -\sinh\omega & & \\ -\sinh\omega & \cosh\omega & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

- INFINITESIMAL BOOST (ALONG X-AXIS)

$$S(a) = \mathbb{1} - \frac{i}{2}\sigma_{01}(\Delta\beta)(I_1)^{01}$$

$$\sigma_{01} = -i\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} = -i\alpha^1$$

$$\underline{\underline{S(a) = \mathbb{1} - \frac{1}{2}(\Delta\beta)\,\alpha^1}}$$

- FINITE BOOST (ALONG X-AXIS)

$$\Delta\beta = \frac{\omega}{N}$$

$$S(a) = \lim_{N\to\infty}\left(\mathbb{1} - \frac{i}{2}\sigma_{01}\frac{\omega}{N}(I_1)^{01}\right)^N$$

$$= \exp\left\{-\frac{i}{2}\sigma_{01}\underbrace{\omega(I_1)^{01}}_{\substack{\parallel \\ (\omega)^{01}}}\right\}$$

$$\underline{\underline{S(a) = \exp\left\{-\frac{1}{2}\omega\,\alpha^1\right\}}}$$

↳ ROTATION AROUND Z-AXIS OVER ANGLE φ

- $$a^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- INFINITESIMAL TF

$\Delta\varphi \ll$ $\qquad \cos\Delta\varphi \simeq 1 \qquad \sin\Delta\varphi \simeq \Delta\varphi$

$$\omega^{\mu}{}_{\nu} = (\Delta\varphi)\, i \begin{pmatrix} 0 & & & \\ & 0 & +i & \\ & -i & 0 & \\ & & & 0 \end{pmatrix} \equiv i(\Delta\varphi)\left(R_3\right)^{\mu}{}_{\nu}$$

$$R_3^{2n} = R_3^{2} = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}$$

$$R_3^{2n+1} = R_3$$

$$\left(R_3\right)^{12} = -i$$

$$\left(R_3\right)^{21} = i$$

- FINITE ROTATION OBTAINED AS SUCCESSION OF INFINITESIMAL ROTATIONS $\Delta\varphi = \frac{\varphi}{N}$

$$a = \lim_{N \to \infty} \left(\mathbb{1} + \frac{i\varphi}{N} R_3 \right)^N$$

$$= \exp\left\{ i \varphi R_3 \right\}.$$

$$= \mathbb{1} - R_3^2 + \sum_{n=0}^{\infty} (-1)^n \frac{\varphi^{2n}}{(2n)!} R_3^2$$

$$+ i \sum_{n=0}^{\infty} (-1)^n \frac{\varphi^{2n+1}}{(2n+1)!} R_3$$

$$= \mathbb{1} + (\cos\varphi - 1) R_3^2 + i \sin\varphi \, R_3$$

$$= \begin{pmatrix} 1 & & & \\ & \cos\varphi & -\sin\varphi & \\ & \sin\varphi & \cos\varphi & \\ & & & 1 \end{pmatrix}$$

- INFINITESIMAL ROTATION (AROUND Z-AXIS)

$$S(a) = \mathbb{1} - \frac{i}{2}\,\sigma_{12}\,(i\,\Delta\varphi)\,(R_3)^{12}$$

$$\sigma_{12} = \frac{i}{2}\left(\gamma^1\gamma^2 - \gamma^2\gamma^1\right) = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}$$

$$\equiv \Sigma^3$$

$$\underline{\underline{S(a) = \mathbb{1} - \frac{i}{2}(\Delta\varphi)\,\Sigma^3}}$$

- FINITE ROTATION (AROUND Z-AXIS) $\Delta\varphi = \frac{\varphi}{N}$

$$S(a) = \lim_{N\to\infty}\left(\mathbb{1} - \frac{i}{2}\,\sigma_{12}\; i\,\frac{\varphi}{N}\,(R_3)^{12}\right)^N$$

$$= \exp\left\{-\frac{i}{2}\,\sigma_{12}\,\underbrace{\varphi\,(i\,R_3)^{12}}_{\equiv\,(\omega)^{12}}\right\}$$

$$\underline{\underline{S(a) = \exp\left\{-\frac{i}{2}\,\varphi\,\Sigma^3\right\}}}$$

- CONSIDER SPINOR WITH SPIN PROJECTION $+\frac{1}{2}$ ALONG Z-AXIS

  i.e. $\Sigma^3 \psi = + \psi$

  UNDER ROTATION, THIS SPINOR TRANSFORMS AS

  $$\psi'(x') = \exp\left(-\frac{i}{2}\varphi\right)\psi(x)$$

  IT TAKES A ROTATION OVER $\underline{\underline{4\pi}}$ (!)

  BEFORE SPINOR TURNS INTO ITSELF

  $\Downarrow$

  PHYSICAL QUANTITIES MUST BE BILINEARS IN ψ

  ↳ TURN INTO THEMSELVES AFTER ROTATION OVER $\underline{\underline{2\pi}}$

- NOTE → FOR ROTATIONS:

  $$\sigma_{ij}^{+} = \sigma_{ij} \quad \Rightarrow \quad \underset{\substack{\uparrow \\ \text{ROTATION}}}{S_R^{+}} = S_R^{-1}$$

  → FOR BOOSTS

  $$S_B = \exp\left\{-\frac{1}{2}\omega\alpha^1\right\}$$

  $$S_B^{+} = S_B$$

FOR PROPER L.T.: $\underline{\underline{S^{-1} = \gamma_0 S^{+} \gamma_0}}$

↳ TRANSFORMATION PROPERTIES UNDER LORENTZ TF OF CURRENT

$$J^{\mu}(x) = \bar{\psi}(x)\,\gamma^{\mu}\,\psi(x)$$

$$\downarrow$$

$$J'^{\mu}(x') = \bar{\psi}'(x')\,\gamma^{\mu}\,\psi'(x')$$

$$= \psi^{+}(x)\,\underbrace{S^{+}\gamma^{0}}\,\gamma^{\mu}\,S\,\psi(x)$$

$$= \psi^{+}(x)\,\gamma^{0}\,\underbrace{S^{-1}\gamma^{\mu}S}_{a^{\mu}{}_{\nu}\gamma^{\nu}}\,\psi(x).$$

$$= a^{\mu}{}_{\nu}\;\bar{\psi}(x)\,\gamma^{\nu}\,\psi(x)$$

$$\underline{\underline{J'^{\mu}(x') = a^{\mu}{}_{\nu}\;J^{\nu}(x)}}$$

J^{μ} TRANSFORMS AS A 4- VECTOR

↳ TRANSFORMATION OF FIELD OPERATORS IN HILBERT SPACE UNDER PROPER L.T. a

- STATE $\quad |\bar{p}, s\rangle = \sqrt{2E_{\bar{p}}}\, a^{+}(\bar{p}, s)\,|0\rangle$

  CONSIDER SPIN QUANTIZATION AXIS EITHER ALONG BOOST AXIS / ROTATION AXIS

$$|\bar{p}, s\rangle \xrightarrow{a} |a\bar{p}, s\rangle = \sqrt{2E_{a\bar{p}}}\, a^{+}(a\bar{p}, s)\,|0\rangle$$

NORM HAS TO BE INVARIANT $\Rightarrow$ UNITARY T.F. U $\quad (U^{+} = U^{-1})$

$$|a\bar{p}, s\rangle = U(a)\,|\bar{p}, s\rangle$$

$$\Downarrow$$

$$\sqrt{2E_{a\bar{p}}}\, a^{+}(a\bar{p}, s) = U(a)\, a^{+}(\bar{p}, s)\, U^{+}(a)\, \sqrt{2E_{\bar{p}}}$$

$\therefore$

$$a(\bar{p}, s) \xrightarrow{a} U(a)\, a(\bar{p}, s)\, U^{+}(a)$$

$$= \sqrt{\frac{E_{a\bar{p}}}{E_{\bar{p}}}}\; a(a\bar{p}, s)$$

- FIELD OPERATOR

$$\psi(x) \xrightarrow{a} U(a)\,\psi(x)\,U^{+}(a)$$

$$= \sum_{s} \int \frac{d^{3}\bar{p}}{(2\pi)^{3}\sqrt{2E_{\bar{p}}}} \Big\{ U(a)\, a(\bar{p}, s)\, U^{+}(a)\, u(\bar{p}, s)\, e^{-ip.x} + U(a)\, b^{+}(\bar{p}, s)\, U^{+}(a)\, v(\bar{p}, s)\, e^{+ip.x} \Big\}$$

$$= \sum_{s} \int \frac{d^{3}\bar{p}}{(2\pi)^{3}\, 2E_{\bar{p}}} \sqrt{2E_{a\bar{p}}} \Big\{ a(a\bar{p}, s)\, u(\bar{p}, s)\, e^{-ip.x} + b^{+}(a\bar{p}, s)\, v(\bar{p}, s)\, e^{+ip.x} \Big\}$$

CHANGE OF VARIABLE $\bar{p}' = a\bar{p}$

$$\frac{d^3\bar{p}'}{2E_{\bar{p}'}} = \frac{d^3\bar{p}}{2E_{\bar{p}}}$$

$$p.x = p'.x' \quad \text{WITH} \quad x' = ax$$

$$U(a)\,\psi(x)\,U^+(a)$$

$$= \sum_s \int \frac{d^3\bar{p}'}{(2\pi)^3\sqrt{2E_{\bar{p}'}}} \left\{ a(\bar{p}',s)\,u(a^{-1}\bar{p}',s)\,e^{-ip'.x'} + b^+(\bar{p}',s)\,v(a^{-1}\bar{p}',s)\,e^{+ip'.x'} \right\}$$

$$\downarrow \quad u(a^{-1}\bar{p}',s) = S^{-1}(a)\,u(\bar{p}',s)$$

(SEE PROOF P. 52-53)

$$= \sum_s \int \frac{d^3\bar{p}'}{(2\pi)^3\sqrt{2E_{\bar{p}'}}} \left\{ a(\bar{p}',s)\,S^{-1}(a)\,u(\bar{p}',s)\,e^{-ip'.x'} + b^+(\bar{p}',s)\,S^{-1}(a)\,v(\bar{p}',s)\,e^{+ip'.x'} \right\}$$

$$= S^{-1}(a)\,\psi(x')$$

$$\therefore \quad \underline{\underline{U(a)\,\psi(x)\,U^+(a) = S^{-1}(a)\,\psi(ax)}}$$

NOTE: THIS IS RELATION FOR AN ACTIVE LORENTZ TF a.

CHECK $\quad \underline{\underline{u(a\bar{p}, s) = S(a)\, u(\bar{p}, s)}}$

FOR LORENTZ BOOST ALONG X - AXIS
WITH SPIN QUANTIZATION AXIS CHOSEN ALONG X - AXIS.
(WE CONSIDER HERE AN <u>ACTIVE</u> TF.)

↳ INVERSE OF PASSIVE TF.
(COORDINATE TF.)

$$a = \begin{pmatrix} \cosh\omega & \sinh\omega & & \\ \sinh\omega & \cosh\omega & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

FOR PARTICLE AT REST, WHICH IS BOOSTED ALONG X- AXIS

$$\begin{pmatrix} E_{\bar{p}} \\ p_x \end{pmatrix} = \begin{pmatrix} \cosh\omega & \sinh\omega \\ \sinh\omega & \cosh\omega \end{pmatrix} \begin{pmatrix} m \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} \cosh\omega &= \frac{E_{\bar{p}}}{m} \\ \sinh\omega &= \frac{p_x}{m} = \frac{|\bar{p}|}{m} \end{aligned}\right\} \tanh\omega = \frac{|\bar{p}|}{E_{\bar{p}}}$$

↓

$$\cosh\frac{\omega}{2} = \sqrt{\frac{1+\cosh\omega}{2}} = \sqrt{\frac{E_{\bar{p}}+m}{2m}}$$

$$\sinh\frac{\omega}{2} = \frac{\cosh\omega}{2\cosh\frac{\omega}{2}} \tanh\omega = \frac{|\bar{p}|}{2m} \cdot \sqrt{\frac{2m}{E_{\bar{p}}+m}}$$

$$= \frac{|\bar{p}|}{E_{\bar{p}}+m} \sqrt{\frac{E_{\bar{p}}+m}{2m}}$$

PROOF $u(\bar{p}, s) = S(a)\, u(\bar{0}; s)$

$\hookrightarrow S(a) = \exp\left\{ \frac{1}{2}\omega \alpha^1 \right\}$ (ACTIVE TF)

$$= \sum_{n=0}^{\infty} \left(\frac{\omega}{2}\right)^{2n} (\alpha^1)^{2n} + \sum_{n=0}^{\infty} \left(\frac{\omega}{2}\right)^{2n+1} (\alpha^1)^{2n+1}$$

$$(\alpha_1)^{2n} = \alpha_1^2 = \mathbb{1}, \quad n = 0, 1, \ldots$$

$$(\alpha_1)^{2n+1} = \alpha_1, \quad n = 0, 1, \ldots$$

$$= \cosh\frac{\omega}{2}\, \mathbb{1} + \sinh\frac{\omega}{2}\, \alpha^1$$

$$= \left(\begin{array}{c|c} \cosh\frac{\omega}{2}\, \mathbb{1}_{2\times 2} & \sinh\frac{\omega}{2}\, \sigma^1 \\ \hline \sinh\frac{\omega}{2}\, \sigma^1 & \cosh\frac{\omega}{2}\, \mathbb{1}_{2\times 2} \end{array} \right)$$

$\hookrightarrow S(a)\, u(\bar{0}, s)$

$$= \left(\begin{array}{c|c} \cosh\frac{\omega}{2}\, \mathbb{1} & \sinh\frac{\omega}{2}\, \sigma^1 \\ \hline \sinh\frac{\omega}{2}\, \sigma^1 & \cosh\frac{\omega}{2}\, \mathbb{1} \end{array} \right) \begin{pmatrix} \chi_s \\ 0 \end{pmatrix} \sqrt{2m}$$

$$= \sqrt{2m}\, \cosh\frac{\omega}{2} \begin{pmatrix} \chi_s \\ \tanh\frac{\omega}{2}\, \sigma^1 \chi_s \end{pmatrix}$$

$$= \sqrt{E_{\bar{p}} + m} \begin{pmatrix} \chi_s \\ \frac{\sigma^1 |\bar{p}|}{E_{\bar{p}} + m} \chi_s \end{pmatrix} \stackrel{!}{=} u(\bar{p}, s)$$

$\hookrightarrow \sigma^1 |\bar{p}| = \bar{\sigma} \cdot \bar{p}$ (BOOST ALONG x)

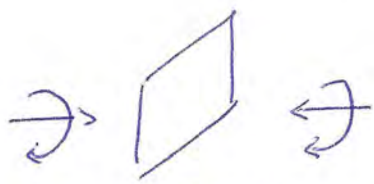
⟹ PARITY (SPATIAL INVERSION)

↳ SPATIAL INVERSION (P) : IMPROPER LORENTZ TF WHICH INVERTS SPATIAL COORDINATES

$$(t, \bar{x}) \xrightarrow{P} (t, -\bar{x})$$

$$\det a = -1$$

- 3-MOMENTUM CHANGES SIGN



MIRROR
P

$$\bar{p} \xrightarrow{P} -\bar{p}$$

- ANGULAR MOMENTUM / SPIN REMAINS UNCHANGED

$$\bar{L} = \bar{x} \times \bar{p} \xrightarrow{P} (-\bar{x}) \times (-\bar{p}) = \bar{x} \times \bar{p} = \bar{L}$$

↳ APPLICATION OF P ON SPINOR

$$\psi'(x') = P\,\psi(x)$$

IF WE APPLY TWICE : STATE GOES BACK TO ITSELF

$$P^2 = 1 \quad \rightarrow \quad P^{-1} = P$$

$$P\,\gamma^\nu\,P = (a_P)^\nu{}_\mu\,\gamma^\mu \qquad \text{WITH } a_P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$P \gamma^0 P = \gamma^0$$

$$P \gamma^i P = -\gamma^i \qquad i = 1, 2, 3$$

$$\Downarrow$$

$$\underline{\underline{P = \eta \gamma^0}} \qquad \eta = \pm 1 \quad \text{PHASE FACTOR}$$

↳ TRANSFORMATION OF FIELD OPERATOR

WE WILL PROVE: $\underline{\underline{\psi(t, \bar{x}) \xrightarrow{P} \eta \gamma^0 \psi(t, -\bar{x})}} \equiv U(a_P)\psi(x)U(a_P)$

$$U^{-1}(a_P) = U^{+}(a_P) = U(a_P)$$

STARTING FROM:

$$a(\bar{p}, s) \xrightarrow{P} \eta_a a(-\bar{p}, s)$$

$$b(\bar{p}, s) \xrightarrow{P} \eta_b b(-\bar{p}, s)$$

$$\psi(x) = \sum_s \int \frac{d^3\bar{p}}{(2\pi)^3 \sqrt{2E_{\bar{p}}}} \left\{ a(\bar{p}, s) u(\bar{p}, s) e^{-ip.x} + b^{+}(\bar{p}, s) v(\bar{p}, s) e^{+ip.x} \right\}$$

$$\psi(t, \bar{x}) \xrightarrow{P} \sum_s \int \frac{d^3\bar{p}}{(2\pi)^3 \sqrt{2E_{\bar{p}}}} \left\{ \eta_a a(-\bar{p}, s) u(\bar{p}, s) e^{-ip.x} + \eta_b^{*} b^{+}(-\bar{p}, s) v(\bar{p}, s) e^{+ip.x} \right\}$$

$$\downarrow \quad \bar{p} \to \bar{p}' = -\bar{p}$$

$$= \sum_s \int \frac{d^3\bar{p}'}{(2\pi)^3 \sqrt{2E_{\bar{p}'}}} \left\{ \eta_a a(\bar{p}', s) u(-\bar{p}', s) e^{-iE_{\bar{p}'}t - i\bar{p}'.\bar{x}} + \eta_b^{*} b^{+}(\bar{p}', s) v(-\bar{p}', s) e^{iE_{\bar{p}'}t + i\bar{p}'.\bar{x}} \right\}$$

$$U(-\bar{p}',s) = N \begin{pmatrix} \chi_s \\ -\frac{\bar{\sigma}.\bar{p}'}{E_{\bar{p}'}+m}\chi_s \end{pmatrix} = \gamma^0 \, U(\bar{p}',s)$$

$$v(-\bar{p}',s) = N \begin{pmatrix} -\frac{\bar{\sigma}.\bar{p}'}{E_{\bar{p}'}+m}\chi_s' \\ \chi_s' \end{pmatrix} = -\gamma^0 \, v(\bar{p}',s)$$

$$\psi(t,\bar{x}) \xrightarrow{P} \sum_s \int \frac{d^3\bar{p}'}{(2\pi)^3\sqrt{2E_{\bar{p}'}}} \Big\{ \eta_a \, a(\bar{p}',s)\, \gamma^0 U(\bar{p}',s)\, e^{-ip'.x'} - \eta_b^* \, b^+(\bar{p}',s)\, \gamma^0 v(\bar{p}',s)\, e^{+ip'.x'} \Big\}$$

WITH $x' = (t, -\bar{x})$

CHOOSE $\underline{\underline{\eta_b^* = -\eta_a}}$

(PARTICLE & ANTI - PARTICLE HAVE OPPOSITE INTRINSIC PARITIES)

$$\therefore \quad \boxed{\psi(t,\bar{x}) \xrightarrow{P} \eta_a \, \gamma^0 \, \psi(t,-\bar{x})}$$

$$\bar{\psi}(t,\bar{x}) \xrightarrow{P} \bar{\psi}(t,-\bar{x})\, \gamma^0 \, \eta_a^*$$

WITH $|\eta_a|^2 = 1 \qquad \eta_a = \pm 1$

WE CAN CONVENIENTLY CHOOSE $\eta_a = +1$

↳ TRANSFORMATION OF BILINEARS

- SCALAR

$$\bar{\psi}(x)\,\psi(x) \xrightarrow{P} |\eta_a|^2\, \bar{\psi}(x')\,\gamma^0\gamma^0\psi(x')$$

$$= \bar{\psi}(x')\,\psi(x')$$

WITH $x' = (t, -\bar{x})$

- PSEUDO - SCALAR

$$\bar{\psi}(x)\,\gamma_5\,\psi(x) \xrightarrow{P} |\eta_a|^2\, \bar{\psi}(x')\,\gamma^0\gamma_5\gamma^0\,\psi(x')$$

$$= -\,\bar{\psi}(x')\,\gamma_5\,\psi(x')$$

⇑

CHANGES SIGN UNDER P

- VECTOR

$$\bar{\psi}(x)\,\gamma^\mu\,\psi(x) \xrightarrow{P} |\eta_a|^2\, \bar{\psi}(x')\,\gamma^0\gamma^\mu\gamma^0\,\psi(x')$$

$$= \begin{cases} +\bar{\psi}(x')\,\gamma^0\,\psi(x') \,, & \mu = 0 \\ -\bar{\psi}(x')\,\gamma^i\,\psi(x') \,, & \mu = i = 1,2,3 \end{cases}$$

TRANSFORMS AS 4-VECTOR UNDER P

↳ TIME COMPONENT DOES NOT CHANGE SIGN

↳ SPACE COMPONENTS CHANGE SIGN UNDER P

- AXIAL - VECTOR

$$\bar{\psi}(x)\,\gamma^\mu\gamma_5\,\psi(x) \xrightarrow{P} |\eta_a|^2\,\bar{\psi}(x')\,\gamma^0\gamma^\mu\gamma_5\,\gamma^0\,\psi(x')$$

$$= \begin{cases} -\,\bar{\psi}(x')\,\gamma^0\gamma_5\,\psi(x')\,, & \mu=0 \\ +\,\bar{\psi}(x')\,\gamma^i\gamma_5\,\psi(x')\,, & \mu=i=1,2,3 \end{cases}$$

↳ TIME COMPONENT CHANGES SIGN

↳ SPACE COMPONENTS DO NOT CHANGE SIGN UNDER P (e.g. ANGULAR MOMENTUM $\bar{L}$)

- TENSOR

$$\bar{\psi}(x)\,\sigma^{\mu\nu}\,\psi(x) \xrightarrow{P} \bar{\psi}(x')\,\gamma^0\,\sigma^{\mu\nu}\,\gamma^0\,\psi(x')$$

$$= \begin{cases} 0 & \mu=0,\ \nu=0 \\ -\,\bar{\psi}(x')\,\sigma^{\mu\nu}\,\psi(x') & \mu=0,\ \nu=i \\ -\,\bar{\psi}(x')\,\sigma^{\mu\nu}\,\psi(x') & \mu=i,\ \nu=0 \\ \bar{\psi}(x')\,\sigma^{\mu\nu}\,\psi(x') & \mu=i,\ \nu=j \\ & i\neq j \end{cases}$$

$$= (-1)^\mu(-1)^\nu\,\bar{\psi}(x')\,\sigma^{\mu\nu}\,\psi(x')$$

$\Rightarrow$ TIME REVERSAL (T)

$\hookrightarrow$ $(t, \bar{x}) \xrightarrow{T} (-t, \bar{x})$ $\qquad a_T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$$\bar{p} \xrightarrow{T} -\bar{p}$$

$$\bar{L} = \bar{x} \times \bar{p} \xrightarrow{T} -\bar{L}$$

$$s \xrightarrow{T} -s$$

$\hookrightarrow$ TRANSFORMATION OF FIELDS

- RECALL: FOR PROPER L.T. & FOR SPACE INVERSION FIELDS TRANSFORM THROUGH UNITARY TF

$$\psi(x) \xrightarrow{a} U(a)\,\psi(x)\,U^{-1}(a) = S^{-1}(a)\,\psi(ax)$$

NOTE $U(a)$: UNITARY OPERATOR WHICH ACTS IN HILBERT SPACE

$S(a)$ 4×4 MATRIX WHICH ACTS IN DIRAC SPACE

SYMMETRY UNDER PROPER L.T. OR SPACE INVERSION REQUIRES MATRIX ELEMENTS TO BE INVARIANT

$$|\psi_1\rangle \xrightarrow{a} U(a)|\psi_1\rangle$$

$$|\psi_2\rangle \xrightarrow{a} U(a)\,|\psi_2\rangle$$

$$\langle\psi_1|\psi_2\rangle \xrightarrow{a} \langle U\psi_1|U\psi_2\rangle \underset{\text{SYMM.}}{=} \langle\psi_1|\psi_2\rangle$$

$$\Downarrow$$

$$U^\dagger U = \mathbb{1} \quad \text{(UNITARY)}$$

- FOR TIME - REVERSAL : INITIAL & FINAL STATES ARE REVERSED

$$|\psi_1\rangle \xrightarrow{T} U(a_T)\,|\psi_1\rangle$$

$$|\psi_2\rangle \xrightarrow{T} U(a_T)\,|\psi_2\rangle$$

$$\langle\psi_1|\psi_2\rangle \xrightarrow{T} \langle U\psi_1|U\psi_2\rangle$$

$$\underset{\text{T - REVERSAL INVARIANCE}}{=} \langle\psi_2|\psi_1\rangle = \langle\psi_1|\psi_2\rangle^*$$

REQUIRES AN ANTI - UNITARY OPERATOR

$U^+ = U^{-1}$ AND $\underline{\underline{U^+(a_T)\,f(x)\,U(a_T) = f^*(x)}}$

COMPLEX FUNCTION

$U^{-1}(a_T) = U(a_T)$

$(U^2(a_T) = 1)$

- $|\bar{p}, s\rangle \xrightarrow{T} |-\bar{p}, -s\rangle$

$$a(\bar{p}, s) \xrightarrow{T} U(a_T)\,a(\bar{p}, s)\,U(a_T) = a(-\bar{p}, -s)$$

$$b(\bar{p}, s) \xrightarrow{T} U(a_T)\,b(\bar{p}, s)\,U(a_T) = b(-\bar{p}, -s)$$

- $\psi(x) \xrightarrow{T} U(a_T)\,\psi(x)\,U(a_T)$

$$= \sum_s \int \frac{d^3\bar{p}}{(2\pi)^3\sqrt{2E_{\bar{p}}}} \left\{ a(-\bar{p},-s)\, u^*(\bar{p},s)\, e^{+ip.x} + b^+(-\bar{p},-s)\, v^*(\bar{p},s)\, e^{-ip.x} \right\}$$

$$\downarrow \quad p' = (E_{\bar{p}}, -\bar{p})$$

$$p.x = E_{\bar{p}}t - \bar{p}.\bar{x} = -p'.x'$$

WITH $x' = (-t, \bar{x})$

$$= \sum_s \int \frac{d^3\bar{p}'}{(2\pi)^3\sqrt{2E_{\bar{p}'}}} \left\{ a(\bar{p}',s')\, u^*(-\bar{p}',-s')\, e^{-ip'.x'} + b^+(\bar{p}',s')\, v^*(-\bar{p}',-s')\, e^{+ip'.x'} \right\}$$

$$\downarrow \quad u^*(-\bar{p}',-s') = \gamma^1\gamma^3\, u(\bar{p}',s')$$

$$= \gamma^1\gamma^3 \sum_s \int \frac{d^3\bar{p}'}{(2\pi)^3\sqrt{2E_{\bar{p}'}}} \left\{ a(\bar{p}',s')\, u(\bar{p}',s')\, e^{-ip'.x'} + b^+(\bar{p}',s')\, v(\bar{p}',s')\, e^{+ip'.x'} \right\}$$

$$\therefore \quad \underline{\underline{U(a_T)\,\psi(t,\bar{x})\,U(a_T) = \gamma^1\gamma^3\,\psi(-t,\bar{x})}}$$

$$\hookrightarrow \quad (\bar{\sigma}.\bar{n})\, \chi_s = \chi_s$$

$$(\bar{\sigma}.\bar{n})\,(-i\sigma^2 \chi_s^*) = -i\sigma^2(-\bar{\sigma}.\bar{n})^* \; \chi_s^*$$

$$= (-1)\;(-i\sigma^2\chi_s^*)$$

$$\therefore \quad \underline{\underline{\chi_{-s} = -i\sigma^2 \chi_s^*}}$$

$$\hookrightarrow \quad -\gamma^1\gamma^3\,(u(\bar{p},s))^*$$

$$= \begin{pmatrix} -i\sigma^2 & \\ & -i\sigma^2 \end{pmatrix} \begin{pmatrix} \chi_s^* \\ \frac{\bar{\sigma}^*.\bar{p}}{E_{\bar{p}}+m}\, \chi_s^* \end{pmatrix} \sqrt{E_{\bar{p}}+m}$$

$$= \sqrt{E_{\bar{p}}+m} \begin{pmatrix} \chi_{-s} \\ -\frac{\bar{\sigma}.\bar{p}}{E_p+m}\,\underbrace{(-i\sigma^2\chi_s^*)}_{\chi_{-s}} \end{pmatrix}$$

$$= u(-\bar{p},-s)$$

$$\therefore \quad \left\| \begin{array}{l} -\gamma^1\gamma^3\,(u(\bar{p},s))^* = u(-\bar{p},-s) \\ -\gamma^1\gamma^3\,(v(\bar{p},s))^* = v(-\bar{p},-s) \end{array} \right.$$

$$\leadsto (u(\bar{p},s))^* = \gamma^1\gamma^3\; u(-\bar{p},-\bar{s})$$

↳ TRANSFORMATION OF BILINEARS

$$U(a_T)\,\psi(x)\,U(a_T) = \gamma^1\gamma^3\,\psi(x') \quad \text{WITH } x' = (-t, \vec{x})$$

$$U(a_T)\,\bar{\psi}(x)\,U(a_T) = +\,\bar{\psi}(x')\,\gamma^3\gamma^1$$

- SCALAR

$$\bar{\psi}(x)\psi(x) \xrightarrow{T} \bar{\psi}(x')\,\gamma^3\gamma^1\gamma^1\gamma^3\,\psi(x')$$
$$= \bar{\psi}(x')\,\psi(x')$$

- PSEUDOSCALAR

$$\bar{\psi}(x)\,i\gamma_5\,\psi(x) \xrightarrow{T} \bar{\psi}(x')\,\gamma^3\gamma^1(-i\gamma_5)\,\gamma^1\gamma^3\psi(x')$$

$$= -\bar{\psi}(x')\,i\gamma_5\,\psi(x')$$

- VECTOR

$$\bar{\psi}(x)\,\gamma^\mu\,\psi(x) \xrightarrow{T} \bar{\psi}(x')\,\gamma^3\gamma^1\,\gamma^{\mu *}\gamma^1\gamma^3\,\psi(x')$$
$$= \begin{cases} \bar{\psi}(x')\,\gamma^\mu\,\psi(x') , & \mu = 0. \\ -\bar{\psi}(x')\,\gamma^\mu\,\psi(x') , & \mu = 1,2,3 \end{cases}$$

- AXIAL-VECTOR

$$\bar{\psi}(x)\,\gamma^\mu\gamma_5\,\psi(x) \xrightarrow{T} \bar{\psi}(x')\,\gamma^3\gamma^1\,\gamma^{\mu *}\gamma_5\,\gamma^1\gamma^3\,\psi(x')$$
$$= \begin{cases} \bar{\psi}(x')\,\gamma^\mu\gamma_5\,\psi(x'), & \mu = 0 \\ -\bar{\psi}(x')\,\gamma^\mu\gamma_5\,\psi(x'), & \mu = 1,2,3 \end{cases}$$

- TENSOR $\qquad \sigma^{\mu\nu} = \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)$

$$\bar{\psi}(x)\,\sigma^{\mu\nu}\,\psi(x) \xrightarrow{T} \bar{\psi}(x')\,\gamma^3\gamma^1\left(\sigma^{\mu\nu}\right)^*\gamma^1\gamma^3\,\psi(x')$$

$$= \begin{cases} 0 & \mu=0,\ \nu=0 \\ +\,\bar{\psi}(x')\,\sigma^{\mu\nu}\,\psi(x') & \mu=0,\ \nu=i \\ +\,\bar{\psi}(x')\,\sigma^{\mu\nu}\,\psi(x') & \mu=i,\ \nu=0 \\ -\,\bar{\psi}(x')\,\sigma^{\mu\nu}\,\psi(x') & \mu=i,\ \nu=j \\ & i\neq j \end{cases}$$

$$= -\,(-1)^\mu(-1)^\nu\,\bar{\psi}(x')\,\sigma^{\mu\nu}\,\psi(x')$$

⟹ CHARGE CONJUGATION (C)

↳ SYMMETRY UNDER PARTICLE ⟷ ANTI-PARTICLE REVERSAL

$$a(\bar{p},s) \xrightarrow{C} U(C)\, a(\bar{p},s)\, U(C) = b(\bar{p},s)$$

$$b(\bar{p},s) \xrightarrow{C} U(C)\, b(\bar{p},s)\, U(C) = a(\bar{p},s)$$

WITH $U(C) = U^{-1}(C)$ UNITARY OPERATOR

↳ RELATION BETWEEN u & v SPINORS

$$v(\bar{p},s) = +i\gamma^2\left(u(\bar{p},s)\right)^*$$

$$u(\bar{p},s) = +i\gamma^2\left(v(\bar{p},s)\right)^*$$

PROOF

$$+i\gamma^2\left(u(\bar{p},s)\right)^* = \sqrt{E_{\bar{p}}+m}\begin{pmatrix} +i\sigma^2 \dfrac{\bar{\sigma}^*\cdot\bar{p}}{E_{\bar{p}}+m}\chi_s^* \\ -i\sigma^2\chi_s^* \end{pmatrix}$$

$$= \sqrt{E_{\bar{p}}+m}\begin{pmatrix} \dfrac{\bar{\sigma}\cdot\bar{p}}{E_{\bar{p}}+m}\left(-i\sigma^2\chi_s^*\right) \\ \left(-i\sigma^2\chi_s^*\right) \end{pmatrix}$$

$$\downarrow \quad -i\sigma^2\chi_s^* = \chi_{-s}$$

$$= \sqrt{E_{\bar{p}}+m}\begin{pmatrix} \dfrac{\bar{\sigma}\cdot\bar{p}}{E_{\bar{p}}+m}\chi_{-s} \\ \chi_{-s} \end{pmatrix} = v(\bar{p},s)$$

↳ TRANSFORMATION OF FIELDS

$$\psi(x) \xrightarrow{C} U(C)\,\psi(x)\,U(C)$$

$$= \sum_s \int \frac{d^3\bar{p}}{(2\pi)^3\sqrt{2E_{\bar{p}}}} \Big\{ b(\bar{p},s)\, u(\bar{p},s)\, e^{-ip\cdot x} + a^{+}(\bar{p},s)\, v(\bar{p},s)\, e^{+ip\cdot x} \Big\}$$

$$= i\gamma^2 \sum_s \int \frac{d^3\bar{p}}{(2\pi)^3\sqrt{2E_p}} \Big\{ b(\bar{p},s)\, v^{*}(\bar{p},s)\, e^{-ip\cdot x} + a^{+}(\bar{p},s)\, u^{*}(\bar{p},s)\, e^{+ip\cdot x} \Big\}$$

$$\underline{\underline{U(C)\psi(x)U(C) = i\gamma^2 \psi^{*}(x) = \left(\bar{\psi}\; i\gamma^0\gamma^2\right)^T}}$$

$$\bar{\psi}(x) \xrightarrow{C} U(C)\,\bar{\psi}(x)\,U(C)$$

$$= \left(U(C)\,\psi(x)\,U(C)\right)^{+} \gamma^0$$

$$= \left(i\gamma^2\psi^{*}\right)^{+}\gamma^0$$

$$= \psi^{T}(-i)\,\gamma^0\gamma^2$$

$$= \left(i\gamma^0\gamma^2\psi\right)^{T}$$

↳ TRANSFORMATION OF BILINEARS

- SCALAR

$$\bar{\psi}(x)\,\psi(x) \xrightarrow{C} \left(i\gamma^0\gamma^2\psi\right)^T \left(\bar{\psi}\, i\gamma^0\gamma^2\right)^T$$

$$= -\left(\bar{\psi}\, i\gamma^0\gamma^2\right)\left(i\gamma^0\gamma^2\psi\right)$$

$\Uparrow$

MINUS SIGN DUE TO
FERMION ANTI-COMMUTATION

$$= +\,\bar{\psi}(x)\,\psi(x)$$

- PSEUDO-SCALAR

$$\bar{\psi}(x)\, i\gamma_5\psi(x) \xrightarrow{C} \left(i\gamma^0\gamma^2\psi\right)^T i\gamma_5 \left(\bar{\psi}\, i\gamma^0\gamma^2\right)^T$$

$$= -\,\bar{\psi}\, i\gamma^0\gamma^2 \left(i\gamma_5\right)^T i\gamma^0\gamma^2\psi$$

$$= \bar{\psi}\, i\gamma_5\,\psi$$

- VECTOR

$$\bar{\psi}(x)\,\gamma^\mu\,\psi(x) \xrightarrow{C} \left(i\gamma^0\gamma^2\psi\right)^T \gamma^\mu \left(\bar{\psi}\, i\gamma^0\gamma^2\right)^T$$

$$= -\,\bar{\psi}\, i\gamma^0\gamma^2\,\gamma^{\mu T}\, i\gamma^0\gamma^2\psi$$

$$\Big\downarrow \quad \gamma^{0T} = \gamma^0, \quad \gamma^{2T} = \gamma^2, \quad \gamma^{1T} = -\gamma^1, \quad \gamma^{3T} = -\gamma^3$$

$$= -\,\bar{\psi}(x)\,\gamma^\mu\,\psi(x)$$

- AXIAL - VECTOR

$$\bar{\psi}(x)\,\gamma^\mu\gamma_5\,\psi(x) \xrightarrow{C} \left(i\gamma^0\gamma^2\psi\right)^T \gamma^\mu\gamma_5 \left(\bar{\psi}\, i\gamma^0\gamma^2\right)^T$$

$$= -\,\bar{\psi}\, i\gamma^0\gamma^2\;\gamma_5\,\gamma^{\mu\,T}\, i\gamma^0\gamma^2\psi$$

$$= -\,\bar{\psi}\;\gamma^0\gamma^2\gamma^{\mu\,T}\gamma^0\gamma^2\gamma_5\,\psi$$

$$= +\;\bar{\psi}(x)\,\gamma^\mu\gamma_5\,\psi(x)$$

- TENSOR

$$\bar{\psi}\;\sigma^{\mu\nu}\;\psi \xrightarrow{C} \left(i\gamma^0\gamma^2\psi\right)^T \sigma^{\mu\nu} \left(\bar{\psi}\, i\gamma^0\gamma^2\right)^T$$

$$= -\;\bar{\psi}\; i\gamma^0\gamma^2\left(\sigma^{\mu\nu}\right)^T i\gamma^0\gamma^2\psi$$

$$= -\;\bar{\psi}(x)\;\sigma^{\mu\nu}\;\psi(x)$$

$\Longrightarrow$ CPT

	$\bar{\psi}\psi$	$\bar{\psi} i \gamma_5 \psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\gamma^\mu\gamma_5\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$
P	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$
T	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$
C	$+1$	$+1$	-1	$+1$	-1
CPT	$+1$	$+1$	-1	-1	$+1$

NOTE

$$\partial_\mu \xrightarrow{P} (-1)^\mu \partial_\mu$$

$$\partial_\mu \xrightarrow{T} -(-1)^\mu \partial_\mu$$

$$\partial_\mu \xrightarrow{C} \partial_\mu$$

$$\partial_\mu \xrightarrow{CPT} -\partial_\mu$$

$\mathcal{L}_{DIRAC} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$ IS INVARIANT UNDER P, T, C