IV INTERACTING FIELDS AND FEYNMAN DIAGRAMS

1) S-MATRIX EXPANSION

2) WICK's THEOREM

3) FEYNMAN DIAGRAMS & FEYNMAN RULES FOR ϕ^4 THEORY

4) FEYNMAN DIAGRAMS & RULES FOR QED

1) S - MATRIX EXPANSION

⇒ TIME EVOLUTION : SCHRÖDINGER VS INTERACTION PICTURE

- SCHRÖDINGER PICTURE

$$\hat{H}\,|\Psi(t)\rangle_S = i\frac{d}{dt}|\Psi(t)\rangle_S$$

↳ SCHRÖDINGER

IN SCHRÖDINGER PICTURE :

↳ ALL TIME DEPENDENCE IN STATE VECTOR $|\Psi(t)\rangle$

↳ OPERATORS (e.g. $\hat{H}$) : TIME INDEPENDENT

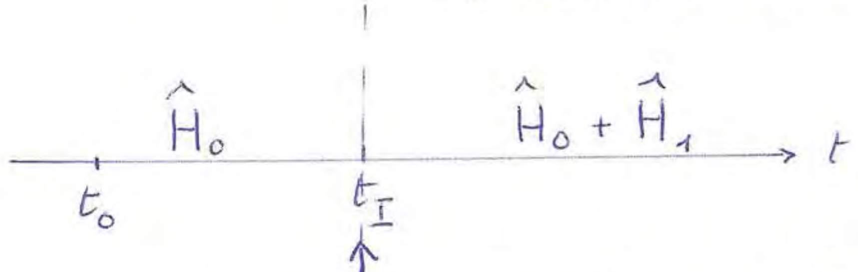
$$|\Psi(t)\rangle_S = e^{-i\hat{H}(t-t_0)}\,|\Psi(t_0)\rangle_S$$

$$= \underbrace{U(t,t_0)}_{\text{UNITARY TIME EVOLUTION OPERATOR}}\,|\Psi(t_0)\rangle_S$$

- INTERACTION PICTURE

↳ FOR $\hat{H} = \hat{H}_0 + \hat{H}_1$

↳ PERTURBATION



INTERACTION ADIABATICALLY SWITCHED ON

FOR $t < t_I$: $|\Psi(t)\rangle_S = e^{-i\hat{H}_0(t-t_0)}\,|\Psi(t_0)\rangle_S$

↳ WE CAN TAKE TIME DEPENDENCE DUE TO $\hat{H}_0$ OUT OF STATE BY DEFINING A NEW STATE $\forall t$ (ALSO $t > t_I$) THROUGH A UNITARY TF.

$$\boxed{\begin{aligned} |\Psi(t)\rangle_I &\equiv U_0^\dagger \, |\Psi(t)\rangle_S \\ &\equiv e^{i\hat{H}_0(t-t_0)} \, |\Psi(t)\rangle_S \end{aligned}}$$

WITH $|\Psi(t_0)\rangle_I = |\Psi(t_0)\rangle_S$

↳ INTERACTION PICTURE

↳ NOTE: FOR $\hat{H} = \hat{H}_0$ (i.e. $\hat{H}_1 = 0$)

$$|\Psi(t)\rangle_I = |\Psi(t_0)\rangle_S = |\Psi(t_0)\rangle_I$$

i.e. IN ABSENCE OF INTERACTION $|\Psi(t)\rangle_I$ IS CONSTANT IN TIME

- **EQUIVALENCE OF MATRIX ELEMENTS**

$$\boxed{{}_S\langle \Psi(t) | \hat{O}^S | \Psi(t) \rangle_S = {}_I\langle \Psi(t) | \hat{O}^I(t) | \Psi(t) \rangle_I}$$

MATRIX ELEMENT SHOULD BE INVARIANT
UNDER UNITARY TRANSFORMATION

$\hat{O}^S$: TIME-INDEPENDENT OPERATOR IN SCHRÖDINGER PICTURE

$\hat{O}^I(t)$: TIME-DEPENDENT OPERATOR IN INTERACTION PICTURE

USING : $|\Psi(t)\rangle_S = U_0 \, |\Psi(t)\rangle_I$

$${}_S\langle \Psi(t) | \hat{O}^S | \Psi(t) \rangle_S = {}_I\langle \Psi(t) | U_0^+ \hat{O}_S U_0 | \Psi(t) \rangle_I$$

$$\Downarrow$$

$$\boxed{\hat{O}^I(t) \equiv U_0^+ \hat{O}^S U_0}$$

- **TIME DEPENDENCE OF OPERATORS IN INT. PICTURE**

IN INTERACTION PICTURE : TIME DEP. DUE TO $\hat{H}_0$ IS TRANSFERED INTO OPERATOR $\hat{O}^I(t)$

$$i \frac{d}{dt} \hat{O}^I(t) = - \hat{H}_0 \hat{O}^I(t) + \hat{O}^I(t) \hat{H}_0$$

$$\boxed{i \frac{d}{dt} \hat{O}^I(t) = \left[\hat{O}^I(t), \hat{H}_0 \right]_-}$$

TRIVIAL CASE $\hat{O}^I = \hat{H}_0 \quad \rightarrow \quad \hat{H}_0^S = \hat{H}_0^I = \hat{H}_0$ (CONSTANT)

- TIME DEPENDENCE OF STATE IN INT. PICTURE

$$i\frac{d}{dt}|\Psi(t)\rangle_I = i\frac{d}{dt}\Big(\underbrace{e^{i\hat{H}_0(t-t_0)}}_{U_0^+}|\Psi(t)\rangle_S\Big)$$

$$= -\hat{H}_0|\Psi(t)\rangle_I$$

$$+\,U_0^+(\hat{H}_0+\hat{H}_1)\underbrace{|\Psi(t)\rangle_S}_{U_0|\Psi(t)\rangle_I}$$

$$\downarrow\quad U_0^+\hat{H}_0U_0=\hat{H}_0$$

$$i\frac{d}{dt}|\Psi(t)\rangle_I = -\hat{H}_0|\Psi(t)\rangle_I + (\hat{H}_0 + U_0^+\hat{H}_1U_0)|\Psi(t)\rangle_I$$

$$\Downarrow$$

$$\boxed{i\frac{d}{dt}|\Psi(t)\rangle_I = \hat{H}_1^I(t)\,|\Psi(t)\rangle_I}$$

WITH $\hat{H}_1^I(t) = U_0^+\hat{H}_1U_0$

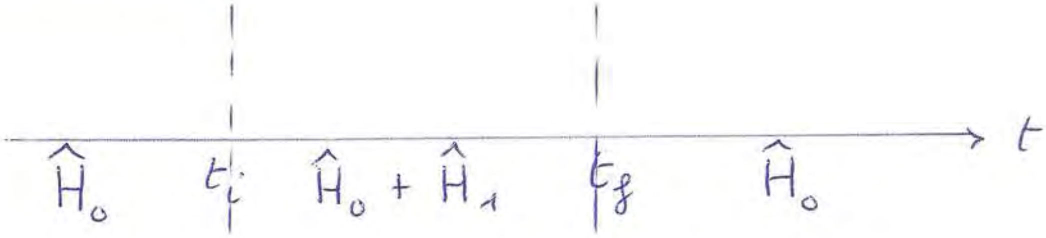
$\therefore$ IN INTERACTION PICTURE:

TIME DEPENDENCE DUE TO $\hat{H}_0$ IS IN OPERATORS.

TIME DEPENDENCE DUE TO $\hat{H}_1$ IS IN STATES $|\Psi(t)\rangle_I$

⟹ S - MATRIX

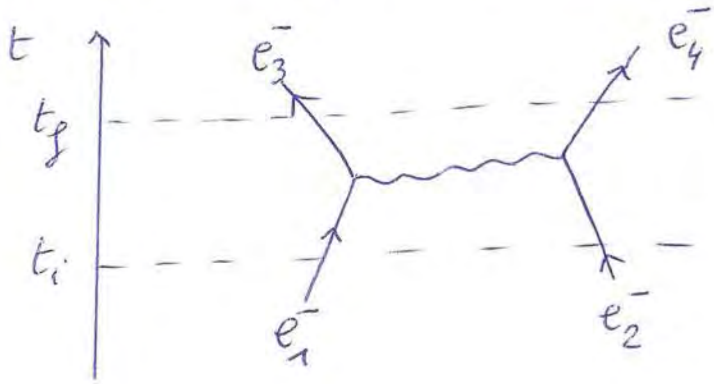
•



AT t_i : INTERACTION SWITCHED ON

AT t_f : " " OFF

BETWEEN t_i ANT t_f : FIELDS INTERACT



↳ FOR $t << t_i$, i.e. $t = -\infty$

SYSTEM IS IN INITIAL STATE

$$|\Psi(t=-\infty)\rangle_I = |i\rangle \qquad (\text{e.g. } |e_1^- e_2^-\rangle)$$

↳ DUE TO INTERACTION :

STATE EVOLVES ACCORDING TO A UNITARY TF.

$$\boxed{|\Psi(t=+\infty)\rangle_I \equiv \underset{\substack{\uparrow \\ \text{S-MATRIX}}}{S} \; |\Psi(t=-\infty)\rangle_I}$$

UNITARY: $S^\dagger S = S S^\dagger = \mathbb{1}$

↳ TRANSITION PROBABILITY AMPLITUDE TO A SPECIFIC FINAL STATE $|f\rangle$ (e.g. $|e_3^- e_4^-\rangle$)

$$\boxed{\langle f | \Psi(t=+\infty)\rangle_I = \langle f | S | i \rangle \equiv S_{fi}}$$

S - MATRIX ELEMENT

- UNITARITY OF S - MATRIX

SUM OF PROBABILITIES OF SCATTERING FROM $|i\rangle$ TO ALL POSSIBLE FINAL STATES $|f\rangle$ SHOULD BE 1

$$\boxed{\sum_f |\langle f | S | i \rangle|^2 = \sum_f |S_{fi}|^2 = 1}$$

PROOF: THIS FOLLOWS FROM UNITARITY OF S

$$S^+ S = 1$$

$$\langle i | S^+ S | i \rangle = \langle i | i \rangle = 1.$$

↓ COMPLETENESS

$$\langle i | S^+ \overbrace{\left(\sum_f |f\rangle\langle f|\right)}^{\mathbb{1}} S | i \rangle = 1$$

$$\sum_f \langle f | S | i \rangle^* \langle f | S | i \rangle = 1 \Rightarrow \sum_f |S_{fi}|^2 = 1$$

⇒ EXPANSION FOR S

- ITERATIVE SOLUTION FOR $|\Psi(t)\rangle_I$.

↳ $$i \frac{d}{dt} |\Psi(t)\rangle_I = \hat{H}_1^I(t)\, |\Psi(t)\rangle_I$$

$$\Big\downarrow \text{ INTEGRATE } \int_{-\infty}^{t}$$

$$i\Big(|\Psi(t)\rangle_I - \underbrace{|\Psi(-\infty)\rangle_I}_{|i\rangle}\Big) = \int_{-\infty}^{t} dt_1\ \hat{H}_1^I(t_1)\, |\Psi(t_1)\rangle_I$$

$$\boxed{|\Psi(t)\rangle_I = |i\rangle - i \int_{-\infty}^{t} dt_1\ \hat{H}_1^I(t_1)\, |\Psi(t_1)\rangle_I}$$

↳ INTEGRAL EQUATION CAN BE SOLVED ITERATIVELY IF $\hat{H}_1^I$ IS A PERTURBATION

↳ CONTAINS SMALL EXPANSION PARAMETER

↳ $$|\Psi(t)\rangle_I = |i\rangle + (-i)\int_{-\infty}^{t} dt_1\ \hat{H}_1^I(t_1)\, |i\rangle + (-i)^2 \int_{-\infty}^{t} dt_1\ \hat{H}_1^I(t_1) \int_{-\infty}^{t_1} dt_2\ \hat{H}_1^I(t_2)\, |\Psi(t_2)\rangle_I$$

$$\hookrightarrow \quad |\Psi(t)\rangle_I = |i\rangle + (-i)\int_{-\infty}^{t} dt_1 \; \hat{H}_1^I(t_1)\,|i\rangle + (-i)^2 \int_{-\infty}^{t} dt_1 \; \hat{H}_1^I(t_1) \int_{-\infty}^{t_1} dt_2 \; \hat{H}_1^I(t_2)\,|i\rangle + (-i)^3 \int_{-\infty}^{t} dt_1 \; \hat{H}_1^I(t_1) \int_{-\infty}^{t_1} dt_2 \; \hat{H}_1^I(t_2) \int_{-\infty}^{t_2} dt_3 \; \hat{H}_1^I(t_3)\,|i\rangle + \dots$$

$\hookrightarrow$ FOR $t \to +\infty$

$$|\Psi(t=+\infty)\rangle_I = S\,|i\rangle$$

THIS YIELDS THE S - MATRIX EXPANSION

$$S = \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \dots \int_{-\infty}^{t_{n-1}} dt_n \cdot \hat{H}_1^I(t_1)\,\hat{H}_1^I(t_2) \dots \hat{H}_1^I(t_n)$$

NOTE: IN THIS PRODUCT, THE INTEGRATION RANGES IMPLY THAT

$$t_1 > t_2 > t_3 > \dots > t_{n-1} > t_n$$

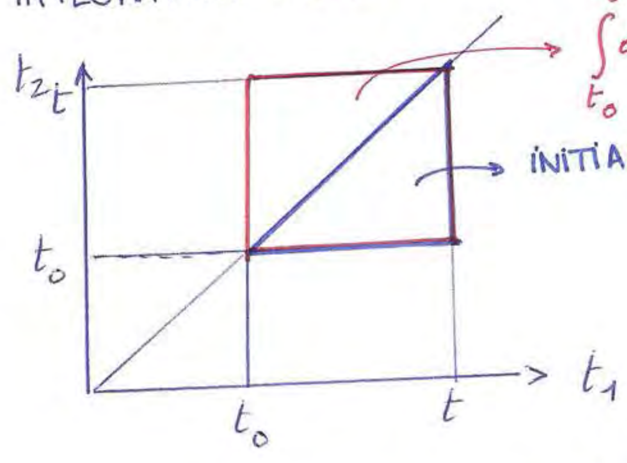
- TIME - ORDERED PRODUCT

↳ $$T\left\{ \hat{H}_1^I(t_1)\ \hat{H}_1^I(t_2)\ \dots\ \hat{H}_1^I(t_n) \right\}$$
$$\equiv \hat{H}_1^I(t_1)\ \hat{H}_1^I(t_2)\ \dots\ \hat{H}_1^I(t_n)$$
IF $t_1 > t_2 > \dots > t_n$

↳ USING T - PRODUCT

$$\boxed{S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} dt_1 \dots \int_{-\infty}^{+\infty} dt_n\ T\left\{ \hat{H}_1^I(t_1) \dots \hat{H}_1^I(t_n) \right\}}$$

NOTE :→ INTEGRATION DOMAIN CAN BE EXTENDED



→ $\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2$

→ INITIAL $\int$ DOMAIN $\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2$.

$$\begin{pmatrix} t_0 \to -\infty \\ t \to +\infty \end{pmatrix}$$

RED DOMAIN : 2× LARGER THAN BLUE DOMAIN
⇒ FACTOR 1/2.

→ FOR n INTEGRATIONS

$n!$ PERMUTATIONS OF

$$\hat{H}_1^I(t_1) \dots \hat{H}_1^I(t_n)$$

⇒ FACTOR $1/n!$ WHEN EXTENDING INTEGRATION DOMAIN

- IN TERMS OF HAMILTONIAN DENSITY

IN QFT $\hat{H}_1^I(t) = \int d^3\bar{x} \; \mathcal{H}_1^I(t, \bar{x})$

$$= \int d^3\bar{x} \; \mathcal{H}_1^I(x)$$

WITH $\mathcal{H}_1^I$: HAMILTONIAN DENSITY IN INTERACTION PICTURE

! FROM NOW ONWARDS :
WE WILL DROP INDEX I (INTERACTION PICTURE)
WHICH IS ALWAYS UNDERSTOOD HOWEVER

$$\boxed{S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \ldots \int d^4x_n \, T\{\mathcal{H}_1(x_1) \ldots \mathcal{H}_1(x_n)\}}$$

↳ DYSON EXPANSION OF S-MATRIX

⟹ EXAMPLE : ϕ^4 THEORY

↳ FREE SPIN 0 FIELD (KLEIN-GORDON)

$$\mathcal{L}_{KG} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$
$$\equiv \mathcal{L}_0$$

↳ INTERACTION BETWEEN FIELDS

$$\mathcal{L}_1 = -\frac{\lambda}{4!} \phi^4$$

λ DESCRIBES STRENGTH OF INTERACTION

↳ $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$

↳ $\pi = \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$

↳ $$\mathcal{H} = \pi \dot{\phi} - \mathcal{L}$$
$$= \underbrace{\frac{1}{2}\left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2 \phi^2\right)}_{\mathcal{H}_0} + \underbrace{\frac{\lambda}{4!}\phi^4}_{\mathcal{H}_1}$$

↳ $$S = \sum_{n=0}^{\infty} (-i)^n \left(\frac{\lambda}{4!}\right)^n \frac{1}{n!} \int_{-\infty}^{+\infty} d^4x_1 \cdots \int_{-\infty}^{+\infty} d^4x_n \, T\left\{ \phi^4(x_1) \ldots \phi^4(x_n) \right\}$$

↳ IF λ IS A SMALL PARAMETER

$\Downarrow$

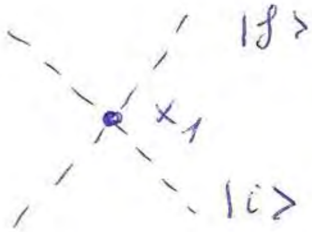
WE CAN MAKE A PERTURBATIVE EXPANSION IN λ

↳ 1° ORDER

$$S^{(1)} = -i \frac{\lambda}{4!} \int d^4x_1 \quad \phi^4(x_1)$$

GRAPHICALLY

$\langle f | S^{(1)} | i \rangle$



4 FIELDS INTERACT IN ONE SPACE-TIME POINT

↳ 2° ORDER

$$S^{(2)} = (-i)^2 \frac{\lambda^2}{2!} \int d^4x_1 \int d^4x_2 \; T\left\{ \phi^4(x_1) \; \phi^4(x_2) \right\}$$

WE HAVE TO WORK OUT THE T - PRODUCT

$\Downarrow$

WICK'S THEOREM

$\Longrightarrow$ EXAMPLE : QED

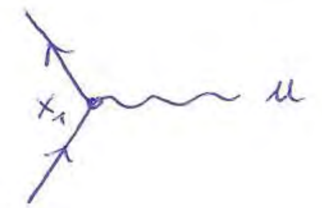
$\hookrightarrow$ $\mathcal{L}_1 = -e\,\bar{\psi}\gamma^\mu\psi\,A_\mu$

$$\mathcal{H}_1 = -\mathcal{L}_1 = e\,N\left(\bar{\psi}\gamma^\mu\psi\,A_\mu\right)$$

NORMAL ORDERING : ENSURES THAT $\langle 0|N(\dots)|0\rangle = 0$

$\hookrightarrow$ 1° ORDER $S^{(1)}$ FOR OBSERVABLES

$$S^{(1)} = -ie\int d^4x_1\;\bar{\psi}(x_1)\gamma^\mu\psi(x_1)\,A_\mu(x_1)$$



$\hookrightarrow$ 2° ORDER $S^{(2)}$

$$S^{(2)} = (-i)^2\frac{e^2}{2!}\int d^4x_1\int d^4x_2$$

$$\cdot\,T\left\{\bar{\psi}(x_1)\gamma^\mu\psi(x_1)\,A_\mu(x_1)\,\bar{\psi}(x_2)\gamma^\nu\psi(x_2)\cdot A_\nu(x_2)\right\}$$

2) WICK'S THEOREM

⇒ CONTRACTIONS (PROPAGATORS)

- NORMAL ORDERED PRODUCTS (CREATION OPERATOR LEFT OF ANNIHILATION)

  $\mathcal{H}_1$ CAN BE WRITTEN AS $N(\dots)$

  THIS ENSURES THAT $\langle 0 | N(\dots) | 0 \rangle = 0$

  FOR OBSERVABLES (e.g. ENERGY)

- HIGHER ORDER TERMS IN S-MATRIX EXPANSION

$$T\left\{A(x_1)\, B(x_2)\right\}$$

$$\text{e.g. QED} \quad A(x_1) = e\, N\left(\bar{\psi}(x_1)\gamma^\mu \psi(x_1) A_\mu(x_1)\right)$$

$$B(x_2) = e\, N\left(\bar{\psi}(x_2)\gamma^\mu \psi(x_2) A_\mu(x_2)\right)$$

- CONTRACTION

$$\boxed{\begin{aligned} &T\left\{A(x_1)\, B(x_2)\right\} \\ &= N\left(A(x_1)\, B(x_2)\right) + \langle 0 | T\left\{A(x_1)\, B(x_2)\right\} | 0 \rangle \end{aligned}}$$

BECAUSE $\langle 0 | N\left(A(x_1)\, B(x_2)\right) | 0 \rangle = 0$

NOTATION : CONTRACTION

$$\boxed{\underbrace{A(x_1)\,B(x_2)} \equiv \langle 0|\, T\{A(x_1)\,B(x_2)\}\,|0\rangle}$$

$$\therefore \quad T\{A(x_1)\,B(x_2)\} = N\big(A(x_1)B(x_2)\big) + \underbrace{A(x_1)\,B(x_2)}$$

- PROPAGATORS

$$\boxed{\begin{aligned}\underbrace{\phi(x_1)\,\phi(x_2)} &= \langle 0|\, T\{\phi(x_1)\,\phi(x_2)\}\,|0\rangle \\ &= i\,\Delta_F(x_1 - x_2)\end{aligned}}$$

$\Uparrow$

FEYNMAN PROPAGATOR

$$\boxed{\begin{aligned}\underbrace{\psi_\alpha(x_1)\,\bar{\psi}_\beta(x_2)} &= \langle 0|\, T\{\psi_\alpha(x_1)\,\bar{\psi}_\beta(x_2)\}\,|0\rangle \\ &= i\,S_F(x_1 - x_2)\end{aligned}}$$

NOTE $\underbrace{\psi_\alpha(x_1)\,\bar{\psi}_\beta(x_2)} = - \underbrace{\bar{\psi}_\beta(x_2)\,\psi_\alpha(x_1)}$

$$\boxed{\begin{aligned}\underbrace{A^\mu(x_1)\,A^\nu(x_2)} &= \langle 0|\, T\{A^\mu(x_1)\,A^\nu(x_2)\}\,|0\rangle \\ &= i\,D_F^{\mu\nu}(x_1 - x_2).\end{aligned}}$$

⟹ WICK'S THEOREM

↳
$$
\begin{aligned}
& T\{ABC\dots YZ\} \\
&= N(ABC\dots YZ) \\
&+ N(\underbrace{AB}_{}C\dots YZ) \\
&+ N(\underbrace{ABC}_{}\dots YZ) \\
&+ \dots \quad N(ABC\dots \underbrace{YZ}_{}) \\
&+ N(\underbrace{AB}_{}\underbrace{CD}_{}\dots YZ) + \dots
\end{aligned}
$$

T - PRODUCT = SUM OF N-ORDERED PRODUCTS WITH ALL POSSIBLE CONTRACTIONS

$$\underbrace{A(x_1)\,B(x_2)}_{}$$

↳ NOTE : PHASE FACTOR FOR FERMIONS

$$N(ABCDE.JKLM\dots)$$

$$= (-1)^P \underbrace{AJ}_{}\ \underbrace{BC}_{}\ \underbrace{DM}_{}\ N(E\dots KL\dots)$$

↓

P : # INTERCHANGES OF FERMION FIELDS TO CHANGE ORDER

↳ EXAMPLE : 4 FIELDS (SCALAR)

USE SHORTHAND NOTATION $\phi_i \equiv \phi(x_i)$

- $T\{\phi_1 \phi_2 \phi_3 \phi_4\}$

$$
\begin{aligned}
= N\Big(& \phi_1 \phi_2 \phi_3 \phi_4 \\
& + \underbracket{\phi_1 \phi_2} \phi_3 \phi_4 + \underbracket{\phi_1 \phi_2 \phi_3} \phi_4 + \underbracket{\phi_1 \phi_2 \phi_3 \phi_4} \\
& + \phi_1 \underbracket{\phi_2 \phi_3} \phi_4 + \phi_1 \underbracket{\phi_2 \phi_3 \phi_4} + \phi_1 \phi_2 \underbracket{\phi_3 \phi_4} \\
& + \underbracket{\phi_1 \phi_2}\, \underbracket{\phi_3 \phi_4} + \phi_1 \phi_2 \phi_3 \phi_4 + \underbracket{\phi_1 \underbracket{\phi_2 \phi_3} \phi_4} \Big)
\end{aligned}
$$

↓

$$
\begin{aligned}
= \; & N(\phi_1 \phi_2 \phi_3 \phi_4) \\
& + i\Delta_F(x_1 - x_2)\, N(\phi_3 \phi_4) + i\Delta_F(x_1 - x_3)\, N(\phi_2 \phi_4) \\
& + i\Delta_F(x_1 - x_4)\, N(\phi_2 \phi_3) + i\Delta_F(x_2 - x_3)\, N(\phi_1 \phi_4) \\
& + i\Delta_F(x_2 - x_4)\, N(\phi_1 \phi_4) + i\Delta_F(x_3 - x_4)\, N(\phi_1 \phi_2) \\
& - \Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4) - \Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4) \\
& - \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3)
\end{aligned}
$$

- $\langle 0 | T\{\phi_1 \phi_2 \phi_3 \phi_4\} | 0 \rangle$

$$= - \Delta_F(x_1 - x_2)\, \Delta_F(x_3 - x_4) - \Delta_F(x_1 - x_3)\, \Delta_F(x_2 - x_4)$$

$$- \Delta_F(x_1 - x_4)\, \Delta_F(x_2 - x_3)$$

⇒ PROOF OF WICK'S THEOREM

↳ FOR $n = 2$

$$\phi(x) = \phi^+(x) + \phi^-(x)$$

WITH $\phi^+(x)|0\rangle = 0$ — ϕ^+ CONTAINS ANNIHILATION OP.

$\langle 0 | \phi^-(x) = 0$ — ϕ^- " CREATION OP.

$$\phi^+(x) = \int \frac{d^3\bar{k}}{(2\pi)^3 \sqrt{2E_{\bar{k}}}}\, a(\bar{k})\, e^{-ik.x}$$

$$\phi^-(x) = \int \frac{d^3\bar{k}}{(2\pi)^3 \sqrt{2E_{\bar{k}}}}\, a^\dagger(\bar{k})\, e^{+ik.x}$$

$$T\{\phi(x_1)\,\phi(x_2)\}$$

$$= \left(\phi^+(x_1) + \phi^-(x_1)\right)\left(\phi^+(x_2) + \phi^-(x_2)\right)$$

FOR $x_1^0 > x_2^0$

$$= \phi^+(x_1)\phi^+(x_2) + \phi^-(x_1)\phi^+(x_2) + \underbrace{\phi^+(x_1)\phi^-(x_2)} + \phi^-(x_1)\phi^-(x_2)$$

$$= \phi^+(x_1)\phi^+(x_2) + \phi^-(x_1)\phi^+(x_2) + \phi^-(x_2)\phi^+(x_1) + \phi^-(x_1)\phi^-(x_2)$$

$$+ \left[\phi^+(x_1), \phi^-(x_2)\right]_-$$

$$T\{\phi(x_1)\,\phi(x_2)\}$$

$$\underset{x_1^0 > x_2^0}{=} N\big(\phi(x_1)\,\phi(x_2)\big) + \left[\phi^+(x_1), \phi^-(x_2)\right]_-$$

$$\underset{x_2^0 > x_1^0}{=} N\big(\phi(x_1)\,\phi(x_2)\big) + \left[\phi^+(x_2), \phi^-(x_1)\right]_-$$

$$\Downarrow$$

$$\langle 0 | T\{\phi(x_1)\,\phi(x_2)\} | 0 \rangle$$

$$= \begin{cases} \left[\phi^+(x_1), \phi^-(x_2)\right]_- , & x_1^0 > x_2^0 \\ \left[\phi^+(x_2), \phi^-(x_1)\right]_- , & x_2^0 > x_1^0 \end{cases}$$

$$= i\,\Delta_F(x_1 - x_2)$$

$$\equiv \contraction{}{\phi}{(x_1)\,}{\phi}\phi(x_1)\,\phi(x_2)$$

↳ PROOF FOR $n > 2$ BY INDUCTION

ASSUME IT HOLDS FOR $n-1$ FIELDS

WITHOUT LOSS OF GENERALITY WE CAN ASSUME $x_1^0 > x_2^0 \geq \cdots \geq x_n^0$

(IF THIS IS NOT CASE : WE CAN RELABEL POINTS)

$$T\{\phi_1 \dots \phi_n\}$$

$$= \phi_1\,\phi_2 \,\cdots\, \phi_n$$

$$= \phi_1 \, N\left(\phi_2 \cdots \phi_n + \begin{array}{l}\text{ALL CONTRACTIONS} \\ \text{NOT INVOLVING } \phi_1\end{array}\right)$$

$$= (\phi_1^+ + \phi_1^-) \, N\left(\phi_2 \cdots \phi_n + \begin{array}{l}\text{ALL CONTRACTIONS} \\ \text{NOT INVOLVING } \phi_1\end{array}\right)$$

$$= N\left(\phi_1^- \phi_2 \cdots \phi_n + \begin{array}{l}\text{ALL CONTRACTIONS NOT} \\ \text{INVOLVING } \phi_1\end{array}\right)$$

$$+ \phi_1^+ \, N\left(\phi_2 \cdots \phi_n + \text{ALL CONTRACTIONS NOT INVOLVING } \phi_1\right)$$

$$\hookrightarrow \phi_1^+ \, N(\phi_2 \cdots \phi_n)$$

$$= N(\phi_2 \cdots \phi_n)\phi_1^+ + \left[\phi_1^+, N(\phi_2 \cdots \phi_n)\right]_-$$

$$= N(\phi_1^+ \phi_2 \cdots \phi_n) + \left[\phi_1^+, N(\phi_2 \cdots \phi_n)\right]_-$$

$$\hookrightarrow \left[\phi_1^+, N(\phi_2 \phi_3)\right]_-$$

$$= \left[\phi_1^+, \phi_2^+ \phi_3^+ + \phi_2^- \phi_3^+ + \phi_3^- \phi_2^+ + \phi_2^- \phi_3^-\right]$$

$$= \phi_1^+ (\phi_2^+ + \phi_2^-)\phi_3^+ - (\phi_2^+ + \phi_2^-)\phi_3^+ \phi_1^+$$

$$+ \phi_1^+ \phi_3^- \phi_2^+ - \phi_3^- \phi_2^+ \phi_1^+ + \phi_1^+ \phi_2^- \phi_3^- - \phi_2^- \phi_3^- \phi_1^+$$

$$= N\left(\left[\phi_1^+, \phi_2^-\right] \phi_3\right) + N\left(\phi_2 \left[\phi_1^+, \phi_3^-\right]\right)$$

$$= N\left(\underbracket{\phi_1 \phi_2} \, \phi_3\right) + N\left(\phi_2 \, \underbracket{\phi_1 \phi_3}\right)$$

↳ $\phi_1^+ \, N(\phi_2 \cdots \phi_n)$

$$= N(\phi_1^+ \phi_2 \cdots \phi_n)$$
$$+ N(\underbrace{\phi_1 \phi_2} \phi_3 \cdots \phi_n) + N(\underbrace{\phi_1 \phi_2 \phi_3} \cdots \phi_n)$$
$$+ \cdots + N(\underbrace{\phi_1 \phi_2 \cdots \phi_n})$$

↳ $N(\phi_1^- \phi_2 \cdots \phi_n) + \phi_1^+ \, N(\phi_2 \cdots \phi_n)$

$$= N(\phi_1 \phi_2 \cdots \phi_n)$$
$$+ N(\underbrace{\phi_1 \phi_2} \cdots \phi_n) + \cdots + N(\underbrace{\phi_1 \phi_2 \cdots \phi_n})$$

↳ ANALOGOUS FOR OTHER TERMS IN N( ) WHICH INVOLVE CONTRACTIONS

$$\therefore \quad T\{\phi_1 \cdots \phi_n\} = N(\phi_1 \cdots \phi_n) + \text{ALL CONTRACTIONS})$$

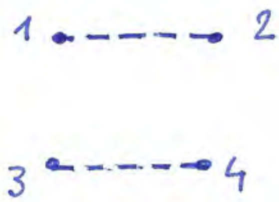
3) FEYNMAN DIAGRAMS & FEYNMAN RULES FOR ϕ^4 THEORY

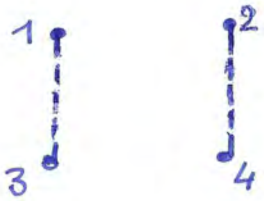
⇒ EXAMPLE: VACUUM EXPECTATION VALUE

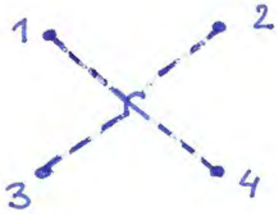
$$\langle 0 | T\{\phi_1 \phi_2 \phi_3 \phi_4\} | 0 \rangle$$

↓ ONLY FULLY CONTRACTED TERMS CONTRIBUTE TO $\langle 0 |$ $| 0 \rangle$

$$= \underbrace{\phi_1 \phi_2}\ \underbrace{\phi_3 \phi_4} + \phi_1 \phi_2 \phi_3 \phi_4 + \phi_1 \phi_2 \phi_3 \phi_4$$







DIAGRAMMATIC INTERPRETATION : FEYNMAN DIAGRAMS

PROPAGATOR $\underbrace{\phi(y)\,\phi(x)}$

$$= i\,\Delta_F(x-y)$$

NOTE : WE WILL INDICATE SCALAR (SPIN 0) PROPAGATOR BY DASHED LINE

⇒ EXAMPLE : 2 → 2 SCATTERING TO 1° ORDER

↳ S - MATRIX ELEMENT TO 1° ORDER

$$S^{(1)} = -i \frac{\lambda}{4!} \int d^4x \, \phi^4(x)$$

↳ INITIAL STATE : 2 SCALAR PARTICLES WITH MOMENTA $\bar{p}_1$ & $\bar{p}_2$

$$|i\rangle = |\bar{p}_1, \bar{p}_2\rangle$$
$$= \sqrt{2E_1}\sqrt{2E_2}\; a^+(\bar{p}_1)\, a^+(\bar{p}_2)\, |0\rangle$$

↳ FINAL STATE : 2 SCALAR PARTICLES WITH MOMENTA $\bar{p}_3$ & $\bar{p}_4$

$$|f\rangle = |\bar{p}_3, \bar{p}_4\rangle$$
$$= \sqrt{2E_3}\sqrt{2E_4}\; a^+(\bar{p}_3)\, a^+(\bar{p}_4)\, |0\rangle$$

↳ TRANSITION AMPLITUDE INDUCED BY $S^{(1)}$ BETWEEN $|i\rangle$ AND $|f\rangle$

$$\langle f | S^{(1)} | i \rangle$$

↳ $\overset{+}{\phi}(x) \mid \bar{p}_1 \rangle$

$$= \int \frac{d^3\bar{k}}{(2\pi)^3\sqrt{2E_k}} \, a(\bar{k})\, e^{-ik.x} \sqrt{2E_{p_1}}\, a^+(\bar{p}_1) \mid 0 \rangle$$

$$= e^{-ip_1 \cdot x} \mid 0 \rangle \quad \Rightarrow \langle 0 \mid \phi(x) \mid \bar{p}_1 \rangle = e^{-ip_1 \cdot x}$$

↳ $\langle \bar{p}_3 \mid \overset{-}{\phi}(x) = \langle 0 \mid e^{+ip_3 \cdot x} \quad \Rightarrow \langle \bar{p}_3 \mid \phi(x) \mid 0 \rangle = e^{+ip_3 \cdot x}$

↳ SYMMETRY FACTOR

TO ANNIHILATE PARTICLE $\bar{p}_1$ ⤳ 4 FIELDS POSSIBLE
" " AFTERWARDS " $\bar{p}_2$ ⤳ 3 " "
" CREATE " " $\bar{p}_3$ ⤳ 2 " "
" " " " $\bar{p}_4$ ⤳ 1 FIELD

TOTAL 4! COMBINATIONS !
SYMMETRY FACTOR 4!

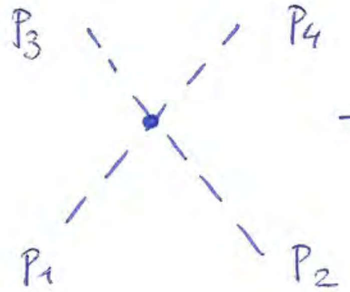
↳ $\langle f \mid S^{(1)} \mid i \rangle$

$$= -i \frac{\lambda}{4!} \underbrace{(4!)}_{\text{SYMMETRY FACTOR}} \int d^4x \; e^{-i(p_1 + p_2 - p_3 - p_4) \cdot x}$$

$$= (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \, (-i\lambda)$$

$$\boxed{\begin{aligned} &\langle f | S^{(1)} | i \rangle \\ &= \underbrace{(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)}_{\text{ENERGY - MOMENTUM CONSERVATION}} \cdot (-i\lambda) \end{aligned}}$$

FEYNMAN DIAGRAM (TREE LEVEL)



$\longrightarrow$ VERTEX : $\underline{\underline{-i\lambda}}$

$\Rightarrow$ EXAMPLE : $2 \to 2$ SCATTERING TO 2° ORDER

↳ S- MATRIX ELEMENT TO 2° ORDER

$$S^{(2)} = \frac{1}{2!}\left(-i\frac{\lambda}{4!}\right)^2 \int d^4x \int d^4y \; T\left\{\phi^4(x)\,\phi^4(y)\right\}$$

↳ CALCULATE

$$\langle \bar{p}_3 \bar{p}_4 | S^{(2)} | \bar{p}_1 \bar{p}_2 \rangle$$

DIFFERENT CONTRIBUTIONS DUE TO WICK'S THEOREM

WE NEED 4 ϕ's TO ANNIHILATE $\bar{p}_1, \bar{p}_2$ & CREATE $\bar{p}_3, \bar{p}_4$

WE CAN CLASSIFY CONTRIBUTIONS THROUGH FEYNMAN DIAGRAMS

↳ 2 FIELDS FROM EITHER $\phi^4(x)$ OR $\phi^4(y)$ ANNIHILATE $\bar{P}_1, \bar{P}_2$
2 " " " " " " CREATE $\bar{P}_3, \bar{P}_4$

e.g. TERM

$$\langle f | N\left(\phi(x)\phi(x)\,\phi(x)\phi(x)\,\phi(y)\,\phi(y)\,\phi(y)\phi(y)\right)| i \rangle$$

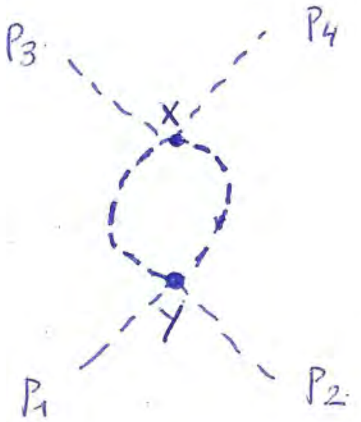


DIAGRAM (a)

- AT LOWER VERTEX 4! CHOICES WHICH FIELD
- AT UPPER VERTEX $4 \cdot 3$ CHOICES WHICH FIELD
- AT VERTEX y : EITHER P_1, P_2 OR P_3, P_4 ⇓ FACTOR 2

$$\langle f | S_a^{(2)} | i \rangle = \frac{2}{2!}\left(\frac{-i\lambda}{4!}\right)\left(\frac{-i\lambda}{4!}\right) \cdot 4!\,(4\cdot 3)$$

↑
DIAGRAM(a)

$$\cdot \int d^4x \int d^4y \; e^{iP_3\cdot x} \, e^{iP_4\cdot x} \, e^{-iP_1\cdot y} \, e^{-iP_2\cdot y}$$

$$\cdot \; i\Delta_F(x-y) \; i\Delta_F(x-y)$$

$$= \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \, i\Delta_F(k) \int \frac{d^4\ell}{(2\pi)^4} \, i\Delta_F(\ell)$$

$$\cdot \int d^4x \; e^{i(P_3 + P_4 - k - \ell)\cdot x}$$

$$\cdot \int d^4y \; e^{-i(P_1 + P_2 - k - \ell)\cdot y}$$

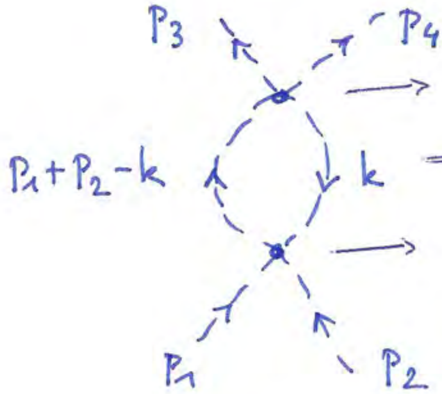
$$\downarrow \quad \int d^4y \; e^{-i(p_1+p_2-k-\ell)\cdot y} = (2\pi)^4\,\delta^4(p_1+p_2-k-\ell)$$

$$\langle f|\,S_a^{(2)}\,|i\rangle = \frac{(-i\lambda)^2}{2}\int \frac{d^4k}{(2\pi)^4}\, i\Delta_F(k)\; i\Delta_F(p_1+p_2-k) \cdot \underbrace{\int d^4x\; e^{+i(p_3+p_4-p_1-p_2)\cdot x}}_{(2\pi)^4\,\delta^4(p_1+p_2-p_3-p_4)}$$

$$\boxed{\langle f|\,S_a^{(2)}\,|i\rangle = (2\pi)^4\,\delta^4(p_1+p_2-p_3-p_4) \cdot \frac{(-i\lambda)^2}{2}\int \frac{d^4k}{(2\pi)^4}\; i\Delta_F(k)\; i\Delta_F(p_1+p_2-k)}$$

WITH $\displaystyle i\Delta_F(k) = \frac{i}{k^2-m^2+i\varepsilon}$

FEYNMAN DIAGRAM IN MOMENTUM SPACE (1-LOOP DIAGRAM)
$\hookrightarrow$ QUANTUM CORRECTION



$\longrightarrow$ VERTEX $(-i\lambda)$

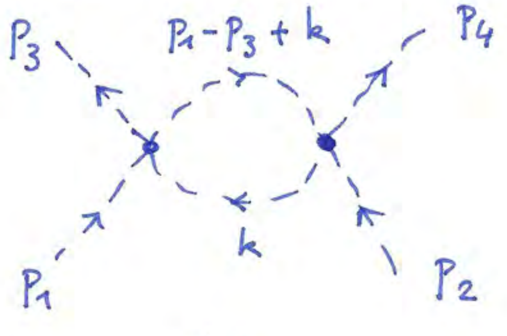
$\Longrightarrow$ PROPAGATORS $i\Delta_F(k)$ AND $i\Delta_F(p_1+p_2-k)$

$\longrightarrow$ VERTEX $(-i\lambda)$

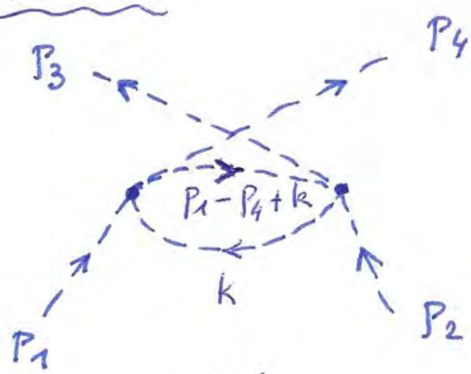
INTERNAL LOOP $\Rightarrow \displaystyle\int \frac{d^4k}{(2\pi)^4}\;\ldots$

$(2\pi)^4\,\delta^4(p_1+p_2-p_3-p_4)$: OVERALL ENERGY-MOMENTUM CONSERVATION

↳ OTHER TOPOLOGIES OF SAME KIND



(b)



(c)

$$\boxed{\langle f | S_b^{(2)} | i \rangle = (2\pi)^4 \, \delta^4(P_1 + P_2 - P_3 - P_4) \cdot \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \, i\Delta_F(k) \; i\Delta_F(P_1 - P_3 + k)}$$

$$\boxed{\langle f | S_c^{(2)} | i \rangle = (2\pi)^4 \, \delta^4(P_1 + P_2 - P_3 - P_4) \cdot \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \, i\Delta_F(k) \; i\Delta_F(P_1 - P_4 + k)}$$

↳ TOPOLOGIES WITH 2 FIELDS WHICH CONTRACT ON THEMSELVES

e.g. TERM

$$\langle f | N\left(\phi(x)\phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y)\phi(y)\right) | i \rangle$$

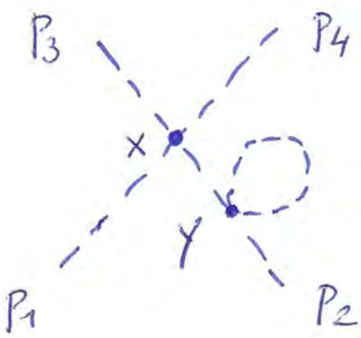


DIAGRAM (d)

$$\langle f | S_d^{(2)} | i \rangle = \frac{(-i\lambda)}{4!} \frac{(-i\lambda)}{4!} \cdot \underset{\substack{\uparrow \\ \text{AT VERTEX } x}}{4!} \, \underset{\nwarrow \text{ AT VERTEX } y}{(4.3)}$$

$$\cdot \int d^4x \int d^4y \; e^{ip_3 \cdot x} e^{ip_4 \cdot x} e^{-ip_1 \cdot x} e^{-ip_2 \cdot y}$$

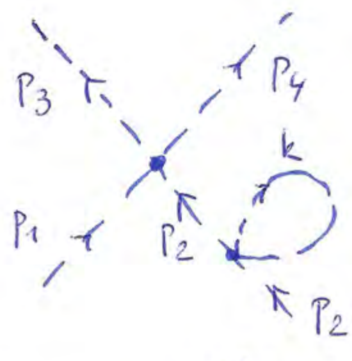
$$\cdot \; i\Delta_F(x-y) \cdot i\Delta_F(0)$$

$\downarrow$ IN MOMENTUM SPACE

$$= \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \, i\Delta_F(k) \int \frac{d^4\ell}{(2\pi)^4} \, i\Delta_F(\ell)$$

$$\cdot \int d^4x \; e^{-i(p_1 - p_3 - p_4 + \ell)\cdot x}$$

$$\cdot \underbrace{\int d^4y \; e^{-i(p_2 - \ell)\cdot y}}_{(2\pi)^4 \delta^4(\ell - p_2)}$$



$$\langle f | S_d^{(2)} | i \rangle = (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$

$$\cdot \; i\Delta_F(p_2)$$

$$\cdot \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \, i\Delta_F(k)$$

BUT EXTERNAL PARTICLES ARE ON-SHELL

$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = m^2$$

$\downarrow$

PROBLEM $\Delta_F(p_2) = \dfrac{1}{p_2^2 - m^2 + i\varepsilon} = \dfrac{1}{0 + i\varepsilon}$

∴ WE NEED TO EXCLUDE DIAGRAMS WHERE LOOPS ARE CONNECTED TO EXTERNAL LINES

(SEE FURTHER : THESE DIAGRAMS WILL "RENORMALIZE" EXTERNAL LINES,
i.e. EXTERNAL LINES HAVE TO BE DEFINED TO BE PHYSICAL IN PRESENCE OF INTERACTION)

↳ DISCONNECTED DIAGRAMS

e.g.

VACUUM BUBBLES

WHEN TURNING ON INTERACTION

TRIVIAL VACUUM $|0\rangle \Longrightarrow$ PHYSICAL VACUUM $|\Omega\rangle$
WHICH CONTAINS
VACUUM GRAPHS

∴ WE WILL ABSORB THESE DISCONNECTED GRAPHS IN DEFINITION OF PHYSICAL VACUUM & DEFINE 1-PARTICLE STATES AS EXCITATIONS IN PHYSICAL VACUUM $|\Omega\rangle$

$$|\vec{p}\rangle \equiv \sqrt{2E_{\vec{p}}}\, a^{\dagger}(\vec{p})\, |\Omega\rangle$$

⇒ FEYNMAN RULES FOR SCATTERING AMPLITUDES

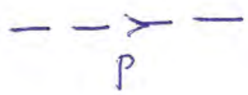
↳ IN MOMENTUM SPACE : CONTRIBUTION TO

$\langle f | S | i \rangle$ → SCATTERING AMPLITUDE

CONSIDER ALL CONNECTED GRAPHS BETWEEN $|i\rangle$ AND $|f\rangle$

FOR SCALAR FIELDS :

1) FOR EACH PROPAGATOR (INTERNAL LINE)



$$\frac{i}{p^2 - m^2 + i\varepsilon}$$

2) FOR EACH VERTEX

$$(-i\lambda)$$

3) FOR EACH EXTERNAL LINE  1
(NO LOOPS ATTACHED TO EXTERNAL LINES)

4) IMPOSE ENERGY - MOMENTUM CONSERVATION AT EACH VERTEX

5) INTEGRATE OVER INDEPENDENT LOOP - MOMENTA $\displaystyle\int \frac{d^4k}{(2\pi)^4}$

6) DETERMINE SYMMETRY FACTOR & DIVIDE BY IT

7) GLOBAL ENERGY MOMENTUM CONSERVATION

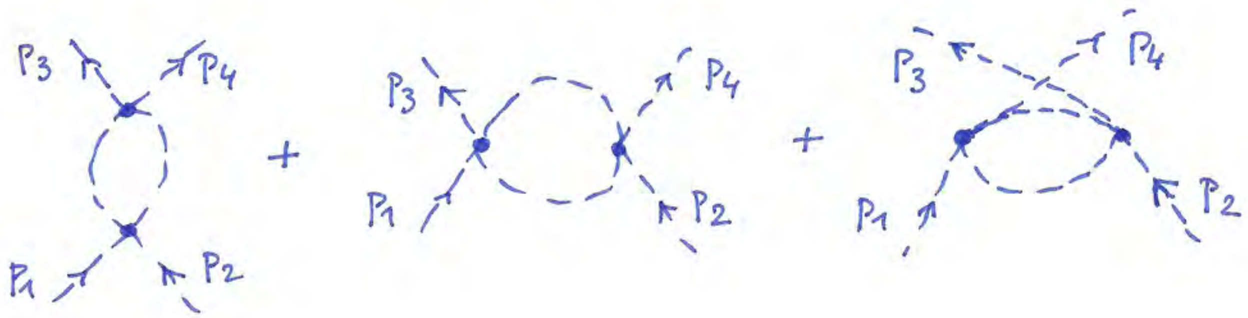
$$(2\pi)^4 \, \delta^4\Big(\sum_i p_i - \sum_f p_f \Big)$$

↳ TO 1° ORDER : 1 DIAGRAM
(TREE LEVEL)

$\sim (-i\lambda)$

↳ TO 2° ORDER : 3 DIAGRAMS (1-LOOP)

$$\sim (-i\lambda)^2$$



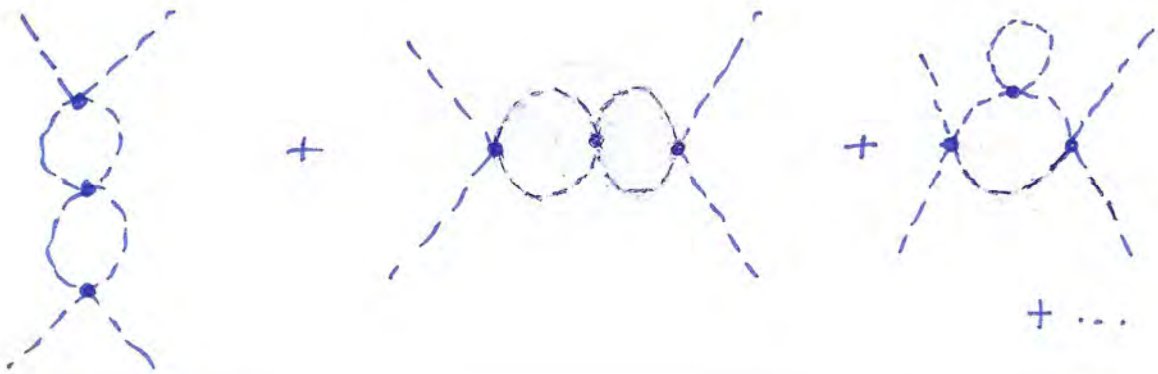
WE NEED TO PERFORM A LOOP INTEGRAL

$$\int \frac{d^4k}{(2\pi)^4} \cdots \quad \Rightarrow \text{ SEE LATER}$$

↳ TO 3° ORDER (2-LOOP)

$$\sim (-i\lambda)^3$$

EXAMPLES OF 2-LOOP DIAGRAMS $\Rightarrow \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4\ell}{(2\pi)^4} \cdots$



4) FEYNMAN DIAGRAMS & RULES FOR QED

⇒ S-MATRIX TO 1° ORDER

↳ $\mathcal{L}_1 = -q\,\bar{\psi}\,\gamma^\mu \psi\, A_\mu$

WITH q : ELECTRIC CHARGE

FOR e^- $q = -e$ $(e > 0)$ WITH $\frac{e^2}{4\pi} \equiv \alpha \simeq \frac{1}{137}$

α: FINE-STRUCTURE CONSTANT

WE WILL CONSIDER PROCESSES WITH e^- IN FOLLOWING

$$\mathcal{L}_1 = e\,\bar{\psi}\,\gamma^\mu \psi\, A_\mu$$

$$\mathcal{H}_1 = -e\,N\left(\bar{\psi}\,\gamma^\mu \psi\, A_\mu\right)$$ NORMAL ORDERED SUCH THAT $\langle 0 | H' | 0 \rangle = 0$

↳ $$S^{(1)} = ie \int d^4x\, N\left(\bar{\psi}(x)\,\gamma^\mu \psi(x)\, A_\mu(x)\right)$$

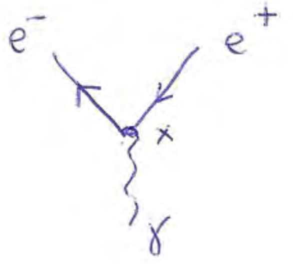
$$\underbrace{\psi(x)}_{} = \underbrace{\psi^+(x)}_{\text{ABSORBS } e^-} + \underbrace{\psi^-(x)}_{\text{CREATES } e^+}$$

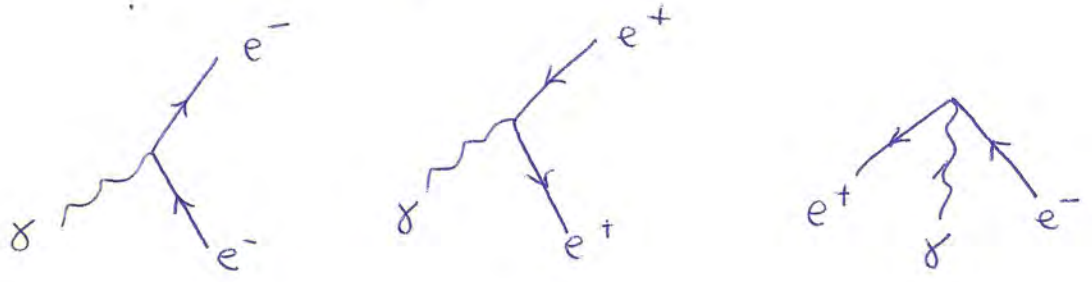
$$\bar{\psi}(x) = \underbrace{\bar{\psi}^-(x)}_{\text{CREATES } e^-} + \underbrace{\bar{\psi}^+(x)}_{\text{ABSORBS } e^+}$$

$$A^\mu(x) = \underbrace{A^+(x)}_{\text{ANNIHILATES } \gamma} + \underbrace{A^-(x)}_{\text{CREATES } \gamma}$$

↳ TO 1° ORDER: 8 PROCESSES

e.g. $N\left(\bar{\psi}^{-}(x)\, \gamma^{\mu} \psi^{-}(x)\, A_{\mu}^{+}(x)\right)$





AND 4 MORE DIAGRAMS WHERE γ IS CREATED

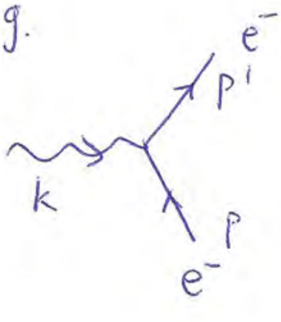
↳ FOR PHYSICAL PROCESSES

$\langle f | S^{(1)} | i \rangle$

e^-, e^+ IN EITHER $|i\rangle$ OR $|f\rangle$ $\quad p^2 = m^2$

γ " " " " " $\quad k^2 = 0$

e.g.



$$p'^2 = (p+k)^2 = m^2$$

$$\Downarrow$$

$$\cancel{m^2} + 2k.p + k^2 = \cancel{m^2}$$

$\underline{k.p = 0}$ ONLY POSSIBLE FOR $k^0 = 0$ ↯

∴ NO PHYSICAL PROCESSES AT 1° ORDER

⟹ S - MATRIX TO 2° ORDER

↳ $$S^{(2)} = \sum_{i=A}^{F} S_i^{(2)}$$

$$S_A^{(2)} = -\frac{e^2}{2!}\int d^4x_1 \int d^4x_2 \; N\left\{ \left(\bar{\psi}\gamma^\mu A_\mu \psi\right)_{x_1} \left(\bar{\psi}\gamma^\nu A_\nu \psi\right)_{x_2} \right\}$$

$$S_B^{(2)} = -\frac{e^2}{2!}\int d^4x_1 \int d^4x_2 \Big[N\left\{ \left(\bar{\psi}\gamma^\mu A_\mu \psi\right)_{x_1} \left(\bar{\psi}\gamma^\nu A_\nu \psi\right)_{x_2} \right\} + N\left\{ \left(\bar{\psi}\gamma^\mu A_\mu \psi\right)_{x_1} \left(\bar{\psi}\gamma^\nu A_\nu \psi\right)_{x_2} \right\} \Big]$$

$$S_C^{(2)} = -\frac{e^2}{2!}\int d^4x_1 \int d^4x_2 \; N\left\{ \left(\bar{\psi}\gamma^\mu A_\mu \psi\right)_{x_1} \left(\bar{\psi}\gamma^\nu A_\nu \psi\right)_{x_2} \right\}$$

$$S_D^{(2)} = -\frac{e^2}{2!}\int d^4x_1 \int d^4x_2 \Big[N\left\{ \left(\bar{\psi}\gamma^\mu A_\mu \psi\right)_{x_1} \left(\bar{\psi}\gamma^\nu A_\nu \psi\right)_{x_2} \right\} + N\left\{ \left(\bar{\psi}\gamma^\mu A_\mu \psi\right)_{x_1} \left(\bar{\psi}\gamma^\nu A_\nu \psi\right)_{x_2} \right\} \Big]$$

$$S_E^{(2)} = -\frac{e^2}{2!}\int d^4x_1 \int d^4x_2 \; N\left\{ \left(\bar{\psi}\gamma^\mu A_\mu \psi\right)_{x_1} \left(\bar{\psi}\gamma^\nu A_\nu \psi\right)_{x_2} \right\}$$

$$S_F^{(2)} = -\frac{e^2}{2!}\int d^4x_1 \int d^4x_2 \; N\left\{ \left(\bar{\psi}\gamma^\mu A_\mu \psi\right)_{x_1} \left(\bar{\psi}\gamma^\nu A_\nu \psi\right)_{x_2} \right\}$$

↳ $S_A^{(2)}$

DISCONNECTED GRAPH $\Rightarrow$ DOES NOT CORRESPOND TO PHYSICAL PROCESS

↳ $S_B^{(2)}$: 2 TERMS ARE IDENTICAL

$x_1 \leftrightarrow x_2 \quad \Rightarrow$ FACTOR 2

$$N\left\{ \left(\bar{\psi}\, \gamma^{\mu} A_{\mu}\, \psi\right)_{x_1} \left(\bar{\psi}\, \gamma^{\nu} A_{\nu}\, \psi\right)_{x_2} \right\}$$

CORRESPONDS e.g. TO COMPTON SCATTERING: $\gamma + e^- \to \gamma + e^-$

$$|i\rangle = |\gamma(\bar{k}_1, \lambda_1)\, e^-(\bar{p}_1, s_1)\rangle$$

$$= \sqrt{2|\bar{k}_1|}\,\sqrt{2E_{p_1}}\;\underset{\substack{\uparrow \\ \gamma}}{a^+(\bar{k}_1, \lambda_1)}\;\underset{\substack{\Uparrow \\ e^-}}{a^+(\bar{p}_1, s_1)}\,|0\rangle$$

$$|f\rangle = |\gamma(\bar{k}_2, \lambda_2)\, e^-(\bar{p}_2, s_2)\rangle$$

$$= \sqrt{2|\bar{k}_2|}\,\sqrt{2E_{\bar{p}_2}}\; a^+(\bar{k}_2, \lambda_2)\; a^+(\bar{p}_2, s_2)\,|0\rangle$$

$$\psi^+(x)\,|e^-(\bar{p}_1, s_1)\rangle = e^{-ip_1 \cdot x}\, u(\bar{p}_1, s_1)\,|0\rangle$$

$$\langle e^-(\bar{p}_2, s_2)|\,\bar{\psi}^-(x) = e^{+ip_2 \cdot x}\, \bar{u}(\bar{p}_2, s_2)\,\langle 0|$$

$$A^{+}_{\nu}(x)\,|\gamma(\bar{k}_1,\lambda_1)\rangle = \varepsilon_{\nu}(\bar{k}_1,\lambda_1)\,e^{-ik_1.x}\,|0\rangle$$

$$\langle\gamma(\bar{k}_2,\lambda_2)|\,A^{-}_{\mu}(x) = \varepsilon^{*}_{\mu}(\bar{k}_2,\lambda_2)\,e^{+ik_2.x}\,\langle 0|$$

$\therefore$ FOR $\langle f|S^{(2)}|i\rangle$ $\quad \gamma + e^- \to \gamma + e^-$

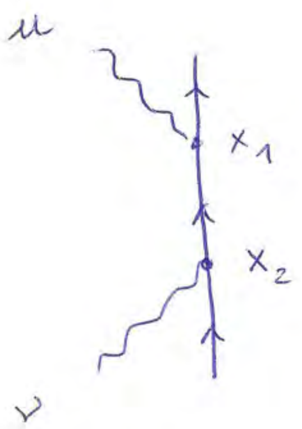
THERE ARE 2 CONTRIBUTIONS:

γ ABSORBED AT x_2 OR x_1

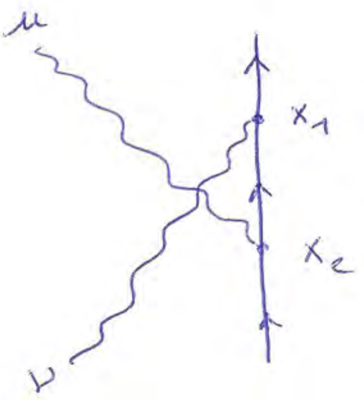
$$\langle f|S^{(2)}|i\rangle = \langle f|S_a + S_b|i\rangle$$

$$S_a = -e^2\int d^4x_1\int d^4x_2\ \bar{\Psi}^{-}(x_1)\,\gamma^{\mu}\,iS_F(x_1-x_2)\,\gamma^{\nu}\,\Psi^{+}(x_2)$$
$$\cdot\ A^{-}_{\mu}(x_1)\,A^{+}_{\nu}(x_2)$$

$$S_b = -e^2\int d^4x_1\int d^4x_2\ \bar{\Psi}^{-}(x_1)\,\gamma^{\nu}\,iS_F(x_1-x_2)\,\gamma^{\mu}\,\Psi^{+}(x_2)$$
$$\cdot\ A^{+}_{\nu}(x_1)\,A^{-}_{\mu}(x_2)$$



(a)



(b)

$$\rightsquigarrow \langle f | S_a | i \rangle = -e^2 \, \varepsilon_\nu(\bar{k}_1, \lambda_1) \, \varepsilon^*_\mu(\bar{k}_2, \lambda_2)$$

$$\cdot \int \frac{d^4 k}{(2\pi)^4} \, \bar{u}(p_2, s_2) \, \gamma^\mu \, \frac{i(\not{k} + m)}{k^2 - m^2 + i\varepsilon} \, \gamma^\nu \, u(p_1, s_1)$$

$$\cdot \int d^4 x_1 \; e^{-i(k - p_2 - k_2)\cdot x_1}$$

$$\cdot \int d^4 x_2 \; e^{-i(p_1 + k_1 - k)\cdot x_2}$$

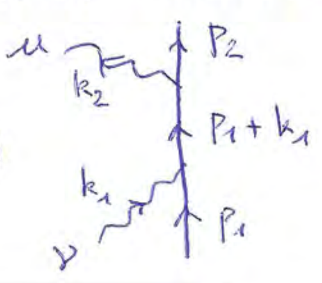
$$\boxed{\begin{aligned} \langle f | S_a | i \rangle = \; & (2\pi)^4 \delta^4(k_1 + p_1 - k_2 - p_2) \\ & \cdot \varepsilon_\nu(\bar{k}_1, \lambda_1) \cdot \varepsilon^*_\mu(\bar{k}_2, \lambda_2) \\ & \cdot \bar{u}(p_2, s_2)(ie\gamma^\mu) \frac{i(\not{p}_1 + \not{k}_1 + m)}{(p_1 + k_1)^2 - m^2 + i\varepsilon} (ie\gamma^\nu) u(p_1, s_1) \end{aligned}}$$

$\rightsquigarrow$ ANALOGOUSLY

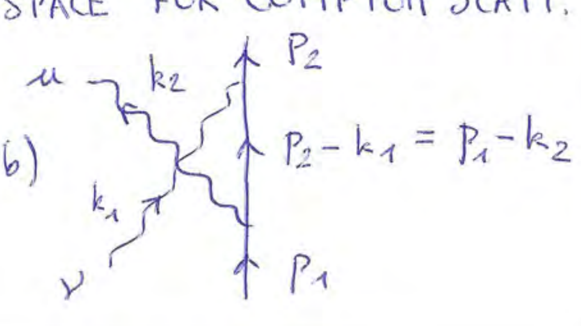
$$\boxed{\begin{aligned} \langle f | S_b | i \rangle = \; & (2\pi)^4 \delta^4(k_1 + p_1 - k_2 - p_2) \\ & \cdot \varepsilon_\nu(\bar{k}_1, \lambda_1) \cdot \varepsilon^*_\mu(\bar{k}_2, \lambda_2) \\ & \cdot \bar{u}(p_2, s_2)(ie\gamma^\nu) \frac{i(\not{p}_1 - \not{k}_2 + m)}{(p_1 - k_2)^2 - m^2 + i\varepsilon} (ie\gamma^\mu) u(p_1, s_1) \end{aligned}}$$

$\rightsquigarrow$ FEYNMAN DIAGRAMS IN MOMENTUM SPACE FOR COMPTON SCATT.

(a)



(b)



$\hookrightarrow$ $S_C^{(2)}$ $N\left\{ (\bar{\psi}\gamma^\mu A_\mu \psi)_{x_1} (\bar{\psi}\gamma^\nu A_\nu \psi)_{x_2} \right\}$

4 UNCONTRACTED FERMION FIELDS :
THIS TERM CONTRIBUTES TO PHYSICAL PROCESSES AS

$e^- + e^- \to e^- + e^-$ (MØLLER SCATTERING)

$e^+ + e^+ \to e^+ + e^+$

$e^- + e^+ \to e^- + e^+$ (BHABHA SCATTERING)

$\rightsquigarrow$ CONSIDER $e^-(p_1, s_1) + e^-(p_2, s_2) \to e^-(p_1', s_1') + e^-(p_2', s_2')$

MØLLER SCATTERING

$$|i\rangle = a^+(p_2, s_2)\, a^+(p_1, s_1) \sqrt{2E_{\bar{p}_1}} \sqrt{2E_{\bar{p}_2}}\, |0\rangle$$

$$|f\rangle = a^+(p_2', s_2')\, a^+(p_1', s_1') \sqrt{2E_{\bar{p}_1'}} \sqrt{2E_{\bar{p}_2'}}\, |0\rangle$$

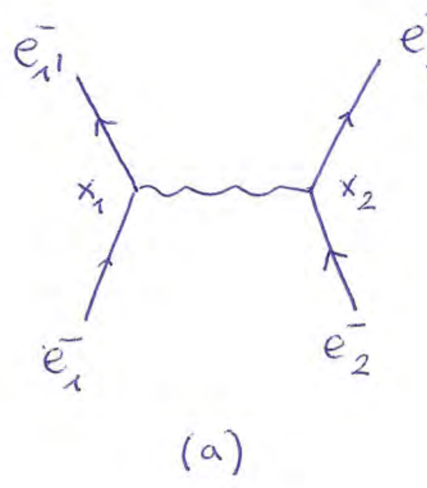
FERMION FIELDS $\psi = \psi^+ + \psi^-$

$\bar{\psi} = \bar{\psi}^- + \bar{\psi}^+$

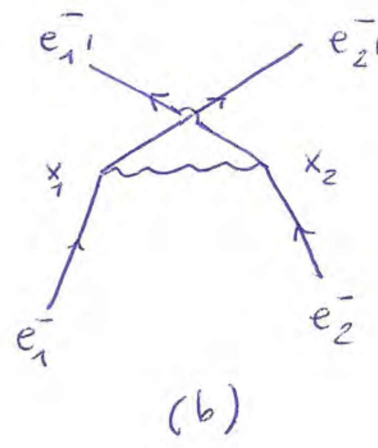
4 TERMS CONTRIBUTING

$\hookrightarrow$ 2 PAIRS WHICH ONLY DIFFER BY INTERCHANGE $x_1 \leftrightarrow x_2$ $\Rightarrow$ FACTOR 2

$\therefore$ IN GENERAL : THE FACTOR $\frac{1}{n!}$ IN $S^{(n)}$ CAN BE OMITTED BY CONSIDERING ONLY TOPOLOGICALLY DIFFERENT FEYNMAN DIAGRAMS



(a)



(b)

$$\langle f | S^{(2)} | i \rangle = \langle f | S_a + S_b | i \rangle$$

$$\rightsquigarrow \quad S_a = -e^2 \int d^4x_1 \int d^4x_2 \, N\{ \bar{\psi}^-(x_1) \gamma^\mu \psi^+(x_1) \cdot i D_{\mu\nu}(x_1 - x_2) \cdot \bar{\psi}^-(x_2) \gamma^\nu \psi^+(x_2) \}$$

$\langle f | S_a | i \rangle$

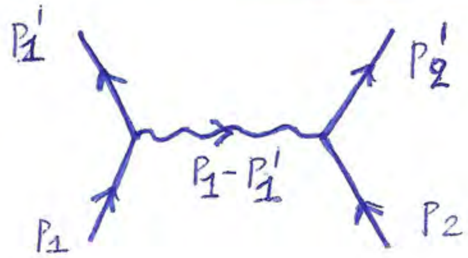
$$= \sqrt{2E_{\bar{p}_1}} \sqrt{2E_{\bar{p}_2}} \sqrt{2E_{\bar{p}_1'}} \sqrt{2E_{\bar{p}_2'}} \cdot (-e^2) \int d^4x_1 \int d^4x_2 \; i D_{\mu\nu}(x_1 - x_2)$$

$$\langle 0 | a(p_1', s_1') \, a(p_2', s_2') \cdot N\{ \bar{\psi}^-(x_1) \gamma^\mu \psi^+(x_1) \; \bar{\psi}^-(x_2) \gamma^\nu \psi^+(x_2) \} \cdot a^\dagger(p_2, s_2) \, a^\dagger(p_1, s_1) | 0 \rangle$$

2 SIGN CHANGES NEEDED $\Rightarrow +1$

$$\boxed{\begin{aligned}\langle f | S_a | i \rangle = {} & (2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2') \\ & \cdot \frac{i(-g_{\mu\nu})}{(p_1 - p_1')^2 + i\varepsilon} \\ & \cdot \bar{u}(p_1', s_1')(+ie\gamma^\mu) u(p_1, s_1) \\ & \cdot \bar{u}(p_2', s_2')(+ie\gamma^\nu) u(p_2, s_2)\end{aligned}}$$

(a)



$\rightsquigarrow$ ANALOGOUSLY

$$\langle f | S_b | i \rangle = \sqrt{2E_{\vec{p}_1}} \sqrt{2E_{\vec{p}_2}} \sqrt{2E_{\vec{p}_1'}} \sqrt{2E_{\vec{p}_2'}}$$

$$\cdot (-e^2) \int d^4x_1 \int d^4x_2 \quad i\, D_{\mu\nu}(x_1 - x_2)$$

$$\cdot \langle 0 | \underline{a(p_1', s_1')}\ \underline{a(p_2', s_2')}$$

$$\cdot N\left\{ \underline{\bar{\psi}^-(x_1)}\, \gamma^\mu\, \underline{\psi^+(x_1)}\ \ \underline{\bar{\psi}^-(x_2)}\, \gamma^\nu\, \underline{\psi^+(x_2)} \right\}$$

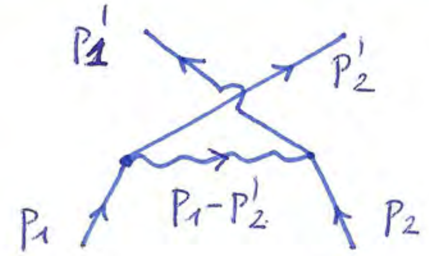
$$\underline{a^+(p_2, s_2)}\ \underline{a^+(p_1, s_1)} | 0 \rangle$$

ODD # SIGN CHANGES NEEDED $\Rightarrow -1$

$\langle f | S_b | i \rangle$ OBTAINED FROM $\langle f | S_a | i \rangle$

BY INTERCHANGING $p_1', s_1' \leftrightarrow p_2', s_2'$

$$\langle f | S_b | i \rangle = (2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2') \cdot i \frac{(-g_{\mu\nu})}{(p_1 - p_2')^2 + i\varepsilon} \cdot (-1)\, \bar{u}(p_2', s_2')\,(ie\gamma^\mu)\, u(p_1, s_1) \cdot \bar{u}(p_1', s_1')\,(ie\gamma^\nu)\, u(p_2, s_2)$$

(b) 

∴ MØLLER SCATTERING (SUM OF (a) + (b))

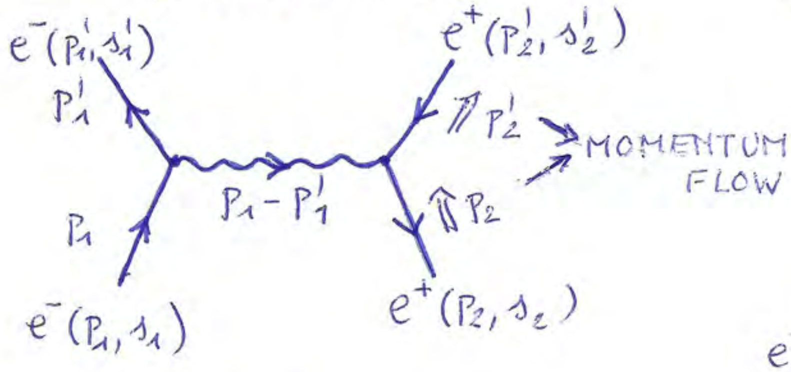
$$\langle f | S^{(2)} | i \rangle = (2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2') \cdot \Big\{ (-i) \frac{1}{(p_1 - p_1')^2 + i\varepsilon} \cdot \bar{u}(p_1', s_1')\,(ie\gamma^\mu)\, u(p_1, s_1) \cdot \bar{u}(p_2', s_2')\,(ie\gamma_\mu)\, u(p_2, s_2) - (1' \leftrightarrow 2') \Big\}$$

DUE TO ANTISYMMETRY OF $e^- e^-$ STATE

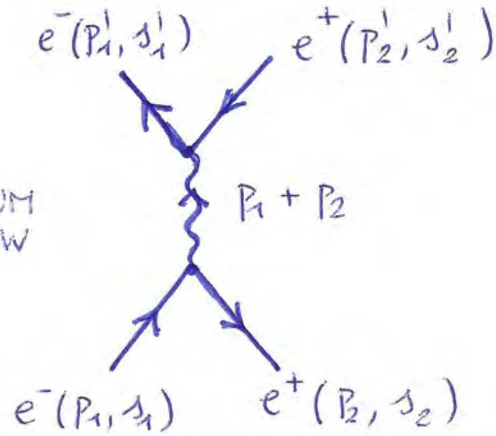
⇝ BHABHA SCATTERING (EXERCISE)

$$e^-(p_1, s_1) + e^+(p_2, s_2) \to e^-(p_1', s_1') + e^+(p_2', s_2')$$

2 CONTRIBUTIONS



(a)



(b)

NOTE: FOR e^+ (ANTI-PARTICLE):
CONVENTION IS TO DRAW FERMION LINE
OPPOSITE TO MOMENTUM FLOW

FOR e^- (PARTICLE):
MOMENTUM FLOW ALONG FERMION LINE

e.g.

$$\boxed{\begin{aligned} \langle f | S_b | i \rangle = {} & (2\pi)^4 \, \delta^4(p_1 + p_2 - p_1' - p_2') \\ & \cdot \frac{(-i g_{\mu\nu})}{(p_1 + p_2)^2 + i\varepsilon} \\ & \cdot \bar{u}(p_1', s_1') (ie\gamma^\mu) v(p_2', s_2') \\ & \cdot \bar{v}(p_2, s_2) (ie\gamma^\nu) u(p_1, s_1) \end{aligned}}$$

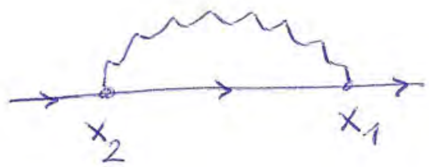
$\hookrightarrow$ $S_D^{(2)}$

$$N\left\{ \left(\bar{\psi}\,\gamma^\mu A_\mu \psi\right)_{x_1} \left(\bar{\psi}\,\gamma^\nu A_\nu \psi\right)_{x_2} \right\}$$

2 TERMS ARE EQUIVALENT $x_1 \leftrightarrow x_2 \Rightarrow$ FACTOR 2

$$S_D^{(2)}(e^- \to e^-) = -e^2 \int d^4x_1 \int d^4x_2 \; i\,D_{\mu\nu}(x_1 - x_2)$$

$$\cdot\; \bar{\psi}^-(x_1)\,\gamma^\mu\, i\,S_F(x_1 - x_2)\,\gamma^\nu\,\psi^+(x_2)$$



FOR $|i\rangle = |e^-(p,s)\rangle = \sqrt{2E_{\bar{p}}}\; a^+(\bar{p},s)\,|0\rangle$

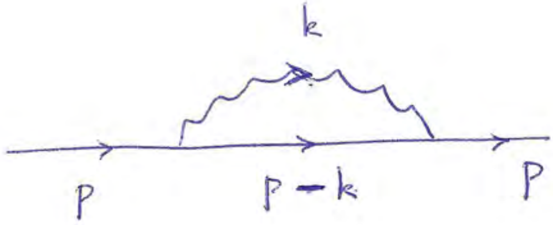
$|f\rangle = |e^-(p',s')\rangle = \sqrt{2E_{\bar{p}'}}\; a^+(\bar{p}',s')\,|0\rangle$

$$\langle f | S^{(2)} | i \rangle = -e^2 \int \frac{d^4k}{(2\pi)^4} \; \frac{i\,(-g_{\mu\nu})}{k^2 + i\varepsilon}$$

$$\cdot \int d^4x_1 \int d^4x_2 \; \bar{u}(p's')\,\gamma^\mu\, i\,S_F(x_1 - x_2)\,\gamma^\nu\, u(p,s)$$

$$\cdot\; \underbrace{e^{-i(k-p')\cdot x_1}\; e^{-i(p-k)\cdot x_2}}$$

$$e^{-i(p-p')\cdot x_1} \qquad e^{+i(p-k)\cdot(x_1 - x_2)}$$



$$\int d^4x_1 \int d^4x_2 \quad e^{-i(p-p')\cdot x_1} \, e^{i(p-k)\cdot(x_1-x_2)} \, S_F(x_1-x_2) \cdots$$

$$= (2\pi)^4 \delta^4(p-p') \; \tilde{S}_F(p-k)$$

$$\therefore \quad \langle f \,|\, S^{(2)} \,|\, i \rangle$$

$$= (2\pi)^4 \delta^4(p-p')$$

$$\cdot\; \bar{u}(p', s') \int \frac{d^4k}{(2\pi)^4} \, (ie\gamma^\mu) \, \frac{i(\not{p} - \not{k} + m)}{(p-k)^2 - m^2 + i\varepsilon} \, (ie\gamma^\nu)$$

$$\cdot\; \frac{i(-g_{\mu\nu})}{k^2 + i\varepsilon} \;\cdot\; u(p, s).$$

1- LOOP INTEGRAL

"e^- SELF-ENERGY"

$\hookrightarrow$ $S_E^{(2)}$

$$N\left\{ (\bar{\Psi}\gamma^\mu A_\mu \Psi)_{x_1} (\bar{\Psi}\gamma^\nu A_\nu \Psi)_{x_2} \right\}$$

EXTERNAL PHOTON

$$S_E^{(2)}(\gamma \to \gamma) = (-e^2)\int d^4x_1 \int d^4x_2 \quad A^-_\mu(x_1)\, A^+_\nu(x_2)$$

$$\left(\gamma^\mu\right)_{\alpha\beta}\left(i\,S_F(x_1 - x_2)\right)_{\beta\gamma}\left(\gamma^\nu\right)_{\gamma\delta}$$

$$\cdot\,(-1)\left(i S_F(x_2 - x_1)\right)_{\delta\alpha}$$

$\uparrow$

ODD # PERMUTATIONS

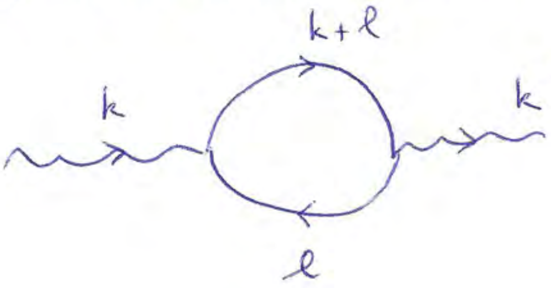
$$\left(\gamma^\mu\right)_{\alpha\beta}\left(S_F\right)_{\beta\gamma}\left(\gamma^\nu\right)_{\gamma\delta}\left(S_F\right)_{\delta\alpha}$$

$$= \mathrm{Tr}\left\{\gamma^\mu S_F \gamma^\nu S_F\right\}$$

$\Uparrow$

TRACE IN DIRAC SPACE

$\rightsquigarrow$ IN MOMENTUM SPACE



$|i\rangle = |\gamma(\bar{k}, \lambda)\rangle$

$|f\rangle = |\gamma(\bar{k}', \lambda')\rangle$

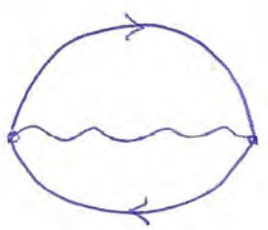
$$\boxed{\begin{aligned} &\langle f | S^{(2)} | i \rangle \\ &= (2\pi)^4 \, \delta^4(k-k') \\ &\quad \cdot \varepsilon_\mu(\bar{k}, \lambda) \, \varepsilon^*_\nu(\bar{k}', \lambda') \\ &\quad \cdot (-1) \int \frac{d^4 l}{(2\pi)^4} \, Tr\left\{ (ie\gamma^\nu) \frac{i(\not{k} + \not{l} + m)}{(k+l)^2 - m^2 + i\varepsilon} (ie\gamma^\mu) \right. \\ &\qquad\qquad \left. \cdot \frac{i(\not{l} + m)}{l^2 - m^2 + i\varepsilon} \right\} \end{aligned}}$$

1-LOOP INTEGRAL

γ-POLARIZATION (SELF-ENERGY)

$\hookrightarrow$ $S_F^{(2)}$ $\quad N\left\{ (\bar{\psi}\gamma^\mu A_\mu \psi)_{x_1} (\bar{\psi}\gamma^\nu A_\nu \psi)_{x_2} \right\}$

FULLY CONTRACTED DIAGRAM



VACUUM DIAGRAM

NO EXTERNAL LINES.

$\hookrightarrow$ NO TRANSITIONS

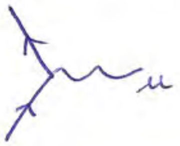
⟹ FEYNMAN RULES FOR QED

↳ IN MOMENTUM SPACE

TO CALCULATE $\langle f | S^{(n)} | i \rangle$

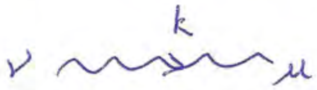
DRAW ALL <u>CONNECTED</u>, <u>TOPOLOGICALLY DIFFERENT</u> DIAGRAMS BETWEEN $|i\rangle$ AND $|f\rangle$ TO ORDER n WITH RULES

1) FOR EACH VERTEX



$-iq\gamma^\mu$ (WITH $q = -e < 0$ FOR e^-)

2) FOR EACH INTERNAL PHOTON LINE

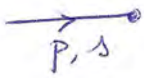


$$i D_F^{\mu\nu}(k) = \frac{i(-g^{\mu\nu})}{k^2 + i\varepsilon}$$

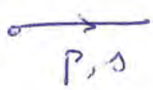
3) FOR EACH INTERNAL FERMION LINE

$$i S_F(p) = \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon}$$

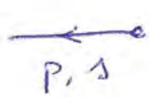
4) FOR EXTERNAL LINES:

↳ INITIAL e^- 

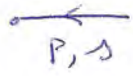
$u(\bar{p}, s)$

↳ FINAL e^- 

$\bar{u}(\bar{p}, s)$

↳ INITIAL e^+ 

$\bar{v}(\bar{p}, s)$

↳ FINAL e^+ 

$v(\bar{p}, s)$

↳ INITIAL γ $\underset{k,\lambda}{\rightsquigarrow}\!\bullet\,\mu$ $\varepsilon_\mu(\vec{k},\lambda)$

↳ FINAL γ $\mu\,\bullet\!\underset{k,\lambda}{\rightsquigarrow}$ $\varepsilon^*_\mu(\vec{k},\lambda)$

5) FERMION LINE

SPINOR FACTORS ARE ORDERED SUCH THAT WHEN READING FROM RIGHT TO LEFT CORRESPONDS TO FOLLOWING FERMION LINE ALONG ARROW

6) FOR CLOSED FERMION LOOP $\Rightarrow$ FACTOR (-1)

7) AT EACH VERTEX: 4-MOMENTUM CONSERVATION

FOR EACH 4-MOMENTUM WHICH IS NOT FIXED BY ENERGY-MOMENTUM CONSERVATION

$$\int \frac{d^4k}{(2\pi)^4}$$

8) GLOBAL ENERGY-MOMENTUM CONSERVATION

$$(2\pi)^4\,\delta^4\left(\sum_i p_i - \sum_f p_f\right)$$

9) PHASE FACTOR $\delta_p = \pm 1$

DEPENDING ON WHETHER ONE NEEDS EVEN (ODD) NUMBER OF INTERCHANGES OF FERMION OPERATORS TO BRING THEM IN NORMAL ORDER

e.g. e^-e^- SCATTERING, e^-e^+ SCATTERING WHERE DIFFERENT DIAGRAMS CONTRIBUTE