Exercise sheet 6
Theoretical Physics 6a (QFT): SS 2019

20.05.2019

Exercise 1. (50 points) : Dirac bilinears

Since a spinor turns into minus itself after a rotation over 2π, physical quantities must be bilinears in ψ, so that physical quantities turn into themselves after a rotation over 2π. These bilinears have the general form $\bar{\psi}\Gamma\psi$. There are 16 independent covariant ones related to 16 complex 4×4 matrices:

- $\Gamma_S = \mathbb{1}$ (scalar);
- $\Gamma_P = \gamma_5$ (pseudoscalar);
- $\Gamma_V^\mu = \gamma^\mu$ (vector);
- $\Gamma_A^\mu = \gamma^\mu \gamma_5$ (axial vector);
- $\Gamma_T^{\mu\nu} = \sigma^{\mu\nu} \equiv \frac{i}{2} [\gamma^\mu, \gamma^\nu]$ (tensor).

Without referring to any explicit representation for the Γ matrices,

(a)(10 points) show that $\Gamma^2 = \pm\mathbb{1}$.

(b)(10 points) show that for any Γ except Γ_S, we have $\mathrm{Tr}[\Gamma] = 0$. *Hint*: show first that for any Γ except Γ_S, there always exists a Γ' such that $\{\Gamma, \Gamma'\} = 0$.

(c)(10 points) check that the product of 2 different Γ's is proportional to some Γ different from Γ_S;

(d)(10 points) show that the Γ's are linearly independent, *i.e.* $\sum_i a_i \Gamma_i =$

$0 \Leftrightarrow a_i = 0$.

(e)(10 points) and using the Lorentz transformation of the Dirac spinor $\psi'(x') = S(a)\psi(x)$ with $x'^\mu = a^\mu{}_\nu x^\nu$, check that the bilinears transform according to their name, *i.e.* $\bar{\psi}'\psi' = \bar{\psi}\psi$, $\bar{\psi}'\gamma_5\psi' = \det(a)\bar{\psi}\gamma_5\psi$, $\bar{\psi}'\gamma^\mu\psi' = a^\mu{}_\nu\bar{\psi}\gamma^\mu\psi$, $\bar{\psi}'\gamma^\mu\gamma_5\psi' = \det(a)a^\mu{}_\nu\bar{\psi}\gamma^\mu\gamma_5\psi$ and $\bar{\psi}'\sigma^{\mu\nu}\psi' = a^\mu{}_\rho a^\nu{}_\sigma\bar{\psi}\sigma^{\rho\sigma}\psi$.

Exercise 2. (50 +10 bonus points) : Lorentz transformation

Under a Lorentz transformation a^μ_ν, a spinor $\psi(x)$ transforms according to:

$$(x')^\mu = a^\mu_\nu x^\nu$$
$$\psi'(x') = S(a)\psi(x)\,.$$

Give the explicit form of the Lorentz transformation matrices a^μ_ν

(a)(5 points) describing a finite boost in y-direction and its infinitesimal generator (K_y),

(b)(5 points) describing a finite boost in z-direction and its infinitesimal generator (K_z),

(c)(5 points) describing a finite rotation around the x-direction and its infinitesimal generator (J_x) and

(d)(5 points) describing a finite rotation around the y-direction and its infinitesimal generator (J_y).

Derive the transformation matrices $S(a)$

(e)(15 points) describing a finite and infinitesimal boost in y-direction and

(f)(15 points) describing a finite and infinitesimal rotation around the x-axis.

(g)(10 bonus points) Show, that the infinitesimal generators of the Lorentz transformation for boosts (K_i) and rotations (J_i) satisfy the algebra relations

$$
\begin{aligned}
[J_i, J_j] &= -i\epsilon_{ijk} J_k \\
[K_i, K_j] &= -i\epsilon_{ijk} J_k \\
[K_i, J_j] &= -i\epsilon_{ijk} K_k.
\end{aligned}
$$