Problem Sheet 1

for the course
„Introduction to Lattice Gauge Theory“
Summer 2019

Lecturer: PD Dr. G. von Hippel

1. *Reading List*

   1. Smit, chapter 2 (p. 8–28)
   2. Gattringer/Lang, chapter 1 (p. 1–23)

2. *Fourier Transform on the Lattice*

   (a) Show that the matrix

   $$U_{mn} = \frac{1}{\sqrt{N}} \mathrm{e}^{ip_m x_n}$$

   with $x_n = na$, $p_m = \frac{2\pi}{La} m$, $m, n \in \{1, \ldots, N\}$, is unitary.

   (b) Conclude that the Fourier transform on the lattice

   $$\Lambda = \{x = an \mid 0 \leq n_\mu < L,\ \mu = 1, \ldots, D\}$$

   is given by

   $$\tilde{\phi}_p = \sum_{x \in \Lambda} \phi_x \mathrm{e}^{-ip.x}$$

   and its inverse by

   $$\phi_x = \frac{1}{L^D} \sum_{p \in \tilde{\Lambda}} \tilde{\phi} \mathrm{e}^{ip.x}\,,$$

   where the reciprocal lattice in momentum space is defined by

   $$\tilde{\Lambda} = \{p = \frac{2\pi}{La} m \mid -L/2 < m_\mu \leq L/2,\ \mu = 1, \ldots, D\}.$$

   (c) Show that in the limit $L \to \infty$ the Fourier transform becomes

   $$\phi(x) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^D p}{(2\pi)^D} \tilde{\phi}(p) \mathrm{e}^{ip.x}\,.$$

3. *The Free Scalar Field on the Lattice*

(a) Show that for periodic boundary conditions $\phi_{x+Le_\mu} = \phi_x$ the identity

$$\sum_{x\in\Lambda} \phi_x \delta_\mu^+ \psi_x = -\sum_x \left(\delta_\mu^- \phi_x\right) \psi_x$$

holds.

(b) Conclude that the Euclidean action for a free scalar field,

$$S = \sum_{x\in\Lambda} \left(\frac{1}{2} \delta_\mu^+ \phi_x \delta_\mu^+ \phi_x + \frac{m^2}{2} \phi_x^2 \right)$$

can be written as

$$S = \frac{1}{2} \sum_{x,y\in\Lambda} \phi_x M_{xy} \phi_y$$

with a matrix M to be determined.

(c) Conclude moreover that the partition function can be written as

$$Z = \int \mathrm{D}\phi \, \mathrm{e}^{-S} = \frac{1}{\sqrt{\det M}} = \mathrm{e}^{-\frac{1}{2}\log\det M} = \mathrm{e}^{-\frac{1}{2}\mathrm{tr}\log M}.$$

(d) Use the generating function

$$Z(J) = \int \mathrm{D}\phi \, \mathrm{e}^{-S + \sum_x J_x \phi_x}$$

and a suitable change of variable to obtain a general expression for the expectation values

$$\langle \phi_{x_1} \cdots \phi_{x_n} \rangle :\equiv \frac{1}{Z} \int \mathrm{D}\phi \, \mathrm{e}^{-S} \phi_{x_1} \cdots \phi_{x_n}.$$

(Hint: use the inverse G of M!)

(e) Visualize this expression for $n = 2, 4, 6$ diagrammatically by sketching a diagram for each summand in which you mark the points x_i and join x_i and x_j by a line if $G_{x_i x_j}$ occurs in the summand.

(f) Use the Fourier transform to obtain an expression for G_{xy}.

(g) Use a suitably chosen contour integral in the complex ($z = \mathrm{e}^{ip_4}$) plane to show that in the the limit $L \to \infty$

$$\sum_{\mathbf{x}} G_{(\mathbf{x},t),(\mathbf{0},0)} \sim \mathrm{e}^{-\widetilde{m}t}, \quad t \to \infty.$$

(h) Interpret the result of the preceding question using the transfer matrix, and conclude that the theory indeed describes free particles of mass $\widetilde{m}$, where $\lim_{a\to 0} \widetilde{m} = m$.

4. *Perturbation Theory and Feynman Diagrams*

   (a) Consider now an expectation value of the form $\langle \phi_{x_1} \cdots \phi_{x_n} \phi^\nu_{z_1} \cdots \phi^\nu_{z_m} \rangle$ in the free theory. In which cases are there how many equivalent summands? For $\nu = 3, 4$ sketch diagrams for the cases $n = 2, 3, 4$, $m = 1, 2$. How can one determine the multiplicity of these diagrams from the diagram itself?

   (b) Consider now the theory of a self-interacting scalar field ϕ with the action $S(\phi) = \frac{1}{2}\phi^t M \phi + \sum_x \frac{\lambda}{\nu!}\phi^\nu_x$, and expand the generating function

$$Z(J) = \int \mathrm{D}\phi \, \mathrm{e}^{-S(\phi) + J^t \phi}$$

   into powers of λ. Give an expression for expectation values in the interacting theory in terms of expectation values in the free theory. Interpret the result in terms of (Feynman-)diagrams using the answer to the preceding question.

   (c) Use the Fourier transform to bring the Feynman diagrams into the momentum-space form familiar from the continuum. What are the crucial differences to the continuum case?

5. *Series Expansion of Integrals*

   (a) Consider the integral

$$Z(m, \lambda) = \int_{-\infty}^{\infty} \mathrm{d}\phi \, \mathrm{e}^{-\frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4}$$

   and use termwise integration to determine its formal expansion in

   i. powers of λ and

   ii. powers of m.

   (b) What can you say based on the coefficients about the convergence and divergence of the respective series?

   (c) Use a computer algebra package to find the exact expression for $Z(m, \lambda)$ and compare the graph of $Z(1, x)$ with the series expansions in x and $1/x$ to first, second, third and fourth order.