Cosmology and General Relativity: HW 1

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Turn in solutions to "Wittig" mailbox in KPH by **noon, 24 Apr 2019**

Problem 1 Space-time diagram: In this problem, consider the simple case of a 1+1 dimensional space-time.

a. Sketch the following in a space-time diagrams (x, t):

  (i) an event,
  (ii) a beam of light,
  (iii) the world line of an object that moves at a velocity $v < 1$,
  (iv) the world line of an object that moves at a velocity $v > 1$,
  (v) the world line of an accelerating object.

b. Consider the space-time diagram (x, t) of a resting observer $\mathcal{O}$. Now add a world line of an observer $\mathcal{O}'$ traveling at velocity v. In this case, v is measured in the stationary reference frame of $\mathcal{O}$. What are the coordinate axes of the $\mathcal{O}'$ space-time diagram?

   Hint: what is the time axis? How would one construct the space-axis?

c. As in the lectures, the measured lengths l and l' of an object in the reference frame $\mathcal{O}$ and $\mathcal{O}'$ are as follows (with light speed $c = 1$):

$$l = \sqrt{1 - v^2}\, l'.$$

   We now want to consider the *ladder-paradox*: consider a car and a garage each with length l at rest. The garage has a front (F) and rear (B) door. The garage opens both doors when the front of the car reaches the front door. Both doors close when the back of the car reaches the front door. The doors open again when the car leaves the garage (i.e., when the front of the car reaches the back of the garage).

   As seen by the garage, the car's length contracts and can thus fit inside. As seen by the car, the garage's length appears to contract and the car will not fit, but be damaged by the doors. Solve this paradox.

   Hint: Consider a space-time diagram with a stationary garage and moving car. In which order do the different events happen for the garage and the car?

Problem 2 Lorentz transformations: In this problem, use the four-dimensional Minkowski space $\mathbb{R}^{1,3}$ with the *Minkowski metric* $\eta = \text{diag}(1, -1, -1, -1)$. The *Lorentz transformation* $x \to \Lambda x$ preserves the invariant line element, i.e., $(x-y)^2 = (\Lambda(x-y))^2$ for $x, y \in \mathbb{R}^{1,3}$. In component notation, this is

$$\eta_{\rho\sigma}\Lambda^{\rho}{}_{\mu}\Lambda^{\sigma}{}_{\nu} = \eta_{\mu\nu}.$$

The *Lorentz group* $O(1,3)$ possesses four connected components that differ by their signs $\text{sgn}(\Lambda^0{}_0) = \pm 1$ and $\det \lambda = \pm 1$. The subgroup $SO^+(1,3)$ with $\Lambda^0{}_0 \geq 1$ and $\det \lambda = 1$ is called the *proper orthochronous Lorentz group.*

a. Write the Lorentz transformation for time- and parity-reversal. How are these related to the four connected components of the Lorentz group?

b. Write down the boosts in the x- and y-direction.

c. Consider only two consecutive boosts. Apply two boosts along the y-axis, then look at a boost along the y-axis and then boost along the x-axis. Write out the resulting boosts and derive a formula for adding relativistic velocities. Do these boosts form a subgroup of the Lorentz group?

d. Show that the the speed of light is constant in all inertial frames.

Problem 3 Electromagnetism: In Gaussian units, the Maxwell equations are

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho, \qquad \vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{j}, \qquad \vec{\nabla} \cdot \vec{B} = 0, \qquad \vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}.$$

The Lorentz covariance can be made explicit by introducing the asymmetric tensors $F^{\mu\nu} = -F^{\nu\mu}$ with

$$F^{i0} = E^i, \qquad F^{ij} = -\sum_{k=1}^{3} \epsilon^{ijk} B^k.$$

a. Show that

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \qquad \text{and,} \qquad \partial^{[\beta} F^{\mu\nu]} = 0 \quad \text{or} \quad \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\partial^\beta F^{\mu\nu} = 0$$

reproduces the Maxwell equations, where $J^\mu = (c\rho, \vec{j})$ and $^{[\cdots]}$ indicates fully anti-symmetric indices.

b. Use the transformation properties of the $F_{\mu\nu}$ tensors to derive the transforming properties of $\vec{E}$ under a boost in the x-direction.

c. Verify that

$$f^\mu \equiv \frac{dp^\mu}{d\tau} = \frac{e}{c} F^\mu{}_\nu \frac{dx^\nu}{d\tau}$$

is the correct equation for the electromagnetic force 4-vector f^μ acting on a particle with charge e and mass m. Use the rest frame of the charged particles. Also show that this equation reproduces the *Lorentz force*, i.e., show that

$$\frac{d\vec{p}}{dt} = e\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right) .$$

Useful definitions and identities:

Levi-Civiti symbol $\epsilon_{\mu_0\mu_1\cdots\mu_{n-1}}$ where $\epsilon_{0\ 1\cdots(n-1)} = +1$, switching any two indices negates the symbol, cyclic permutations do nothing, and $\epsilon_{\mu_0\mu_1\cdots\mu_{n-1}} = 0$ if any indices are identical. For example:

$$\epsilon_{00} = \epsilon_{11} = 0,\ \epsilon_{01} = +1,\ \text{and}\ \epsilon_{10} = -1\ .$$

Note that the indices are raised using the metric $g^{\mu\nu}$.

A useful identity (in 3 dimensions):

$$\epsilon_{ij\ell}\epsilon^{ijk} = 2\delta^k_\ell\ .$$

The Levi-Civiti symbol can be used to define the curl (in 3 dimensions):

$$\left(\vec{\nabla} \times \vec{B}\right)_k = \epsilon_{ijk}\partial_i B_j\ .$$

Anti-symmetric indices: This is simply the condition that switching any two indices can be done with negating the tensor. By convention,

$$T^{[\mu_1\cdots\mu_n]} \equiv \frac{1}{n!}\epsilon_{\mu_1\cdots\mu_n} \sum_{\sigma\in\text{perm.}} T^{\mu_{\sigma(1)}\cdots\mu_{\sigma(n)}}\ ,$$

where the sum is over permutations σ which bijectively map all the indices to new indices. For example,

$$T^{[\mu\nu\lambda]} = \frac{1}{3!}\left(T^{\mu\nu\lambda} + T^{\nu\lambda\mu} + T^{\lambda\mu\nu} - T^{\mu\lambda\nu} - T^{\nu\mu\lambda} - T^{\lambda\nu\mu}\right)\ .$$

The resulting tensor is fully antisymmetric, e.g., $T^{[012]} = -T^{[102]}$ and $T^{[112]} = 0$.