I. KLEIN-GORDON FIELD

1) INTRODUCTION AND MOTIVATION

2) ELEMENTS OF CLASSICAL FIELD THEORY

3) QUANTIZATION OF KLEIN-GORDON FIELD

4) FEYNMAN PROPAGATOR FOR KLEIN-GORDON FIELD

1) INTRODUCTION AND MOTIVATION

⇒ **SPECIAL RELATIVITY**

↳ SPACE-TIME POINT DENOTED BY

$$x^\mu(t, \bar{x}) \quad \text{"CONTRAVARIANT"}$$

NOTATION:	THROUGHOUT THESE NOTES WE USE NATURAL UNITS $\hbar = c = 1$

OR $x_\mu(t, -\bar{x})$ "COVARIANT"

CONNECTED BY

$$x_\mu = g_{\mu\nu} x^\nu \quad \text{OR} \quad x^\mu = g^{\mu\nu} x_\nu$$

FOR MINKOWSKI SPACE-TIME (FLAT SPACE-TIME)

WE USE METRIC

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

↳ RELATIVISTIC ENERGY-MOMENTUM : 4-MOMENTUM

$$p^\mu(E, \bar{p})$$

$$p^2 \equiv p^\mu p_\mu = E^2 - \bar{p}^2 = m^2 \iff E^2 = \bar{p}^2 + m^2$$

m ↳ REST MASS OF PARTICLE

p^2 IS LORENTZ INVARIANT

⇒ RELATIVISTIC QUANTUM MECHANICS

↳ NON-RELATIVISTIC : SCHRÖDINGER EQ.

$$H\,\Psi(\bar{x},t) = i\,\frac{\partial \Psi}{\partial t}$$

DESCRIBES QUANTUM MECHANICAL PARTICLE

$$H = T + V = -\frac{1}{2m}\nabla^2 + V$$

↳ RELATIVISTIC EQUATION SPIN 0 : KLEIN-GORDON

$$\boxed{(\partial_\mu \partial^\mu + m^2)\,\phi(x) = 0}$$

WITH x STANDS FOR x^μ

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \bar{\nabla}\right)$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\bar{\nabla}\right)$$

$$\partial_\mu \partial^\mu \equiv \Box = \frac{\partial^2}{\partial t^2} - \nabla^2 \quad \text{(D' ALEMBERTIAN)}$$

FOR FREE PARTICLE : SOLUTIONS → PLANE WAVES

$$\phi(x) = C\,e^{-i p.x} \quad \text{WITH } p.x = p_\mu x^\mu$$

$$\downarrow$$

$$p^2 = m^2$$

↳ RELATIVISTIC EQUATION SPIN 1/2 : DIRAC

FOR FREE PARTICLE :

$$\left(i \gamma^\mu \partial_\mu - m \right) \psi(x) = 0$$

WITH γ^μ : DIRAC MATRICES (4×4 MATRICES)

$$\gamma^\mu = (\gamma^0, \bar{\gamma}) \qquad \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}$$

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix} \qquad \bar{\gamma} = \begin{pmatrix} 0 & \bar{\sigma} \\ -\bar{\sigma} & 0 \end{pmatrix}$$

$$\bar{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$$

PAULI MATRICES

AND $\psi(x)$: 4-COMPONENT SPINOR

⇝ POSITIVE ENERGY SOLUTIONS

$$\psi_+(x) = C \, e^{-i p . x} \, u(\bar{p}, s) \qquad s = \pm \frac{1}{2}$$

↳ $(\not{p} - m) \, u(\bar{p}, s) = 0$

$$\Rightarrow u(\bar{p}, s) = \sqrt{E+m} \begin{pmatrix} \chi_s \\ \dfrac{\bar{\sigma} . \bar{p}}{E+m} \chi_s \end{pmatrix}$$

FOR SPIN QUATIZED ALONG Z-AXIS

$$\chi_{+\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

NORMALIZATION $\bar{u}(\bar{p}, s) \, u(\bar{p}, s') = 2m \, \delta_{ss'}$

⇝ NEGATIVE ENERGY SOLUTIONS.

$$\psi_-(x) = C e^{+ip\cdot x} v(\bar{p}, s) \quad \ldots \quad s = \pm \frac{1}{2}$$

↳ $(\not{p} + m) v(\bar{p}, s) = 0$

$$\Rightarrow v(\bar{p}, s) = \sqrt{E+m} \begin{pmatrix} \frac{\bar{\sigma}\cdot\bar{p}}{E+m} \chi'_s \\ \chi'_s \end{pmatrix}$$

DIRAC INTERPRETED NEGATIVE-ENERGY SOLUTIONS AS ANTI-PARTICLES $e^- \leftrightarrow e^+$ WHICH WERE THEREAFTER CONFIRMED EXPERIMENTALLY

IF ENOUGH ENERGY IS AVAILABLE $E > 2m$

$e^- e^+$ PAIR CAN BE CREATED

↳ LIMITATION OF EQUATION DESCRIBING SINGLE PARTICLES

⇒ PARTICLE CREATION / ANNIHILATION

↳ e.g. KLEIN-GORDON EQ.

DESCRIBES FREE PARTICLES OF MOMENTUM $\bar{p}$

ENERGY $E = \sqrt{\bar{p}^2 + m^2}$

$$\phi(x) = \sum_{\bar{p}} \left[a(\bar{p})\, e^{-ip.x} + a^*(\bar{p})\, e^{+ip.x} \right]$$

ϕ : REAL , $a(\bar{p}) \in \mathbb{C}$ (COMPLEX)

$a(\bar{p})$ DESCRIBES WEIGHT OF MODE $\bar{p}$ IN SOLUTION $\phi(x)$

↳ SECOND QUANTIZATION

$$\left.\begin{array}{l} a(\bar{p}) \Rightarrow \hat{a}(\bar{p}) \\ a^*(\bar{p}) \Rightarrow \hat{a}^+(\bar{p}) \end{array}\right\} \text{OPERATORS.}$$

$\hat{a}(\bar{p})$ DESCRIBES ANNIHILATION OF PARTICLE WITH MOMENTUM $\bar{p}$

$\hat{a}^+(\bar{p})$ DESCRIBES CREATION OF PARTICLE WITH MOMENTUM $\bar{p}$

$\hat{a}(\bar{p})$ & $\hat{a}^+(\bar{p})$ SATISFY COMMUTATION RELATIONS

(SEE FURTHER)

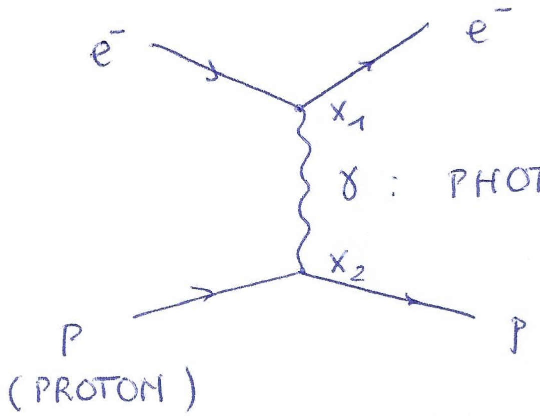
⇒ RELATIVISTIC QUANTUM FIELD THEORY

$\phi(x)$: WILL THEN ALSO BE INTERPRETED AS FIELD OPERATOR WHICH CAN CREATE / ANNIHILATE PARTICLES IN SPACE-TIME POINT x.

IMPORTANCE :

↳ DESCRIPTION OF SCATTERING PROCESSES.

e.g. $e^- p$ SCATTERING



γ : PHOTON ⤳ COULOMB INTERACTION.

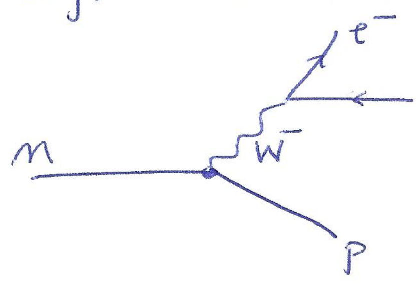
in MODERN PHYSICS :

INTERACTIONS = EXCHANGE OF "FORCE-CARRYING" PARTICLES

1) ELECTROMAGNETIC : PHOTON (MASSLESS)
↳ LONG RANGE

2) WEAK INTERACTIONS : W^+, W^-, Z GAUGE BOSONS (MASSIVE)
↳ SHORT RANGE

e.g. NEUTRON β-DECAY



$\bar{\nu}_e$ (ANTI-NEUTRINO)

3) STRONG INTERACTIONS

MATTER : SPIN 1/2 QUARKS

UP u $\quad e_u = +\frac{2}{3}e$

DOWN d $\quad e_d = -\frac{1}{3}e$

e : PROTON CHARGE

+ 4 HEAVIER s, c, b, t

- PROTON p CONSISTS OF uud

  (BUT ALSO $uud\,u\bar{u}$, $uud\,d\bar{d}$, $uud\,u\bar{u}\,u\bar{u}$, ...).

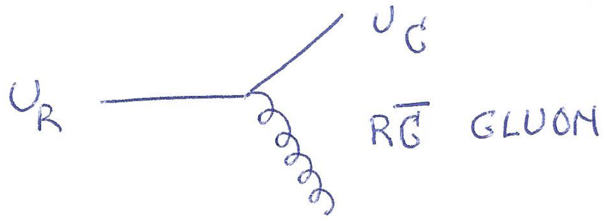
  RELATIVISTIC BOUND STATE SYSTEM STILL NOT SOLVED !

- QUARKS COME IN 3 COLORS (RED, GREEN, BLUE)

  u_R, u_G, u_B

FORCE CARRIERS : SPIN 1 MASSLESS GLUONS

COLOR IS CONSERVED



8 GLUONS $\quad R\bar{G}, R\bar{B}, G\bar{R}, G\bar{B}, B\bar{R}, B\bar{G},$

$R\bar{R} - G\bar{G}, \quad R\bar{R} + G\bar{G} - 2B\bar{B}$

4) GRAVITATIONAL : GRAVITON (SPIN 2) IS EXCHANGED BETWEEN 2 MASSES

↳ DESCRIPTION OF QUANTUM CORRECTIONS

e.g. e^- ANOMALOUS MAGNETIC MOMENT

$$\bar{\mu} = \frac{e}{2m} \cdot g \cdot \bar{S} \qquad \bar{S} = \frac{\bar{\sigma}}{2} \text{ SPIN VECTOR}$$

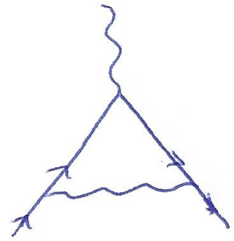
DIRAC THEORY g-FACTOR $g = 2$

QUANTUM FIELD THEORY LEADS TO CORRECTIONS

$$g = 2(1 + a)$$

↑ ANOMALOUS MAGN. MOMENT

- 1° ORDER CORRECTION TO e^- a



1 LOOP DIAGRAM

$$a^{(1)} = \frac{\alpha}{2\pi} \quad \text{(SCHWINGER) 1948}$$

$$\simeq 0.00116$$

$$\simeq 1\ 10^{-3}$$

$$\alpha = \frac{e^2}{4\pi} \simeq \frac{1}{137}$$

FINE-STRUCTURE CONSTANT

- a is CALCULATED NOWADAYS UP TO 5th ORDER α^5 (5 LOOPS !)

THEORY PRECISION ON a : $\sim 5\ 10^{-10}$!

EXP. " " " : $\sim 6\ 10^{-10}$

2) ELEMENTS OF CLASSICAL FIELD THEORY

⇒ LAGRANGIAN FIELD THEORY

↳ SYSTEM SPECIFIED BY FIELDS $\phi_r(x)$

x DENOTES x^μ $(t, \bar{x})$

r DENOTES COMPONENTS OF FIELD

↳ SCALAR FIELD $(r = 1)$ $\phi(x)$

e.g. TEMPERATURE AT GIVEN SPACE-TIME POINT

↳ VECTOR FIELD $(r = 1,2,3,4)$ $A^\mu(x)$

e.g. ELECTROMAGNETIC 4-VECTOR POTENTIAL

↳ DYNAMICS DESCRIBED BY LAGRANGIAN DENSITY

$$\mathcal{L}(\phi_r, \partial_\mu \phi_r)$$

LAGRANGIAN $L \equiv \int d^3\bar{x}\ \mathcal{L}$

THROUGH VARIATIONAL PRINCIPLE

ACTION $$S = \int_\Omega d^4x\ \mathcal{L}(\phi_r, \partial_\mu \phi_r).$$

CONSIDER VARIATIONS $\delta\phi_r$ OF ϕ_r

SUCH THAT $\delta\phi_r = 0$ ON BOUNDARY $\Gamma(\Omega)$

REQUIRE THAT S IS MINIMIZED $\delta S = 0$

$\Downarrow$

FIELD EQUATIONS (EULER-LAGRANGE EQUATIONS)

FOR ϕ_r

UNDER VARIATIONS $\delta\phi_r$

$$\delta S = \int_\Omega d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_r} \delta\phi_r + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \delta(\partial_\mu \phi_r) \right\}$$

$$= \int_\Omega d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \right) \right\} (\delta\phi_r)$$

$$+ \underbrace{\int_\Omega d^4x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \delta\phi_r \right)}_{\text{BOUNDARY TERM}}$$

BOUNDARY TERM

VANISHES BECAUSE $\delta\phi_r = 0$ ON $\Gamma(\Omega)$

$\therefore$ BY REQUIRING THAT

$\delta S = 0$ FOR ARBITRARY $\delta\phi_r$

$\Updownarrow$

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \right) = 0}$$

EULER-LAGRANGE EQUATIONS

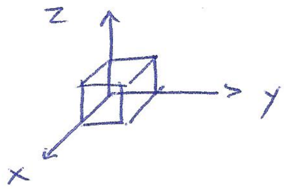
⇒ HAMILTONIAN FIELD THEORY

↳ CONJUGATE MOMENTA :

- $\phi_r(t, \bar{x})$ FIELD

  DIVIDE SPACE IN CELLS (DISCRETIZATION)

  POSITION OF CELL : $i = 1, 2, 3, 4, \dots$



  GENERALIZED COORDINATES

$$q_{r\,i}(t) \equiv \phi_r(t, i) \qquad i = 1, 2, 3, 4, \dots$$

- LAGRANGIAN (DISCRETIZED)

$$L = \int d^3\bar{x} \;\; \mathcal{L}$$

$$= \sum_i \underset{\substack{\uparrow \\ \text{VOLUME ELEMENT}}}{(\delta \bar{x}_i)} \cdot \mathcal{L}\left(\phi_r(t,i), \dot{\phi}_r(t,i)\right)$$

- CONJUGATE MOMENTA

$$p_{r\,i}(t) = \frac{\partial L}{\partial \dot{q}_{r,i}} = \frac{\partial L}{\partial \dot{\phi}_r(t,i)}$$

$$\equiv (\delta \bar{x}_i) \cdot \pi_r(t, i)$$

$$\pi_r(t, i) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r(t,i)}$$

↳ HAMILTONIAN

$$H = \sum_r \sum_i p_{ri} \dot{q}_{ri} - L$$

$$= \sum_i (\delta \bar{x}_i) \left\{ \sum_r \Pi_r(t,i) \dot{\phi}_r(t,i) - \mathcal{L} \right\}$$

↓ CONTINUUM

$$\equiv \int d^3\bar{x} \quad \mathcal{H}(\phi_r(x), \Pi_r(x))$$

FIELD THEORY : CONTINUUM LIMIT $\delta \bar{x}_i \to 0$

$$\boxed{\Pi_r(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r}}$$

$$L(t) = \int d^3\bar{x} \quad \mathcal{L}(\phi_r, \partial_\mu \phi_r)$$

$$H(t) = \int d^3\bar{x} \quad \mathcal{H}(\phi_r, \Pi_r)$$

$$\boxed{\mathcal{H}(x) = \Pi_r(x) \dot{\phi}_r(x) - \mathcal{L}}$$

↳ HAMILTONIAN DENSITY

$\Longrightarrow$ EXAMPLE : CLASSICAL KLEIN-GORDON FIELD

$\hookrightarrow$ $$\underline{\underline{\mathcal{L}_{KG} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2}}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

$\hookrightarrow$ E-L EQUATIONS : $(\partial_\mu \partial^\mu + m^2)\phi = 0$

KLEIN-GORDON EQ.

$\hookrightarrow$ CONJUGATE MOMENTUM

$$\underline{\underline{\Pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}}}$$

$\hookrightarrow$ HAMILTONIAN

$$\underline{\underline{\mathcal{H}_{KG} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2}}$$

⇒ NOETHER'S THEOREM

↳ CONNECTION BETWEEN SYMMETRIES & CONSERVATION LAWS

CONSIDER CONTINUOUS TRANSFORMATION OF FIELDS

$$\phi_r(x) \longrightarrow \phi_r'(x) = \phi_r(x) + \delta\phi_r(x)$$

THIS TF. INDUCES A CHANGE $\delta\mathcal{L}$ TO $\mathcal{L}$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_r}\,\delta\phi_r + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\,\delta(\partial_\mu\phi_r)$$

↓ E-L EQUATION

$$= \left(\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\right)\delta\phi_r + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\,\delta(\partial_\mu\phi_r)$$

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\,\delta\phi_r\right)$$

THE ABOVE TF. IS SYMMETRY TRANSFORMATION IF IT LEAVES ACTION INVARIANT $\delta S = 0$

$\Updownarrow$

LAGRANGIAN IS INVARIANT UP TO A 4-DIVERGENCE

$$\mathcal{L} \longrightarrow \mathcal{L}' = \mathcal{L} + \partial_\mu \mathcal{J}^\mu$$

$$\therefore \quad \partial_\mu \mathcal{J}^\mu = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\,\delta\phi_r\right)$$

$$\Downarrow$$

$$\underline{\underline{\partial_\mu j^\mu = 0}} \qquad \text{FOR} \quad \underline{\underline{j^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \delta\phi_r - \mathcal{J}^\mu}}$$

$$\Downarrow$$

CONSERVED CURRENT j^μ

$$\Downarrow$$

CONSERVED CHARGE (CONSTANT IN TIME)

$$\underline{\underline{Q \equiv \int d^3\bar{x} \;\; j^0(t, \bar{x})}}$$

$$Q = \int d^3\bar{x} \left\{ \Pi_r(x) \, \delta\phi_r(x) - \mathcal{J}^0 \right\}$$

↳ <u>EXAMPLE</u> : SYMMETRY UNDER SPACE-TIME TRANSLATION

$$x^\mu \to x'^\mu = x^\mu + a^\mu$$

↑ CONSTANT TRANSLATION

$$\phi'_r(x') = \phi_r(x)$$

TF. FIELD AT SHIFTED POSITION = ORIGINAL FIELD AT ORIGINAL POSITION

$$\begin{aligned}\delta\phi_r(x) &= \phi'_r(x) - \phi_r(x) \\ &= \phi'_r(x) - \phi'_r(x+a) \\ &= -(\partial_\mu \phi_r)\, a^\mu\end{aligned}$$

$\mathcal{L}$ IS SCALAR $\Rightarrow$ MUST TRANSFORM IN SAME WAY

$$\underline{\underline{\delta\mathcal{L} = -a^\mu(\partial_\mu \mathcal{L})}}$$

EXPLICIT PROOF: $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2$

$$\phi \to \phi' = \phi - a^\nu(\partial_\nu \phi)$$

$$\partial_\mu \phi \longrightarrow \partial_\mu \phi - a^\nu(\partial_\mu \partial_\nu \phi)$$

$$\mathcal{L} \to \mathcal{L} - a^\nu(\partial_\mu \partial_\nu \phi)(\partial^\mu \phi)$$
$$- a^\nu m^2 (\partial_\nu \phi)\phi + O(a^2)$$

$$\delta\mathcal{L} \stackrel{!}{=} -a^\nu(\partial_\nu \mathcal{L})$$

$$\therefore \quad J^\mu = -a^\mu \mathcal{L}$$

$\hookrightarrow$ CONSERVED CURRENT

$$j^\mu \equiv -(\partial^\mu \phi_r)\, a_\nu (\partial^\nu \phi_r) + a_\nu g^{\mu\nu}\mathcal{L}$$

$$\partial_\mu j^\mu = 0 \iff \partial_\mu\left[(\partial^\mu \phi_r)(\partial^\nu \phi_r) - g^{\mu\nu}\mathcal{L}\right]a_\nu$$
$$= 0 \quad \underline{\underline{\forall a_\nu}}$$

$$\Downarrow$$

$$\partial_\mu T^{\mu\nu} = 0$$

WITH
$$\boxed{T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} (\partial^\nu \phi_r) - g^{\mu\nu} \mathcal{L}}$$

↳ CONSERVED ENERGY-MOMENTUM TENSOR

$$T^{\mu\nu} = (\partial^\mu \phi_r)(\partial^\nu \phi_r) - g^{\mu\nu} \mathcal{L}$$

↳ CONSERVED CHARGE

4-MOMENTUM

$$\boxed{\begin{aligned} P^\nu &\equiv \int d^3\bar{x}\; T^{0\nu} \\ &= \int d^3\bar{x} \left\{ \Pi_r(x)\, \partial^\nu \phi_r(x) - g^{0\nu} \mathcal{L} \right\} \end{aligned}}$$

$$\begin{aligned} P^0 &= \int d^3\bar{x} \left\{ \Pi_r(x)\, \dot{\phi}_r(x) - \mathcal{L} \right\} \\ &= \int d^3\bar{x}\; \mathcal{H}(x) = H \end{aligned}$$

(ENERGY)

$$\bar{P} = - \int d^3\bar{x}\; \Pi_r(x)\, \bar{\nabla} \phi_r(x)$$

(3-MOMENTUM) CARRIED BY FIELD

3) QUANTIZATION OF KLEIN - GORDON FIELD

⇒ CANONICAL COMMUTATION RELATIONS

↳ IN QUANTUM THEORY:

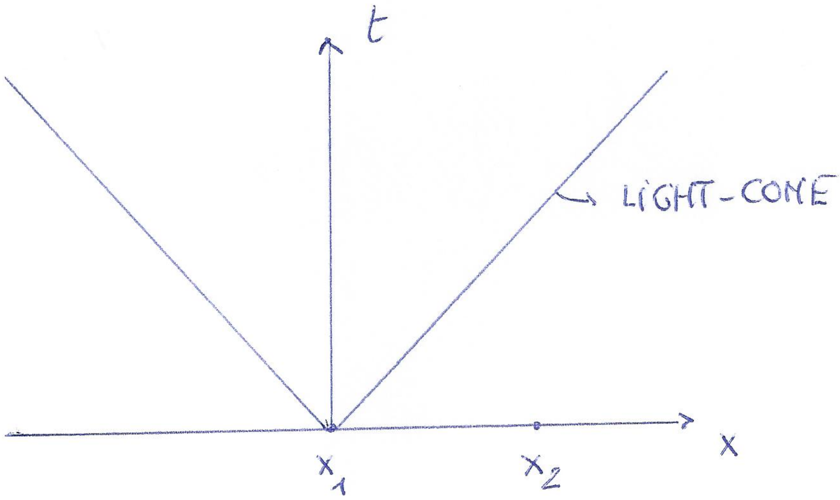
WE POSTULATE EQUAL TIME COMMUTATION RELATIONS (ETCR)

FOR FIELDS & CONJUGATE MOMENTA:

$$[\phi_r(\bar{x},t),\ \phi_s(\bar{x}',t)] = 0$$

$$[\pi_r(\bar{x},t),\ \pi_s(\bar{x}',t)] = 0$$

$$[\phi_r(\bar{x},t),\ \pi_s(\bar{x}',t)] = i\delta_{rs}\,\delta^3(\bar{x}-\bar{x}')$$



FIELDS AT SPACELIKE SEPARATED POINTS $(x_1 - x_2)^2 < 0$ CANNOT INFLUENCE EACH OTHER (OUTSIDE EACH OTHER'S LIGHT CONE)

↳ COMMUTATORS VANISH

↳ FOR REAL KLEIN-GORDON FIELD

(DESCRIBING NEUTRAL SPIN-0 PARTICLE)

$$[\phi(\bar{x},t), \phi(\bar{x}',t)] = 0$$

$$[\dot{\phi}(\bar{x},t), \dot{\phi}(\bar{x}',t)] = 0$$

$$[\phi(\bar{x},t), \dot{\phi}(\bar{x}',t)] = i\,\delta^3(\bar{x}-\bar{x}')$$

⇒ NORMAL MODE EXPANSION FOR FIELD ϕ $(\phi = \phi^+)$

↳ $$\boxed{\phi(\bar{x},t) = \int \frac{d^3\bar{k}}{(2\pi)^3}\, N(\bar{k}) \left\{ a(\bar{k})\, e^{-ik.x} + a^+(\bar{k})\, e^{+ik.x} \right\}}$$

NORMAL MODE $\bar{k}$ (CONTINUOUS MOMENTUM)

SOLUTIONS OF K-G EQUATION : $k^2 = m^2$

$$\Downarrow$$

$$k^0 = \sqrt{\bar{k}^2 + m^2} \equiv E_{\bar{k}}$$

$a(\bar{k})$: ANNIHILATES A PARTICLE OF MOM. $\bar{k}$ & ENERGY $E_{\bar{k}}$

$a^+(\bar{k})$: CREATES A PARTICLE OF MOM. $\bar{k}$ & ENERGY $E_{\bar{k}}$

$N(\bar{k})$: CONVENIENTLY CHOSEN NORMALIZATION FACTOR

$\hookrightarrow \quad i\dot{\phi}(\bar{x},t) = \int \frac{d^3\bar{k}}{(2\pi)^3} \, N(\bar{k}) \, E_{\bar{k}} \left\{ a(\bar{k}) . e^{-ik.x} - a^+(\bar{k}) \, e^{+ikx} \right\}$

$\hookrightarrow \quad \int d^3\bar{x} \; e^{-i\bar{k}'.\bar{x}} \; \phi(\bar{x},t)$

$$= \int \frac{d^3\bar{k}}{(2\pi)^3} N(\bar{k}) \left\{ a(\bar{k}) \, e^{-iE_{\bar{k}}t} \underbrace{\int d^3\bar{x} \; e^{i(\bar{k}-\bar{k}').\bar{x}}}_{(2\pi)^3 \delta^3(\bar{k}-\bar{k}')} + a^+(\bar{k}) \, e^{+iE_{\bar{k}}t} \underbrace{\int d^3\bar{x} \; e^{-i(\bar{k}+\bar{k}').\bar{x}}}_{(2\pi)^3 \delta^3(\bar{k}+\bar{k}')} \right\}$$

$$= N(\bar{k}') \, a(\bar{k}') \, e^{-iE_{\bar{k}'}t} + N(-\bar{k}') \, a^+(-\bar{k}') \, e^{+iE_{\bar{k}'}t}$$

$\downarrow \quad E_{-\bar{k}'} = E_{\bar{k}'}$

CHOOSE NORMALIZATION SUCH THAT $N(-\bar{k}') = N(\bar{k}')$

$$= N(\bar{k}') \left\{ a(\bar{k}') \, e^{-iE_{\bar{k}'}t} + a^+(-\bar{k}') \, e^{+iE_{\bar{k}'}t} \right\}$$

$\hookrightarrow$ ANALOGOUSLY

$\int d^3\bar{x} \; e^{-i\bar{k}'.\bar{x}} \; i\dot{\phi}(\bar{x},t)$

$$= N(\bar{k}') \, E_{\bar{k}'} \left\{ a(\bar{k}') \, e^{-iE_{\bar{k}'}t} - a^+(-\bar{k}') \, e^{+iE_{\bar{k}'}t} \right\}$$

$$\hookrightarrow \int d^3\bar{x}\; e^{-i\bar{k}.\bar{x}} \left\{ E_{\bar{k}}\, \phi + i\, \dot{\phi} \right\}$$

$$= N(\bar{k})\; 2E_{\bar{k}}\; a(\bar{k})\, e^{-iE_{\bar{k}} t}$$

$$\int d^3\bar{x}\; e^{-i\bar{k}.\bar{x}} \left\{ E_{\bar{k}}\, \phi - i\, \dot{\phi} \right\}$$

$$= N(\bar{k})\; 2E_{\bar{k}}\; a^{+}(-\bar{k})\; e^{+iE_{\bar{k}} t}$$

IN 2° EQUATION WE CHANGE $\bar{k} \rightarrow -\bar{k}$

$\Downarrow$

$$\boxed{a(\bar{k}) = \frac{1}{2E_{\bar{k}}\, N(\bar{k})} \int d^3\bar{x}\; e^{ik.x} \left\{ E_{\bar{k}}\, \phi + i\dot{\phi} \right\}}$$

WITH $k^0 = E_{\bar{k}}$

$$\boxed{a^{+}(\bar{k}) = \frac{1}{2E_{\bar{k}}\, N(\bar{k})} \int d^3\bar{x}\; e^{-ik.x} \left\{ E_{\bar{k}}\, \phi - i\dot{\phi} \right\}}$$

WITH $k^0 = E_{\bar{k}}$

$\Rightarrow$ COMMUTATION RELATIONS FOR a, a^+

$\hookrightarrow$ $\left[a(\bar{k}),\, a^+(\bar{k}')\right] = a(\bar{k})a^+(\bar{k}') - a^+(\bar{k}')a(\bar{k})$

$$= \frac{1}{2E_{\bar{k}}\,N(\bar{k})}\,\frac{1}{2E_{\bar{k}'}\,N(\bar{k}')}$$

$$\cdot \int d^3\bar{x} \int d^3\bar{x}'\; e^{ik.x}\, e^{-ik'.x'}$$ WITH $\underline{x'^0 = x^0}$

$$\underbrace{\left[E_{\bar{k}}\,\phi(\bar{x},t) + i\dot{\phi}(\bar{x},t),\; E_{\bar{k}'}\,\phi(\bar{x}',t) - i\dot{\phi}(\bar{x}',t)\right]}$$

APPLY ETCR's

ONLY $\left[\phi(\bar{x},t),\, \dot{\phi}(\bar{x}',t)\right]$

IS NON-VANISHING

$$E_{\bar{k}}\,\delta^3(\bar{x}-\bar{x}') + E_{\bar{k}'}\,\delta^3(\bar{x}-\bar{x}')$$

$$= \frac{1}{2E_{\bar{k}}\,N(\bar{k})}\,\frac{1}{2E_{\bar{k}'}\,N(\bar{k}')}$$

$$\cdot \int d^3\bar{x}\; e^{i(E_{\bar{k}} - E_{\bar{k}'})t} \cdot (E_{\bar{k}} + E_{\bar{k}'})\, \underbrace{e^{-i(\bar{k}-\bar{k}').\bar{x}}}_{\downarrow}$$

$$(2\pi)^3\,\delta^3(\bar{k}-\bar{k}')$$

$$\left[a(\bar{k}), a^{+}(\bar{k}') \right] = \frac{1}{2E_{\bar{k}}(N(\bar{k}))^2} (2\pi)^3 \delta^3(\bar{k} - \bar{k}')$$

WE CAN CHOOSE (COVARIANT NORMALIZATION CONVENTION)

$$\boxed{N(\bar{k}) = \frac{1}{\sqrt{2E_{\bar{k}}}}}$$

$$\left[a(\bar{k}), a^{+}(\bar{k}') \right] = (2\pi)^3 \delta^3(\bar{k} - \bar{k}')$$

$$\left[a(\bar{k}), a(\bar{k}') \right] = 0$$

$$\left[a^{+}(\bar{k}), a^{+}(\bar{k}') \right] = 0$$

ANALOGOUS RELATIONS
AS FOR HARMONIC OSCILLATOR

$\Rightarrow$ HAMILTONIAN IN TERMS OF a, a^{+}

$$\hookrightarrow \quad \mathcal{H} = \frac{1}{2}\left\{ \dot{\phi}^2 + (\bar{\nabla}\phi)^2 + m^2\phi^2 \right\}$$

$$H = \int d^3\bar{x} \;\; \mathcal{H}(\bar{x}, t)$$

$\Downarrow$

SHOW THAT (EXERCISE !)

$$\boxed{H = \int \frac{d^3\bar{k}}{(2\pi)^3} \; E_{\bar{k}} \left(a^{+}(\bar{k})\, a(\bar{k}) + \frac{1}{2} \right)}$$

$a^{+}(\bar{k})\, a(\bar{k})$ IS NUMBER OPERATOR

\# PARTICLES WITH MOMENTUM $\bar{k}$

$\Rightarrow$ MOMENTUM IN TERMS OF a, a^{+}

$$\bar{P} = -\int d^3\bar{x} \;\; \dot{\phi}\,(\bar{\nabla}\phi)$$

$\Downarrow$

SHOW THAT (EXERCISE !)

$$\boxed{\bar{P} = \int \frac{d^3\bar{k}}{(2\pi)^3} \; \bar{k} \;\; a^{+}(\bar{k})\, a(\bar{k})}$$

⇒ SINGLE PARTICLE STATES

↳ VACUUM $|0\rangle$ NO PARTICLES.

$a(\bar{k})|0\rangle = 0$

$\langle 0|0\rangle = 1$ NORMALIZATION

↳ SINGLE PARTICLE STATE

$$\boxed{|\bar{k}\rangle \equiv \sqrt{2E_{\bar{k}}}\; a^{+}(\bar{k})\,|0\rangle}$$

$E_{\bar{k}} = \sqrt{\bar{k}^2 + m^2}$

CREATES ONE PARTICLE OF MOMENTUM $\bar{k}$ IN VACUUM

NORMALIZATION CHOSEN SUCH THAT

$$\langle \bar{k}'|\bar{k}\rangle = \sqrt{2E_{\bar{k}'}}\sqrt{2E_{\bar{k}}}\;\langle 0|\,a(\bar{k}')a^{+}(\bar{k})\,|0\rangle$$

$$= \sqrt{2E_{\bar{k}'}}\sqrt{2E_{\bar{k}}}\;\langle 0|\,[a(\bar{k}'),a^{+}(\bar{k})]\,|0\rangle$$

BECAUSE $a(\bar{k}')|0\rangle = 0$

$$\underline{\underline{\langle \bar{k}'|\bar{k}\rangle = (2\pi)^3\, 2E_{\bar{k}}\, \delta^3(\bar{k}-\bar{k}')}}$$

$\hookrightarrow \quad \phi(\bar{x},t)|0\rangle$

$$= \int \frac{d^3\bar{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\bar{k}}}} \left\{ \underbrace{\cancel{a(\bar{k})e^{-ik.x}}}_{0} + a^+(\bar{k})e^{+ik.x} \right\} |0\rangle$$

$$= \int \frac{d^3\bar{k}}{(2\pi)^3} \frac{e^{ik.x}}{2E_{\bar{k}}} |\bar{k}\rangle$$

CREATES ONE-PARTICLE STATE AT POSITION $\bar{x}$

AND TIME t FROM VACUUM

$\hookrightarrow$ WAVE FUNCTION INTERPRETATION AS IN QM

$$\langle \bar{x}|\bar{k}\rangle \sim e^{i\bar{k}.\bar{x}}$$

$$\therefore \quad \langle \bar{k}|\phi(\bar{x},t)|0\rangle = e^{ik.x}$$

$$\langle 0|\phi(\bar{x},t)|\bar{k}\rangle = e^{-ik.x}$$

ANNIHILATES A PARTICLE

AT POSITION $\bar{x}$ & TIME t

FROM SINGLE PARTICLE STATE $|\bar{k}\rangle$

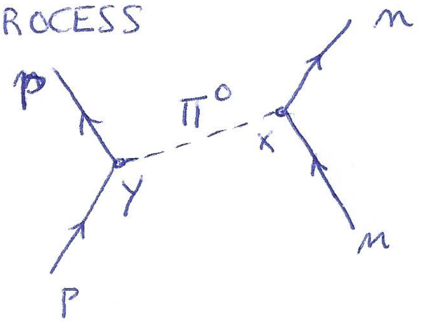
4) FEYNMAN PROPAGATOR FOR KLEIN-GORDON FIELD

⇒ 2-POINT CORRELATION FUNCTION

↳ AMPLITUDE FOR A PARTICLE TO BE CREATED AT POINT y AND ANNIHILATED AT POINT x

AT DIFFERENT $x_0 \neq y_0$

e.g. IN PROCESS



π^0: NEUTRAL PION (SPIN 0)

$$\langle 0 | \phi(x) \, \phi(y) | 0 \rangle$$

ONLY $a^\dagger(\bar{k})$ CONTRIBUTES IN $\phi(y)$

↓ " $a(\bar{k})$ CONTRIBUTES IN $\phi(x)$

$$= \langle 0 | \int \frac{d^3\bar{k}}{(2\pi)^3\sqrt{2E_{\bar{k}}}} \int \frac{d^3\bar{k}'}{(2\pi)^3\sqrt{2E_{\bar{k}'}}} \, e^{-ik.x} \, e^{+ik'.y} \, a(\bar{k}) a^\dagger(\bar{k}') | 0 \rangle$$

↓ $\langle 0 | a(\bar{k}) a^\dagger(\bar{k}') | 0 \rangle = (2\pi)^3 \delta^3(\bar{k} - \bar{k}')$

$$= \int \frac{d^3\bar{k}}{(2\pi)^3 \, 2E_{\bar{k}}} \, e^{-ik.(x-y)} \equiv i\Delta(x-y)$$

WITH $E_{\bar{k}} = \sqrt{\bar{k}^2 + m^2}$

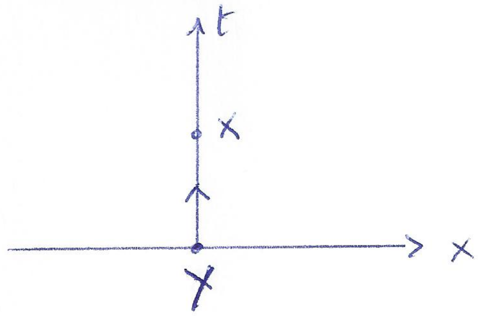
↳ NOTE $\langle 0 | \phi(x)\, \phi(y) | 0 \rangle \equiv i\Delta(x-y)$

IS ONLY A FUNCTION OF DIFFERENCE $x - y$

(TRANSLATIONAL INVARIANCE)

↳ SPECIAL CASES

① $x^0 - y^0 = t \,, \quad \bar{x} = \bar{y}$: PARTICLE AT REST



$$i\Delta(x-y)$$

$$= \int \frac{d^3\bar{k}}{(2\pi)^3 2E_{\bar{k}}}\, e^{-iE_{\bar{k}} t}$$

$$= \frac{1}{4\pi^2} \int_0^\infty d|\bar{k}|\, \frac{|\bar{k}|^2}{E_{\bar{k}}}\, e^{-iE_{\bar{k}} t}$$

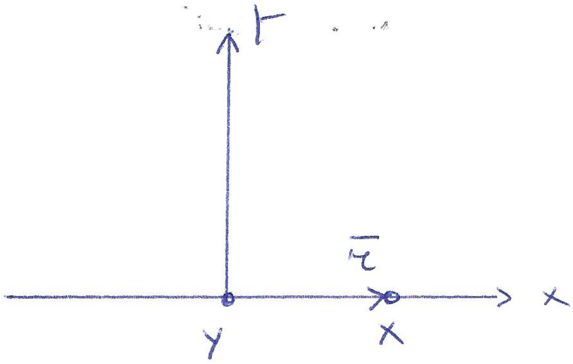
$$E_{\bar{k}} = \sqrt{\bar{k}^2 + m^2} \qquad \frac{dE_{\bar{k}}}{d|\bar{k}|} = \frac{|\bar{k}|}{E_{\bar{k}}}$$

$$= \frac{1}{4\pi^2} \int_m^\infty dE\, \sqrt{E^2 - m^2}\, e^{-iEt}$$

$$\xrightarrow[t\to\infty]{} e^{-imt}$$

⇝ PARTICLE AT REST WITH MASS m PROPAGATING IN TIME

(2) $x^0 = y^0$, $\bar{x} - \bar{y} = \bar{r}$: PURE SPATIAL SEPARATION



$$i\Delta(x-y)$$

$$= \int \frac{d^3\bar{k}}{(2\pi)^3\, 2E_{\bar{k}}}\, e^{i\bar{k}.\bar{r}}$$

$k \equiv |\bar{k}|$

$$= \frac{1}{4\pi^2}\int_0^\infty dk\, \frac{k^2}{2E_k}\int_{-1}^{1} d\cos\Theta\; e^{ikr\cos\Theta}$$

$$= \frac{1}{4\pi^2}\,\frac{1}{ir}\int_0^\infty dk\, \frac{k}{2E_k}\left(e^{ikr} - e^{-ikr}\right)$$

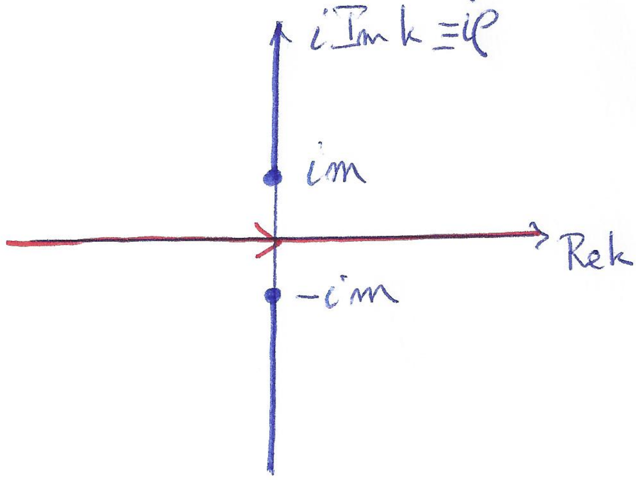
$$= \frac{1}{8\pi^2 ir}\int_{-\infty}^{+\infty} dk\, \frac{k}{\sqrt{k^2+m^2}}\, e^{ikr}$$

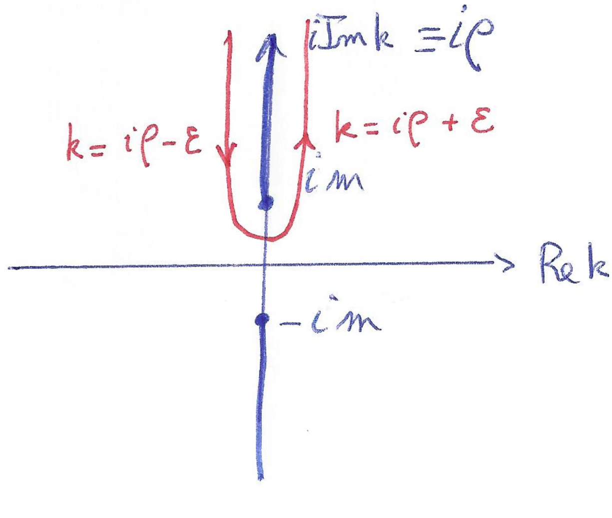
$\dfrac{1}{\sqrt{k^2+m^2}}$ HAS BRANCH CUTS IN COMPLEX k-PLANE

ALONG IM k-AXIS

FOR $k > im$ AND $k < -im$

PERFORM INTEGRAL BY CONTOUR DEFORMATION





$$i\Delta(x-y)\Big|_{x^0=y^0,\ \bar{x}-\bar{y}=\bar{r}}$$

$$= \frac{1}{8\pi^2 i r}\left\{ i^2\int_\infty^m d\rho \frac{\rho\, e^{-\rho r}}{\sqrt{m^2-\rho^2-i\varepsilon\rho}} + i^2\int_m^\infty d\rho \frac{\rho\, e^{-\rho r}}{\sqrt{m^2-\rho^2+i\varepsilon\rho}}\right\}$$

ε INFINITESIMAL

$$\sqrt{m^2-\rho^2-i\varepsilon\rho} = \sqrt{\rho^2-m^2}\ \underbrace{e^{-i\frac{\pi}{2}}}_{-i}$$

$$\sqrt{m^2-\rho^2+i\varepsilon\rho} = \sqrt{\rho^2-m^2}\ \underbrace{e^{+i\frac{\pi}{2}}}_{i}$$

$$= \frac{1}{4\pi^2 r}\int_m^\infty d\rho \frac{\rho\, e^{-\rho r}}{\sqrt{\rho^2-m^2}} \xrightarrow[r\to\infty]{} \frac{e^{-mr}}{r}$$

$\therefore$ WE SEE THAT OUTSIDE LIGHT-CONE CORRELATION IS EXPONENTIALLY VANISHING

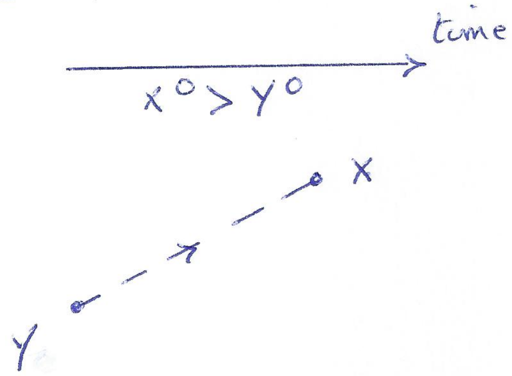
⇒ TIME-ORDERED PRODUCT

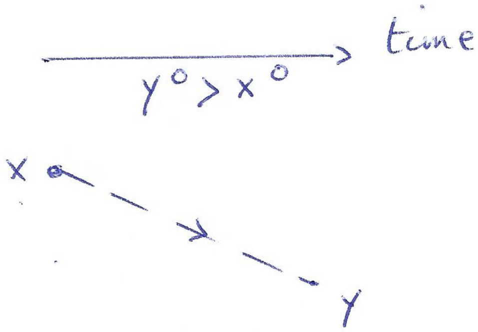
$$T\,\phi(x)\,\phi(y) \equiv \theta(x^0 - y^0)\,\phi(x)\,\phi(y) + \theta(y^0 - x^0)\,\phi(y)\,\phi(x)$$

WITH $\theta(\alpha) = \begin{cases} 1 , & \alpha > 0 \\ 0 , & \alpha < 0 \end{cases}$

TIME-ORDERED PRODUCT :
FIELD WITH LARGEST TIME ARGUMENT STANDS ON LEFT

SCHEMATICALLY :





INTERPRETATION :
PARTICLE PROPAGATING FROM y TO x
(FORWARD IN TIME)

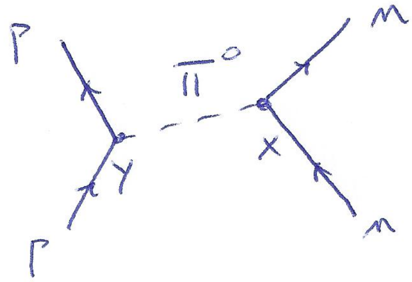
PARTICLE PROPAGATING FROM x TO y

⇒ FEYNMAN PROPAGATOR FUNCTION

$$\hookrightarrow \boxed{\langle 0 | T\, \phi(x)\, \phi(y) | 0 \rangle \equiv i \Delta_F(x-y)}$$

THIS FUNCTION WILL ENTER PHYSICAL PROCESSES.

→ FEYNMAN DIAGRAMS



$$i\Delta_F(x-y) = \theta(x^0-y^0) \langle 0 | \phi(x)\, \phi(y) | 0 \rangle$$

$$+ \theta(y^0-x^0) \langle 0 | \phi(y)\, \phi(x) | 0 \rangle$$

$$= \theta(x^0-y^0)\; i\Delta(x-y)$$

$$+ \theta(y^0-x^0)\; i\Delta(y-x).$$

$$\boxed{i\Delta_F(x-y) = \int \frac{d^3\bar{k}}{(2\pi)^3\, 2E_{\bar{k}}} \left\{ \theta(x^0-y^0)\, e^{-ik\cdot(x-y)} + \theta(y^0-x^0)\, e^{+ik\cdot(x-y)} \right\}}$$

WITH $k^0 = E_{\bar{k}} = \sqrt{\bar{k}^2+m^2}$

↳ PROPAGATOR IN MOMENTUM SPACE

$$\boxed{\Delta_F(x) \equiv \int \frac{d^4k}{(2\pi)^4} \, e^{-ik.x} \, \Delta_F(k^2)}$$

WITH $k^2 = k_\mu k^\mu$

FOURIER TRANSFORM $\Delta_F(k^2)$ IS GIVEN BY

$$\boxed{\Delta_F(k^2) = \frac{1}{k^2 - m^2 + i\varepsilon}}$$

$\varepsilon > 0$ INFINITESIMAL

<u>PROOF</u>: BY CONTOUR DEFORMATION IN COMPLEX k^0 PLANE

$$\begin{aligned} & k^2 - m^2 + i\varepsilon \\ &= (k^0)^2 - (\vec{k}^2 + m^2) + i\varepsilon \\ &= (k^0)^2 - E_{\vec{k}}^2 + i\varepsilon \\ &= (k^0 - E_{\vec{k}} + i\varepsilon)(k^0 + E_{\vec{k}} - i\varepsilon) \end{aligned}$$

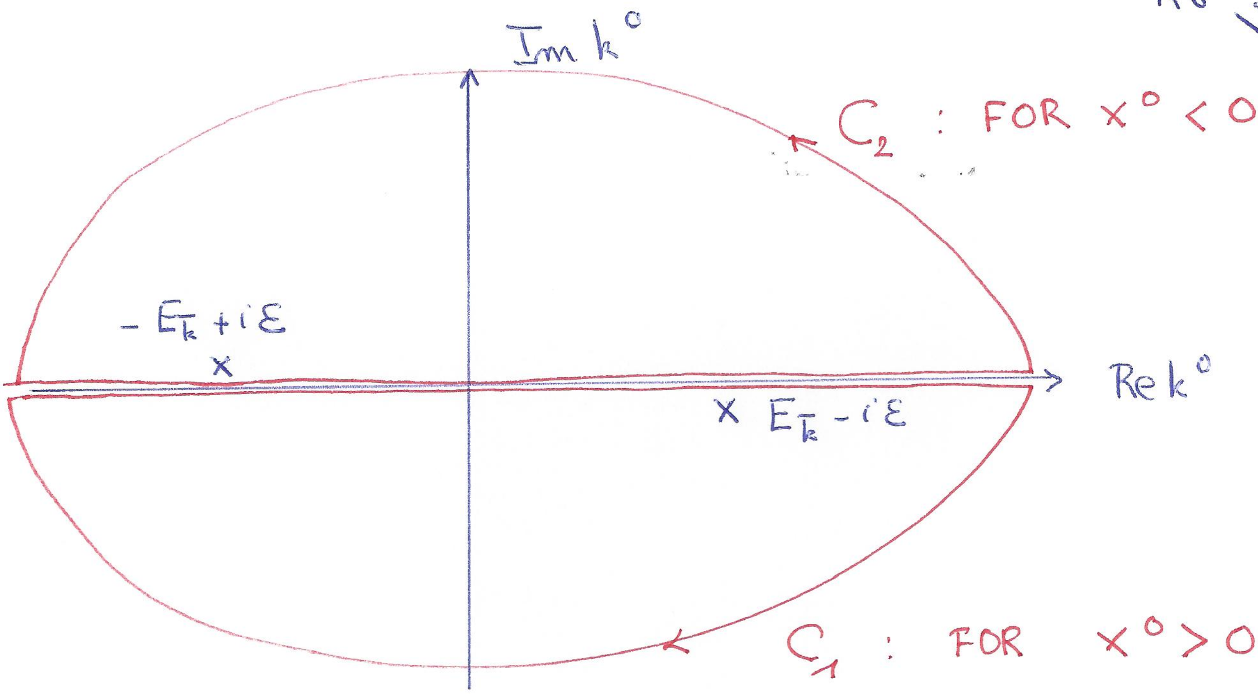
$E_{\vec{k}} = \sqrt{\vec{k}^2 + m^2} > 0$

FOR ε INFINITESIMAL

$\Delta_F(k^2)$ HAS 2 POLES IN COMPLEX k^0 PLANE

↗ $k^0 = E_{\vec{k}} - i\varepsilon$.

↘ $k^0 = -E_{\vec{k}} + i\varepsilon$



- FOR $\underline{x^0 > 0}$

CLOSE CONTOUR IN LOWER HALF-PLANE

$e^{-ik^0x^0}$ REQUIRES $\mathrm{Im}\, k^0 < 0$ IN ORDER TO NEGLECT SEMI-CIRCLE AT INFINITY

$x^0 > 0$:

$$\Delta_F(x) = \int \frac{d^3\bar{k}}{(2\pi)^4} e^{i\bar{k}.\bar{x}} \cdot \int_{-\infty}^{+\infty} dk^0 \frac{e^{-ik^0.x^0}}{(k^0 - E_{\bar{k}} + i\varepsilon)(k^0 + E_{\bar{k}} - i\varepsilon)}$$

$$= \int \frac{d^3\bar{k}}{(2\pi)^4} e^{i\bar{k}.\bar{x}} \underbrace{\int_{C_1} dk^0 \frac{e^{-ik^0x^0}}{(k^0 - E_{\bar{k}} + i\varepsilon)(k^0 + E_{\bar{k}} - i\varepsilon)}}_{\text{RESIDUE THEOREM}}$$

$$-2\pi i \frac{1}{2E_{\bar{k}}} e^{-iE_{\bar{k}}x^0}$$

$$x^0 > 0 \; : \quad \Delta_F(x) = -i \int \frac{d^3\bar{k}}{(2\pi)^3\, 2E_{\bar{k}}}\, e^{-ik.x} \Big|_{k^0 = E_{\bar{k}}}$$

- FOR $x^0 < 0$

CLOSE CONTOUR IN UPPER HALF-PLANE

$$e^{-ik^0x^0} \qquad \mathrm{Im}\, k^0 > 0.$$

$$x^0 < 0 \; : \quad \Delta_F(x) = \int \frac{d^3\bar{k}}{(2\pi)^4}\, e^{i\bar{k}.\bar{x}} \underbrace{\int_{C_2} dk^0 \frac{e^{-ik^0x^0}}{(k^0 - E_{\bar{k}} + i\varepsilon)(k^0 + E_{\bar{k}} - i\varepsilon)}}_{+2\pi i \frac{1}{(-2E_{\bar{k}})} \cdot e^{iE_{\bar{k}}x^0}}$$

$$= -i \int \frac{d^3\bar{k}}{(2\pi)^3\, 2E_{\bar{k}}}\, e^{iE_{\bar{k}}x^0 + i\bar{k}.\bar{x}}$$

$\downarrow$ CHANGE INTEGRATION VARIABLE $\bar{k} \to -\bar{k}$

$$= -i \int \frac{d^3\bar{k}}{(2\pi)^3\, 2E_{\bar{k}}}\, e^{iE_{\bar{k}}x^0 - i\bar{k}.\bar{x}}$$

$$x^0 < 0 \; : \quad \Delta_F(x) = -i \int \frac{d^3\bar{k}}{(2\pi)^3\, 2E_{\bar{k}}}\, e^{ik.x} \Big|_{k^0 = E_{\bar{k}}}$$

- IN TOTAL

$$i\,\Delta_F(x) = \int \frac{d^3\bar{k}}{(2\pi)^3\, 2E_{\bar{k}}} \left\{ \theta(x^0)\, e^{-ik.x} + \theta(-x^0)\, e^{+ik.x} \right\}_{k^0 = E_{\bar{k}}}$$

THIS AGREES WITH EXPRESSION OBTAINED ABOVE WHICH COMPLETES PROOF THAT 4D FOURIER TF IS GIVEN BY $\Delta_F(k^2) = \dfrac{1}{k^2 - m^2 + i\varepsilon}$

↳ FEYNMAN PROPAGATOR AS GREEN'S FUNCTION

$$\Delta_F(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik.x}\, \frac{1}{k^2 - m^2 + i\varepsilon}$$

$$(\Box_x + m^2)\,\Delta_F(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik.x}\, \frac{(-k^2 + m^2)}{k^2 - m^2 + i\varepsilon}$$

$$= - \int \frac{d^4k}{(2\pi)^4}\, e^{-ik.x}$$

$$= - \delta^4(x)$$

$$\boxed{(\Box_x + m^2)\,\Delta_F(x) = - \delta^4(x)}$$

↓

GREEN'S FUNCTION OF KG EQUATION

↓

SOLUTION OF FIELD EQUATION WITH $-\delta^4(x)$ SOURCE TERM