Theoretical Physics 6a (QFT): SS 2019
Exercise sheet 3

29.04.2019

Exercise 1 (100 points): Klein-Gordon Green's function

Consider the differential equation for the Klein-Gordon Green's function:

$$(\partial_\mu \partial^\mu + m^2) G(x - x') = -\delta^4(x - x'). \tag{1}$$

(a)(20 points) Show that, using a Fourier transformation, the solution of Eq. (1) is formally given as

$$G(x - x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{k^2 - m^2 \pm i\epsilon}. \tag{2}$$

Using the Green's function, show that

$$\phi(x) = -\int d^4x G(x - x')\rho(x') \tag{3}$$

is a solution of

$$(\partial_\mu \partial^\mu - m^2)\phi(x) = \rho(x). \tag{4}$$

The expression of Eq. (2) can only be evaluated using a contour deformation for the integration over k^0, due to the poles on the real axes at

$$k^0 = \pm\sqrt{k^2 + m^2} \tag{5}$$

Consider the three contours in figures 1, 2 and 3.

(b)(20 points) Show, that the retarded Green's function given by

$$G_{\text{ret}}(x - x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{(k_0 + i\epsilon)^2 - \vec{k}^2 - m^2},$$

is zero for $t < t'$ and that the advanced Green's function given by

$$G_{\text{adv}}(x - x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{(k_0 - i\epsilon)^2 - \vec{k}^2 - m^2},$$

is zero for $t > t'$.

(c)(20 points) Show, that the Feynman Green's function,

$$G_{\mathrm{F}}(x-x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{k^2 - m^2 + i\epsilon},$$

defines the integration contour of figure 3, i.e. has its poles at

$$k_0 = \pm(\sqrt{k^2 + m^2} - i\epsilon).$$

Show, that after evaluating the contour integration:

$$\begin{aligned} G_{\mathrm{F}}(x-x') &= (-i) \int \frac{d^3k}{(2\pi)^3 2E_k} \left(\Theta(t-t')e^{-ik(x-x')} + \Theta(t'-t)e^{ik(x-x')} \right) \\ &= (-i) \left\langle 0 \,\middle|\, T[\phi(x)\phi(x')] \,\middle|\, 0 \right\rangle . \end{aligned}$$

(d)(20 points) Evaluate the spacelike Klein-Gordon propagator, i.e. for $(x-y)^2 < 0$, explicitly in terms of Bessel functions.

(e)(20 points) Can the Klein-Gordon field be interpreted as a one-particle wave-function in quantum mechanics?

Figure 1: Contour for retarded Green's function with poles at $k_0 = \pm\sqrt{k^2+m^2} - i\epsilon$

Figure 2: Contour for advanced Green's function with poles at $k_0 = \pm\sqrt{k^2+m^2} + i\epsilon$

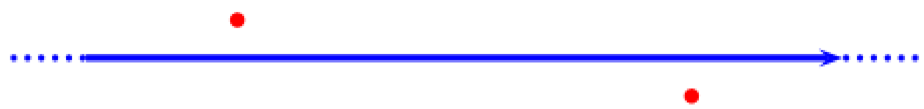

Figure 3: Contour for Feynman Green's function with poles at $k_0 = \pm(\sqrt{k^2+m^2} - i\epsilon)$