Theoretical Physics 6a (QFT): SS 2019
Exercise sheet 2

24.04.2019

Exercise 1 (100 points) : Complex Klein-Gordon field

The complex Klein-Gordon field is used to describe charged bosons. Its Lagrangian is given by

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi, \tag{1}$$

where the field ϕ has the following normal mode expansion

$$\phi(\vec{x}, t) \quad = \quad \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \left[a(\vec{k})\, e^{-ik\cdot x} + b^\dagger(\vec{k})\, e^{ik\cdot x} \right]$$

and satifies the equal-time commutation relations

$$\begin{aligned} \left[\phi(\vec{x}, t), \Pi_\phi(\vec{x}', t)\right] &= i\,\delta^{(3)}(\vec{x} - \vec{x}'), \\ \left[\phi^\dagger(\vec{x}, t), \Pi_{\phi^\dagger}(\vec{x}', t)\right] &= i\,\delta^{(3)}(\vec{x} - \vec{x}'), \end{aligned}$$

and all other commutators vanishing. In the following, you can conveniently consider the fields ϕ and $\phi^\dagger$ as independent.

(a)(20 points) Show that (1) is equivalent to the Lagrangian of two independent real scalar fields with same mass and satisfying the standard equal-time commutation relations. *Hint*: Decompose the complex field in real components $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$.

(b)(20 points) Write down the conjugate momentum fields Π_ϕ and $\Pi_{\phi^\dagger}$ in terms of ϕ and $\phi^\dagger$, and derive the equal-time commutation relations of a,

$a^\dagger$, b and $b^\dagger$.

(c)(20 points) Show that the Hamiltonian $H = \int d^3\vec{x}\, \frac{1}{2}\left[\dot{\phi}\,\dot{\phi}^\dagger + (\vec{\nabla}\phi)(\vec{\nabla}\phi^\dagger) + m^2\phi\,\phi^\dagger\right]$ takes the form

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3} E_k \left[a^\dagger(k)a(k) + b^\dagger(k)b(k) + 1\right],$$

(d)(20 points) Show that (1) is invariant under any global phase transformation of the field $\phi \to \phi' = e^{-i\alpha}\phi$ with α real. Write down the associated conserved Noether current J^μ and express the conserved charge $Q = \int d^3x\, J^0$ in terms of creation and annihilation operators.

(e)(20 points) Compute the commutators $[Q, \phi]$ and $[Q, \phi^\dagger]$. Using these commutators and the eigenstates $|q\rangle$ of the charge operator Q, show that the field operators ϕ and $\phi^\dagger$ modify the charge of the system. How would you interpret the operators a, $a^\dagger$, b and $b^\dagger$?