Noether theorem

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Preliminary remark

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Noether theorem

Preliminary remark

Noether's theorem provides a link between the continuous symmetries of a dynamical system and the corresponding conserved quantities (constants of motion).

1. Here, only internal symmetries
2. Discussion of Poincaré invariance, see advanced quantum mechanics class

Noether theorem

Lagrangian and Euler-Lagrange equations

Consider Lagrangian $\mathcal{L}$ depending on n independent fields Φ_i:

$$\mathcal{L} = \mathcal{L}(\Phi, \partial_\mu \Phi), \quad \Phi = (\Phi_1, \ldots, \Phi_n);$$

n equations of motion (EOM):

$$\frac{\partial \mathcal{L}}{\partial \Phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} = 0, \quad i = 1, \cdots, n.$$

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Noether theorem

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Method of Gell-Mann and Lévy

Consider transformations which depend on r real local parameters $\epsilon_a(x)$:

$$\Phi_i(x) \mapsto \Phi_i'(x) = \Phi_i(x) + \delta\Phi_i(x) = \Phi_i(x) - i\epsilon_a(\boldsymbol{x})F_{ai}[\Phi(x)].$$

Remark: functions F_{ai} are not necessarily linear in the fields, i.e., nonlinear realizations are also allowed.

Noether theorem

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Noether theorem

Variation of the Lagrangian

$$\delta\mathcal{L} = \mathcal{L}(\Phi_i', \partial_\mu\Phi_i') - \mathcal{L}(\Phi_i, \partial_\mu\Phi_i) = \frac{\partial\mathcal{L}}{\partial\Phi_i}\delta\Phi_i + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\partial_\mu\delta\Phi_i + \mathcal{O}(\epsilon^2)$$

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$$\partial_\mu \delta\Phi_i = -i[\partial_\mu \epsilon_a(x)]F_{ai} - i\epsilon_a(x)\partial_\mu F_{ai}$$

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Noether theorem

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$$\partial_\mu\delta\Phi_i = -i[\partial_\mu\epsilon_a(x)]F_{ai} - i\epsilon_a(x)\partial_\mu F_{ai}$$

$$\cdots = \epsilon_a(x)\left(-i\frac{\partial\mathcal{L}}{\partial\Phi_i}F_{ai} - i\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\partial_\mu F_{ai}\right) + \partial_\mu\epsilon_a(x)\left(-i\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}F_{ai}\right)$$
$$\equiv \epsilon_a(x)D_a + \partial_\mu\epsilon_a(x)J_a^\mu.$$

Noether theorem

Four-vector currents and divergences of currents

Define

$$J_a^\mu = \frac{\partial \delta \mathcal{L}}{\partial \partial_\mu \epsilon_a},$$

$$D_a = \frac{\partial \delta \mathcal{L}}{\partial \epsilon_a}.$$

Four-vector currents and divergences of currents

Define

$$J^\mu_a = \frac{\partial \delta \mathcal{L}}{\partial \partial_\mu \epsilon_a},$$
$$D_a = \frac{\partial \delta \mathcal{L}}{\partial \epsilon_a}.$$

For solutions to the EOM

$$\partial_\mu J^\mu_a = -i \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} \right) F_{ai} - i \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} \partial_\mu F_{ai}$$
$$\overset{\text{EOM}}{=} -i \frac{\partial \mathcal{L}}{\partial \Phi_i} F_{ai} - i \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} \partial_\mu F_{ai}$$
$$= D_a.$$

Noether theorem

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Assume Lagrangian to be invariant under a global transformation:

$$\delta\mathcal{L} = 0 \quad \wedge \quad \partial_\mu \epsilon_a(x) J_a^\mu = D_a = 0.$$

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$\Rightarrow$ Total charge

$$Q_a(t) \equiv \int d^3x\, J_a^0(t, \vec{x})$$

is time independent, i.e., a constant of the motion:

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$$\frac{dQ_a}{dt} = \frac{d}{dt}\int d^3x\, J_a^0(t,\vec{x}) = \int d^3x \frac{\partial J_a^0(t,\vec{x})}{\partial t}$$
$$\stackrel{(*)}{=} \int d^3x \left[\frac{\partial J_a^0(t,\vec{x})}{\partial t} + \vec{\nabla}\cdot\vec{J}_a(t,\vec{x})\right] = \int d^3x\, \partial_\mu J_a^\mu = 0.$$

$$(*): \quad \int d^3x\, \vec{\nabla}\cdot\vec{J}_a = \oint da\, \vec{J}_a\cdot\hat{n} = \lim_{R\to\infty} R^2 \int d\Omega\, \vec{J}_a\cdot\hat{e}_r = 0.$$

Noether theorem

Example

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu\Phi_1\partial^\mu\Phi_1 + \partial_\mu\Phi_2\partial^\mu\Phi_2 - m^2(\Phi_1^2+\Phi_2^2)\right] - \frac{\lambda}{4}\left(\Phi_1^2+\Phi_2^2\right)^2,$$

with $m^2 > 0$ and $\lambda > 0$.

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with $m^2 > 0$ and $\lambda > 0$.
Consider active infinitesimal rotation by $\epsilon(x)$ (see Handout 2),

$$D(\epsilon) = \begin{pmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{pmatrix}.$$

Noether theorem

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$$D(\epsilon) = \begin{pmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{pmatrix}.$$

$\Rightarrow$ transformation of fields

$$\Phi_1' = \Phi_1 + \delta\Phi_1 = \Phi_1 - \epsilon(x)\Phi_2,$$
$$\Phi_2' = \Phi_2 + \delta\Phi_2 = \Phi_2 + \epsilon(x)\Phi_1.$$

Noether theorem

Noether theorem

Variation of Lagrangian

$$
\begin{aligned}
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\Phi_i}\delta\Phi_i + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\partial_\mu\delta\Phi_i \\
&= \underbrace{-m^2\Phi_1[-\epsilon(x)]\Phi_2 - m^2\Phi_2\epsilon(x)\Phi_1}_{=0} \\
&\quad - \underbrace{\lambda(\Phi_1^2+\Phi_2^2)\{\Phi_1[-\epsilon(x)]\Phi_2 + \Phi_2\epsilon(x)\Phi_1\}}_{=0} \\
&\quad + \partial^\mu\Phi_1\partial_\mu[-\epsilon(x)\Phi_2] + \partial^\mu\Phi_2\partial_\mu[\epsilon(x)\Phi_1] \\
&= \partial_\mu\epsilon(x)(-\partial^\mu\Phi_1\Phi_2 + \Phi_1\partial^\mu\Phi_2).
\end{aligned}
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&= \partial_\mu\epsilon(x)(-\partial^\mu\Phi_1\Phi_2 + \Phi_1\partial^\mu\Phi_2).
\end{aligned}
$$

Current and divergence

$$
J^\mu = \frac{\partial\delta\mathcal{L}}{\partial\partial_\mu\epsilon} = \Phi_1\partial^\mu\Phi_2 - \partial^\mu\Phi_1\Phi_2, \quad \partial_\mu J^\mu = \frac{\partial\delta\mathcal{L}}{\partial\epsilon} = 0.
$$

one parameter $\Rightarrow$ one conserved current

Noether theorem

Transition to a quantum (field) theory

Noether theorem

Transition to a quantum (field) theory

1. Classical mechanics
   Consider point mass m in a central potential $V(|\vec{r}|) = V(r)$.
   Lagrange and Hamilton functions are rotationally invariant.
   Consequence: angular momentum $\vec{l} = \vec{r} \times \vec{p}$ is a constant of the motion.

Noether theorem

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2. Transition to quantum mechanics (see chapter 2) $\Rightarrow$ operators

$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij}, \quad [\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{p}_i, \hat{p}_j] = 0.$$

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   Components of the angular momentum operator,

$$\hat{l}_i = \epsilon_{ijk}\hat{x}_j\hat{p}_k = -i\hat{p}_j \underbrace{(-i\epsilon_{ijk})}_{(L_i^{\mathrm{ad}})_{jk}} \hat{x}_k.$$

   iL_i^{ad}: 3×3 matrices of the adjoint representation of so(3) (see section 4.1)

Commutation relations (angular-momentum algebra)

$$[\hat{l}_i, \hat{l}_j] = i\epsilon_{ijk}\hat{l}_k, \quad [L_i^{\mathrm{ad}}, L_j^{\mathrm{ad}}] = i\epsilon_{ijk}L_k^{\mathrm{ad}}.$$

Noether theorem

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Angular momentum operators are generators of rotations:

$$|\Psi'\rangle = \exp(-i\alpha_i\hat{l}_i)|\Psi\rangle.$$

Noether theorem

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Rotational invariance of the quantum system

$$[\hat{H}, \hat{l}_i] = 0$$

i.e., $\hat{l}_i$ are still constants of the motion.

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Rotational invariance of the quantum system

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Simultaneously diagonalize $\hat{H}$, $\hat{l}_1^{\,2} + \hat{l}_2^{\,2} + \hat{l}_3^{\,2}$, and $\hat{l}_3$.
Multiplets with eigenvalues $l(l+1)$ and $m = -l, \cdots, l$ $(l = 0, 1, 2, \cdots)$.
Energy eigenvalues depend on V (dynamics).

Noether theorem

Classify operators according to their transformation behavior.

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Example: Components $\hat{A}_i$ of a vector operator

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Example: Components $\hat{A}_i$ of a vector operator

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Introduce spherical tensor operator of rank 1 via

$$\hat{A}_0^{(1)} = \hat{A}_3, \quad \hat{A}_{\pm 1}^{(1)} = \frac{\mp 1}{\sqrt{2}}(\hat{A}_1 \pm i\hat{A}_2).$$

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Use Wigner-Eckart theorem to calculate matrix elements:

$$\langle l', m'|\hat{A}_\nu^{(1)}|l, m\rangle = \left(\begin{array}{cc|c} l & 1 & l' \\ m & \nu & m' \end{array} \right) \frac{\langle l'||\hat{A}^{(1)}||l\rangle}{\sqrt{2l'+1}}.$$

Analogous case in quantum field theory

Canonical quantization: Fields Φ_i and their conjugate momenta $\Pi_j = \partial\mathcal{L}/\partial(\partial_0\Phi_j) \Rightarrow$ operators (symbol hat omitted).

Noether theorem

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Equal-time commutation relations in the Heisenberg picture

$$
\begin{aligned}
\Phi_i(t,\vec{x}) &\leftrightarrow \hat{x}_i, \\
\Pi_j(t,\vec{x}) &\leftrightarrow \hat{p}_j, \\
[\Phi_i(t,\vec{x}),\Pi_j(t,\vec{y})] = i\delta^3(\vec{x}-\vec{y})\delta_{ij} &\leftrightarrow [\hat{x}_i,\hat{p}_j] = i\delta_{ij}, \\
[\Phi_i(t,\vec{x}),\Phi_j(t,\vec{y})] = 0 &\leftrightarrow [\hat{x}_i,\hat{x}_j] = 0, \\
[\Pi_i(t,\vec{x}),\Pi_j(t,\vec{y})] = 0 &\leftrightarrow [\hat{p}_i,\hat{p}_j] = 0.
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[\Pi_i(t,\vec{x}),\Pi_j(t,\vec{y})] = 0 &\leftrightarrow [\hat{p}_i,\hat{p}_j] = 0.
\end{aligned}$$

Consider infinitesimal transformations which are *linear* in the fields,

$$\Phi_i(x) \mapsto \Phi_i'(x) = \Phi_i(x) - i\epsilon_a(x) t_{a,ij}\Phi_j(x) \leftrightarrow \hat{x}_i \mapsto \hat{x}_i - i\epsilon_k \underbrace{(-i\epsilon_{ijk})}_{(L_k^{\mathrm{ad}})_{ij}} \hat{x}_j$$

Noether theorem

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$t_{a,ij}$ are constants generating a mixing of the fields

Noether theorem

Charge operators

Current operator

$$J_a^\mu(x) = -it_{a,ij}\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\Phi_j.$$

$$Q_a(t) = -i\int d^3x\,\Pi_i(x)t_{a,ij}\Phi_j(x) \quad \leftrightarrow \quad \hat{l}_k = -i\hat{p}_i(-i\epsilon_{kij})\hat{x}_j,$$

where $J_a^\mu(x)$ and $Q_a(t)$ are now operators.

Charge operators

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Transformation behavior of field operators

$$[Q_a(t),\Phi_k(t,\vec{y})] \;=\; -it_{a,ij}\int d^3x\,[\Pi_i(t,\vec{x})\Phi_j(t,\vec{x}),\Phi_k(t,\vec{y})]$$
$$=\; -t_{a,kj}\Phi_j(t,\vec{y}) \quad\leftrightarrow\quad [\hat{l}_k,\hat{x}_i]=i\epsilon_{kij}\hat{x}_j$$

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$$= -t_{a,kj}\Phi_j(t,\vec{y}) \quad \leftrightarrow \quad [\hat{l}_k,\hat{x}_i] = i\epsilon_{kij}\hat{x}_j$$

Q_a are generators of the transformations acting on the states of Hilbert space.

Noether theorem

Example: U(1) invariance

$$\Phi = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2), \quad \Phi^\dagger = \frac{1}{\sqrt{2}}(\Phi_1 - i\Phi_2).$$

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Transformation behavior (Gell-Mann and Lévy)

$$\begin{pmatrix} \Phi \\ \Phi^\dagger \end{pmatrix} \mapsto \mathbb{1} - i\epsilon(x) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Phi \\ \Phi^\dagger \end{pmatrix}.$$

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Corresponding charge operator

$$Q = \int d^3x\, i(\underbrace{\dot{\Phi}^\dagger}_{=\Pi}\Phi - \underbrace{\dot{\Phi}}_{=\Pi^\dagger}\Phi^\dagger)$$
$$= i\int d^3x\,(\Pi\Phi - \Phi^\dagger\Pi^\dagger)$$
$$'' ='' -i\int d^3x\,(\Pi \quad \Pi^\dagger) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Phi \\ \Phi^\dagger \end{pmatrix}.$$

Noether theorem

Commutation relations

$$
\begin{aligned}
[Q, \Phi(x)] &= \Phi(x), \\
[Q, \Pi(x)] &= -\Pi(x), \\
[Q, \Phi^\dagger(x)] &= -\Phi^\dagger(x), \\
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$$

Example

$$
\begin{aligned}
[Q, \Phi(x)] &= \Big[i \int d^3y \big(\Pi(t,\vec{y})\Phi(t,\vec{y}) - \Phi^\dagger(t,\vec{y})\Pi^\dagger(t,\vec{y})\big), \Phi(t,\vec{x})\Big] \\
&= i \int d^3y \big([\Pi(t,\vec{y})\Phi(t,\vec{y}), \Phi(t,\vec{x})] \\
&\qquad - [\Phi^\dagger(t,\vec{y})\Pi^\dagger(t,\vec{y}), \Phi(t,\vec{x})]\big)
\end{aligned}
$$

Noether theorem

Example cont'd

$$\cdots = i\int d^3y\Big(\Pi(t,\vec y)\underbrace{[\Phi(t,\vec y),\Phi(t,\vec x)]}_{=0}+\underbrace{[\Pi(t,\vec y),\Phi(t,\vec x)]}_{=-i\delta^3(\vec y-\vec x)}\Phi(t,\vec y)$$

$$-\Phi^\dagger(t,\vec y)\underbrace{[\Pi^\dagger(t,\vec y),\Phi(t,\vec x)]}_{=0}-\underbrace{[\Phi^\dagger(t,\vec y),\Phi(t,\vec x)]}_{=0}\Pi^\dagger(t,\vec y)\Big)$$

$$=\int d^3y\,\delta^3(\vec y-\vec x)\Phi(t,\vec y)=\Phi(x).$$

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Example cont'd

$$\cdots = i \int d^3y \Big(\Pi(t,\vec{y}) \underbrace{[\Phi(t,\vec{y}),\Phi(t,\vec{x})]}_{=0} + \underbrace{[\Pi(t,\vec{y}),\Phi(t,\vec{x})]}_{=-i\delta^3(\vec{y}-\vec{x})} \Phi(t,\vec{y})$$

$$- \Phi^\dagger(t,\vec{y}) \underbrace{[\Pi^\dagger(t,\vec{y}),\Phi(t,\vec{x})]}_{=0} - \underbrace{[\Phi^\dagger(t,\vec{y}),\Phi(t,\vec{x})]}_{=0} \Pi^\dagger(t,\vec{y}) \Big)$$

$$= \int d^3y\, \delta^3(\vec{y}-\vec{x})\Phi(t,\vec{y}) = \Phi(x).$$

Interpretation

Let $|\alpha\rangle$ be an eigenstate of Q with eigenvalue q_α:

$$Q\Phi(x)|\alpha\rangle = \big([Q,\Phi(x)] + \Phi(x)Q\big)|\alpha\rangle$$
$$= \big(\Phi(x) + \Phi(x)q_\alpha\big)|\alpha\rangle = (1+q_\alpha)\Phi(x)|\alpha\rangle.$$

Noether theorem

Noether theorem

Interpretation cont'd

Conclusion: the operators $\Phi(x)$ and $\Pi^\dagger(x)$ [$\Phi^\dagger(x)$ and $\Pi(x)$] increase (decrease) the (Noether) charge of a system by one unit.

Interpretation cont'd

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Generalization (in analogy to the definition of a tensor operator of section 4.3.6): an operator A_ν ($\nu \in \mathbb{Z}$) with $[Q, A_\nu] = \nu A_\nu$ changes the Noether charge of a system with charge q_α by ν to $q_\alpha + \nu$.

Noether theorem

Example: isospin

Isospin algebra

$$[I_i, I_j] = i\epsilon_{ijk} I_k.$$

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$$\Psi = \begin{pmatrix} p \\ n \end{pmatrix},$$

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transformation behavior

$$\Psi \mapsto \Psi' = \left(\mathbb{1} - i\vec{\epsilon}(x) \cdot \frac{\vec{\tau}}{2}\right)\Psi.$$

Noether theorem

Noether theorem

Example: isospin cont'd

Adjoint representation

$$T_1^{\rm ad}=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\ T_2^{\rm ad}=\begin{pmatrix}0&0&i\\0&0&0\\-i&0&0\end{pmatrix},\ T_3^{\rm ad}=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}.$$

Example: isospin cont'd

Adjoint representation

$$T_1^{\mathrm{ad}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \; T_2^{\mathrm{ad}} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \; T_3^{\mathrm{ad}} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

carrier space (pion fields)

$$\vec{\Phi} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix},$$

spherical notation

$$\pi^+ := \frac{1}{\sqrt{2}}(\Phi_1 - i\Phi_2) = \Phi^{(1)}_{-1},$$

$$\pi^0 := \Phi_3 = \Phi^{(1)}_0,$$

$$\pi^- := \frac{-1}{\sqrt{2}}(\Phi_1 + i\Phi_2) = \Phi^{(1)}_{+1}.$$

Example: isospin cont'd

Charge operators

$$Q_i = \int d^3x \left(\Psi^\dagger(x) \frac{\tau_i}{2} \Psi(x) + \epsilon_{ijk} \Phi_j(x) \Pi_k(x) \right),$$

Example: isospin cont'd

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$$[Q_i, Q_j] = i\epsilon_{ijk} Q_k.$$

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Example: isospin cont'd

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Q_i: generators of isospin transformations on the Hilbert space of the system

$$|A'\rangle = \exp(-i\Theta_j Q_j)|A\rangle.$$

Noether theorem

Example: isospin cont'd

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- the equal-time commutation relations for bosons in terms of commutators: $[\Phi_j(t,\vec{x}),\Pi_k(t,\vec{y})] = i\delta^3(\vec{x}-\vec{y})\delta_{jk}$, etc.

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- the equal-time commutation relations for fermions in terms of anticommutators: $\{\Psi_{\alpha,r}(t,\vec{x}),\Psi^\dagger_{\beta,s}(t,\vec{y})\}=\delta^3(\vec{x}-\vec{y})\delta_{\alpha\beta}\delta_{rs}$, etc.

Noether theorem

Example: isospin cont'd

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  in the case of bosons;
- $$[ab,cd] = a\{b,c\}d - ac\{b,d\} + \{a,c\}db - c\{a,d\}b$$
  in the case of fermions;

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  in the case of bosons;
- $$[ab,cd]=a\{b,c\}d-ac\{b,d\}+\{a,c\}db-c\{a,d\}b$$

  in the case of fermions;
- explicit calculation, see book pp 275-277