Gauge theories

Gauge principle

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5. Introduce additional so-called gauge fields to warrant invariance
6. This generates interactions between gauge fields and elementary particles

Gauge theories

Example: Quantum electrodynamics (QED, U(1), Abelian)

Starting point: Lagrangian of a free electron

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Dirac equation

$$(i\partial\!\!\!/ - m)\Psi = 0.$$

Gauge theories

Invariance of $\mathcal{L}_0$

$\mathcal{L}_0$ is invariant under global U(1) transformations:

$$\Psi(x) \mapsto \Psi'(x) = e^{-i\alpha}\Psi(x),$$
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$\alpha \in [0, 2\pi[$ does not depend on x:

$$\begin{aligned} \bar{\Psi}\Psi &\mapsto \bar{\Psi}\underbrace{e^{i\alpha}e^{-i\alpha}}_{=1}\Psi = \bar{\Psi}\Psi, \\ \bar{\Psi}\gamma^\mu\partial_\mu\Psi &\mapsto \bar{\Psi}e^{i\alpha}\gamma^\mu\partial_\mu e^{-i\alpha}\Psi = \bar{\Psi}e^{i\alpha}e^{-i\alpha}\gamma^\mu\partial_\mu\Psi = \bar{\Psi}\gamma^\mu\partial_\mu\Psi. \end{aligned}$$

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Remark

All components Ψ_α are multiplied by the same phase.

Current density

Infinitesimal transformation

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Identify conserved current using Gell-Mann-Lévy trick, $\epsilon \to \epsilon(x)$:

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\delta\mathcal{L}_0 &= -i\partial_\mu\epsilon(x) i\bar{\Psi}(x)\gamma^\mu\Psi(x) = \partial_\mu\epsilon(x)\bar{\Psi}(x)\gamma^\mu\Psi(x) \\
\Rightarrow \quad J^\mu &= \frac{\partial\delta\mathcal{L}_0}{\partial\partial_\mu\epsilon} = \bar{\Psi}\gamma^\mu\Psi, \\
\partial_\mu J^\mu &= \frac{\partial\delta\mathcal{L}_0}{\partial\epsilon} = 0.
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Charge operator (electron number operator)

$$Q(t) = \int d^3x\, J^0(t,\vec{x}) = \int d^3x\, \Psi^\dagger(t,\vec{x})\Psi(t,\vec{x}), \quad \frac{dQ}{dt} = 0.$$

Gauge theories

Transformation behavior

Convention: electron has negative electric charge ($q_e = -1$)

$$\mathrm{U}(1) \ni e^{-i\alpha} \mapsto e^{-i\alpha q_e} = e^{i\alpha}$$

$\Rightarrow$ convention for local transformation

$$\Psi(x) \mapsto e^{i\alpha(x)}\Psi(x).$$

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Covariant derivative

$$D_\mu \Psi(x) \mapsto [D_\mu \Psi(x)]' = D'_\mu \Psi'(x) \stackrel{!}{=} e^{i\alpha(x)} D_\mu \Psi(x).$$

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Introduce

gauge four-vector potential $\mathcal{A}_\mu(x)$ with transformation behavior

$$\mathcal{A}_\mu(x) \mapsto \mathcal{A}'_\mu(x) = \mathcal{A}_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x), \quad e > 0.$$

Gauge theories

Define covariant derivative

$$
\begin{aligned}
D_\mu \Psi(x) &:= [\partial_\mu - ie\mathcal{A}_\mu(x)]\Psi(x) \\
\mapsto D'_\mu \Psi'(x) &= [\partial_\mu - ie\mathcal{A}_\mu(x) - i\partial_\mu\alpha(x)] \left[e^{i\alpha(x)}\Psi(x)\right] \\
&= e^{i\alpha(x)}[\partial_\mu + i\partial_\mu\alpha(x) - ie\mathcal{A}_\mu(x) - i\partial_\mu\alpha(x)]\Psi(x) \\
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New Lagrangian

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\mathcal{L}_0(\Psi, D_\mu\Psi) = \bar{\Psi}(i\not{D} - m)\Psi = \mathcal{L}_0(\Psi, \partial_\mu\Psi) + e\bar{\Psi}\gamma^\mu\Psi\mathcal{A}_\mu.
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Invariant under so-called gauge transformation of the second kind

$$\Psi(x) \mapsto e^{i\alpha(x)}\Psi(x),$$

$$\mathcal{A}_\mu(x) \mapsto \mathcal{A}_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x).$$

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Interpret $\mathcal{A}_\mu$ as a dynamical variable.

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$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$$

and introduce in addition a „kinetic“ term for the vector field:

$$\mathcal{L}_{\mathrm{QED}} = \bar{\Psi} i \gamma^\mu (\partial_\mu - ie\mathcal{A}_\mu)\Psi - m\bar{\Psi}\Psi - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}.$$

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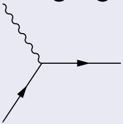
- After quantization, the gauge field is identified with the photon.
- Interaction between the matter field and the gauge field

$$\mathcal{L}_{\rm int} = -(-e)\bar{\Psi}\gamma^\mu\Psi\mathcal{A}_\mu = -J^\mu_{\rm em}\mathcal{A}_\mu$$



Remarks

1. A mass term

$$\frac{1}{2}M^2\mathcal{A}_\mu\mathcal{A}^\mu \mapsto \frac{1}{2}M^2(\mathcal{A}_\mu\mathcal{A}^\mu + \frac{2}{e}\partial_\mu\alpha\mathcal{A}^\mu + \frac{1}{e^2}\partial_\mu\alpha\partial^\mu\alpha)$$
$$\neq \frac{1}{2}M^2\mathcal{A}_\mu\mathcal{A}^\mu$$

would spoil gauge invariance.
Gauge bosons are massless (no spontaneous symmetry breaking).

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Consider matter field Ψ_q for a particle with charge q

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Why charge is quantized cannot be explained solely from QED.

3. The requirement of renormalizability in the traditional sense excludes further gauge-invariant couplings such as the interaction with an anomalous magnetic moment,

$$-\frac{e\kappa}{4m}\mathcal{F}_{\mu\nu}\bar{\Psi}\sigma^{\mu\nu}\Psi, \quad \sigma^{\mu\nu}=\frac{i}{2}[\gamma^\mu,\gamma^\nu].$$

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4. Due to the Abelian nature of U(1), photons do not have a direct self coupling.

Non-Abelian case

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- Assume $\mathcal{L}_0$ to be invariant under a global transformation of the matter fields Φ.
- Let the corresponding symmetry group G be a compact Lie group with r abstract infinitesimal generators X_a and structure constants C_{abc} of the Lie algebra:

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- Examples: SU(N) and SO(N) with $r = N^2 - 1$ and $r = N(N-1)/2$, respectively.
- Also: direct products (Standard model)

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- Commutation relations: $[T_a, T_b] = iC_{abc} T_c$.
- Group elements in the neighborhood of the identity e with corresponding infinitesimal linear transformation:

$$g = e - i\epsilon_a X_a,$$
$$U(g) = (1 - i\epsilon_a T_a) : \Phi(x) \mapsto (1 - i\epsilon_a T_a)\Phi(x)$$

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- Local $\epsilon_a(x) \Rightarrow$ additional terms in $\delta\mathcal{L}$, because

$$\partial_\mu \delta\Phi(x) = \underbrace{-i\partial_\mu\epsilon_a(x) T_a \Phi(x)}_{\text{"problematic" term}} - i\epsilon_a(x) T_a \partial_\mu \Phi(x)$$

Gauge principle

- Demand invariance of Lagrangian under local transformations, i.e., $g \rightarrow g(x)$.
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- Analogy to QED: introduce covariant derivative with the property

$$D_\mu \Phi(x) \mapsto [D_\mu \Phi(x)]' = D'_\mu \Phi'(x) \stackrel{!}{=} [1 - i\epsilon_a(x) T_a] D_\mu \Phi(x),$$

i.e., the covariant derivative of the fields transforms as the fields.

Covariant derivative

Ansatz: introduce for each generator X_a of the abstract group a gauge field $\mathcal{A}_{a\mu}$,

$$D_\mu \Phi(x) = [\partial_\mu + igT_a\mathcal{A}_{a\mu}(x)]\,\Phi(x).$$

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Transformation behavior of the gauge fields (in detail)

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Transformation behavior of the gauge fields (in detail)

- Define (summation over a from 1 to r implied)

$$\widetilde{O} = T_a O_a.$$

With a suitable choice of the T_a, O_a may be projected from $\widetilde{O}$. For

$$\kappa \mathrm{Tr}(T_a T_b) = \delta_{ab},$$

we have

$$O_a = \kappa \mathrm{Tr}(T_a \widetilde{O}).$$

- Example: Let $\widetilde{O}$ be a Hermitian traceless 2×2 matrix,

$$\begin{aligned} \widetilde{O} &= O_a \tau_a \quad O_a \in \mathbb{R}, \\ \frac{1}{2}\mathrm{Tr}(\tau_a \tau_b) &= \delta_{ab}, \\ \Rightarrow \quad O_a &= \frac{1}{2}\mathrm{Tr}(\tau_a \widetilde{O}). \end{aligned}$$

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- Write covariant derivative of Φ as

$$D_\mu \Phi(x) = [\partial_\mu + ig\widetilde{\mathcal{A}}_\mu(x)]\Phi(x).$$

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- Requirement for transformation behavior $\Rightarrow$

$$(\partial_\mu + ig\widetilde{\mathcal{A}}_\mu + ig\widetilde{\delta\mathcal{A}}_\mu)[(1 - i\widetilde{\epsilon})\Phi(x)] = (1 - i\widetilde{\epsilon})(\partial_\mu + ig\widetilde{\mathcal{A}}_\mu)\Phi(x)$$

- Comparison of small terms of linear order:

$$-i\partial_\mu\widetilde{\epsilon}+g\widetilde{\mathcal{A}}_\mu\widetilde{\epsilon}+ig\widetilde{\delta\mathcal{A}}_\mu=g\widetilde{\epsilon}\widetilde{\mathcal{A}}_\mu$$

or

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  No: The transformation behavior is determined in terms of the structure constants C_{abc}:

$$\delta\mathcal{A}_{a\mu} = C_{bca}\epsilon_b\mathcal{A}_{c\mu} + \frac{1}{g}\partial_\mu\epsilon_a.$$

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$$\mathcal{L}_0(\Phi, D_\mu\Phi) \text{ with } D_\mu\Phi = (\partial_\mu + ig\widetilde{\mathcal{A}}_\mu)\Phi$$

is invariant under the (simultaneous) local transformations

$$\Phi(x) \mapsto \exp\left[-i\Theta_a(x)T_a\right]\Phi(x) = \underbrace{\exp\left[-i\widetilde{\Theta}(x)\right]}_{=:\, U[g(x)]}\Phi(x),$$

$$\widetilde{\mathcal{A}}_\mu(x) = T_a\mathcal{A}_{a\mu}(x) \mapsto U\widetilde{\mathcal{A}}_\mu(x)U^\dagger + \frac{i}{g}\partial_\mu U U^\dagger.$$

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- Gauge principle ⇒ interaction of matter fields with gauge fields.
- However, so far gauge bosons are no dynamical degrees of freedom.

- Analogy to QED: add

$$-\frac{1}{4}\mathcal{F}_{a\mu\nu}\mathcal{F}_a^{\mu\nu}.$$

Prerequisite: $\mathcal{F}_{a\mu\nu}$ transforms under the adjoint representation.

Gauge theories

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The fields F_a, $a = 1, \ldots, r$, transform under the adjoint representation iff

$$\begin{pmatrix} F_1 \\ \vdots \\ F_r \end{pmatrix} =: F \mapsto (1 - i\epsilon_c T_c^{\mathrm{ad}})F,$$

$$F_a \mapsto F_a - i\epsilon_c (T_c^{\mathrm{ad}})_{ab} F_b = F_a + C_{abc}\epsilon_b F_c.$$

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Lagrangian of a gauge theory (Yang-Mills theory)

$$\mathcal{L} = \mathcal{L}_0(\Phi, D_\mu \Phi) - \frac{1}{4} \mathcal{F}_{a\mu\nu} \mathcal{F}_a^{\mu\nu}$$

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Remarks

- Mass terms $\frac{1}{2} M_a^2 \mathcal{A}_{a\mu} \mathcal{A}_a^\mu$ violate gauge invariance

  gauge principle $\Rightarrow$ gauge bosons are massless (without spontaneous symmetry breaking)

Gauge theories

Remarks cont'd

- Non-Abelian group $\Rightarrow$ interaction terms with three and four gauge fields

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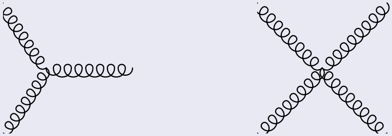

Remarks cont'd

- Non-Abelian group $\Rightarrow$ interaction terms with three and four gauge fields

- G direct product, $G = G_1 \times \cdots \times G_k$ $\Rightarrow$ introduce for each subgroup G_i an independent coupling constant g_i

Remarks cont'd

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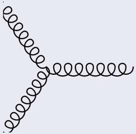

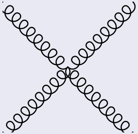

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- Example: gauge group of the Standard Model

$$\underbrace{\mathrm{SU}(3)_c}_{\text{strong int.}} \times \underbrace{\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y}_{\text{electroweak int.}}$$

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⇒ 3 gauge couplings

$$g_3 \leftrightarrow \mathrm{SU}(3)_c, \quad g \leftrightarrow \mathrm{SU}(2)_L, \quad g' \leftrightarrow \mathrm{U}(1)_Y$$