Handout 12 (read by Jan. 22)

Theorem 4.3.9 (Wigner-Eckart theorem). The matrix elements of an irreducible tensor operator of rank n satisfy

$$\langle j', m'|A^{(n)}_\nu|j, m\rangle = \left(\begin{array}{cc|c} j & n & j' \\ m & \nu & m' \end{array}\right) \frac{\langle j'||A^{(n)}||j\rangle}{\sqrt{2j'+1}},$$

where $\langle j'||A^{(n)}||j\rangle$ denotes the so-called *reduced matrix element*. The dependence on the magnetic quantum numbers m', ν, and m is entirely given in terms of a Clebsch-Gordan coefficient. Because of the selection rules for the Clebsch-Gordan coefficients (see subsection 4.3.4 of Handout 11), the Wigner-Eckart theorem implies

$$\langle j', m'|A^{(n)}_\nu|j, m\rangle = 0 \quad \text{for } j' > j+n, j' < |j-n|, m+\nu \neq m',$$

i.e., $A^{(n)}_\nu$ changes the projection of the initial state by ν and transfers the angular momentum n to the system.

Proof: We consider the $(2n+1)\cdot(2j+1)$ states $A^{(n)}_\nu|j, m\rangle$. Under "rotations", these states transform as $|n, \nu; j, m\rangle$, because

$$\begin{aligned} J_3 A^{(n)}_\nu|j, m\rangle &= \left([J_3, A^{(n)}_\nu] + A^{(n)}_\nu J_3\right)|j, m\rangle = (\nu+m)A^{(n)}_\nu|j, m\rangle, \\ J_\pm A^{(n)}_\nu|j, m\rangle &= \left([J_\pm, A^{(n)}_\nu] + A^{(n)}_\nu J_\pm\right)|jm\rangle \\ &= \sqrt{n(n+1)-\nu(\nu\pm1)}A^{(n)}_{\nu\pm1}|j, m\rangle \\ &\quad + \sqrt{j(j+1)-m(m\pm1)}A^{(n)}_\nu|j, m\pm1\rangle. \end{aligned}$$

Using the Clebsch-Gordan coefficients, we construct states transforming under an irreducible representation. Let $|n-j| \leq j'' \leq n+j$. We define a state

$$|\phi_A(j, n, j'', m'')\rangle := \sum_{m',\nu'} \left(\begin{array}{cc|c} j & n & j'' \\ m' & \nu' & m'' \end{array}\right) A^{(n)}_{\nu'}|j, m'\rangle, \tag{4.32}$$

(the identifier A in the ket refers to the operator A) with

$$\begin{aligned} \vec{J}^2|\phi_A(j, n, j'', m'')\rangle &= j''(j''+1)|\phi_A(j, n, j'', m'')\rangle, \\ J_3|\phi_A(j, n, j'', m'')\rangle &= m''|\phi_A(j, n, j'', m'')\rangle, \\ J_\pm|\phi_A(j, n, j'', m'')\rangle &= \sqrt{j''(j''+1)-m''(m''\pm1)}|\phi_A(j, n, j'', m''\pm1)\rangle. \end{aligned}$$

As in the proof of Example 4.3.5 (scalar operator) one obtains

$$\langle j', m'|\phi_A(j, n, j'', m'')\rangle = \delta_{j'j''}\delta_{m'm''}c_A(j', n, j). \tag{4.33}$$

Using Eq. (4.27), we now invert Eq. (4.32) to obtain

$$\begin{aligned} &\sum_{j'',m''} \left(\begin{array}{cc|c} j & n & j'' \\ m & \nu & m'' \end{array}\right)|\phi_A(j, n, j'', m'')\rangle \\ &= \sum_{j'',m'',m',\nu'} \left(\begin{array}{cc|c} j & n & j'' \\ m & \nu & m'' \end{array}\right)\left(\begin{array}{cc|c} j & n & j'' \\ m' & \nu' & m'' \end{array}\right) A^{(n)}_{\nu'}|j, m'\rangle \\ &\overset{(4.27)}{=} \sum_{m',\nu'} \delta_{m'm}\delta_{\nu'\nu}A^{(n)}_{\nu'}|j, m'\rangle = A^{(n)}_\nu|j, m\rangle. \end{aligned}$$

Finally, we "multiply" from the left by $\langle j', m'|$:

$$\begin{aligned}
\langle j', m'|A^{(n)}_\nu|j, m\rangle &= \sum_{j'',m''} \left(\begin{array}{cc|c} j & n & j'' \\ m & \nu & m'' \end{array} \right) \langle j'm'|\phi_A(j,n,j'',m'')\rangle \\
&\overset{(4.33)}{=} \sum_{j'',m''} \left(\begin{array}{cc|c} j & n & j'' \\ m & \nu & m'' \end{array} \right) \delta_{j'j''}\delta_{m'm''} c_A(j',n,j) \\
&= \left(\begin{array}{cc|c} j & n & j' \\ m & \nu & m' \end{array} \right) c_A(j',n,j), \Rightarrow \quad \text{proposition.}
\end{aligned}$$

Identifying

$$c_A(j',n,j) = \frac{\langle j'||A^{(n)}||j\rangle}{\sqrt{2j'+1}},$$

we have proven the proposition. The additional factor $1/\sqrt{2j'+1}$ is a matter of convention.

Theorem 4.3.10 Spherical tensor operators may be coupled as angular momenta. Let $A^{(n_1)}$ and $B^{(n_2)}$ be irreducible tensors of rank n_1 and n_2, respectively. We may then define a "tensor product of rank n" ($|n_1 - n_2| \leq n \leq n_1 + n_2$) as

$$[A^{(n_1)} \times B^{(n_2)}]^{(n)}_\nu := \sum_{\nu_1,\nu_2} \left(\begin{array}{cc|c} n_1 & n_2 & n \\ \nu_1 & \nu_2 & \nu \end{array} \right) A^{(n_1)}_{\nu_1} B^{(n_2)}_{\nu_2}.$$

Compare with the coupling of two angular momenta,

$$|(j_1,j_2)j,m\rangle = \sum_{m_1,m_2} \left(\begin{array}{cc|c} j_1 & j_2 & j \\ m_1 & m_1 & m \end{array} \right) |j_1,m_1;j_2,m_2\rangle.$$

Remark: The coupling of two irreducible tensors $A^{(n_1)}$ and $B^{(n_2)}$ may refer to two different "particles" (two different vector spaces). However, this does not necessarily have to be the case. The important property refers to the commutation relations with the components of the total angular momentum operator.

Proof: We investigate

$$[J_3, [A^{(n_1)} \times B^{(n_2)}]^{(n)}_\nu] = \sum_{\nu_1,\nu_2} \left(\begin{array}{cc|c} n_1 & n_2 & n \\ \nu_1 & \nu_2 & \nu \end{array} \right) [J_3, A^{(n_1)}_{\nu_1} B^{(n_2)}_{\nu_2}]$$

$$[J_3, A^{(n_1)}_{\nu_1} B^{(n_2)}_{\nu_2}] = A^{(n_1)}_{\nu_1}[J_3, B^{(n_2)}_{\nu_2}] + [J_3, A^{(n_1)}_{\nu_1}]B^{(n_2)}_{\nu_2} = (\nu_2 + \nu_1) A^{(n_1)}_{\nu_1} B^{(n_2)}_{\nu_2}$$

$$\begin{aligned}
&= \sum_{\nu_1,\nu_2} \left(\begin{array}{cc|c} n_1 & n_2 & n \\ \nu_1 & \nu_2 & \nu \end{array} \right) \nu A^{(n_1)}_{\nu_1} B^{(n_2)}_{\nu_2} \\
&= \nu [A^{(n_1)} \times B^{(n_2)}]^{(n)}_\nu,
\end{aligned}$$

$$
\begin{aligned}
[J_\pm, [A^{(n_1)} \times B^{(n_2)}]^{(n)}_\nu] &= \sum_{\nu_1,\nu_2} \left(\begin{array}{cc|c} n_1 & n_2 & n \\ \nu_1 & \nu_2 & \nu \end{array} \right) [J_\pm, A^{(n_1)}_{\nu_1} B^{(n_2)}_{\nu_2}] \\
&= \sum_{\nu_1,\nu_2} \left(\begin{array}{cc|c} n_1 & n_2 & n \\ \nu_1 & \nu_2 & \nu \end{array} \right) \Big(\sqrt{n_1(n_1+1) - \nu_1(\nu_1 \pm 1)} A^{(n_1)}_{\nu_1 \pm 1} B^{(n_2)}_{\nu_2} \\
&\quad + \sqrt{n_2(n_2+1) - \nu_2(\nu_2 \pm 1)} A^{(n_1)}_{\nu_1} B^{(n_2)}_{\nu_2 \pm 1} \Big) \\
&\overset{*}{=} \sum_{\nu_1,\nu_2} \Bigg[\left(\begin{array}{cc|c} n_1 & n_2 & n \\ \nu_1 \mp 1 & \nu_2 & \nu \end{array} \right) \sqrt{n_1(n_1+1) - \nu_1(\nu_1 \mp 1)} \\
&\quad + \left(\begin{array}{cc|c} n_1 & n_2 & n \\ \nu_1 & \nu_2 \mp 1 & \nu \end{array} \right) \sqrt{n_2(n_2+1) - \nu_2(\nu_2 \mp 1)} \Bigg] A^{(n_1)}_{\nu_1} B^{(n_2)}_{\nu_2} \\
&\overset{**}{=} \sum_{\nu_1,\nu_2} \sqrt{(n \mp \nu)(n \pm \nu + 1)} \left(\begin{array}{cc|c} n_1 & n_2 & n \\ \nu_1 & \nu_2 & \nu \pm 1 \end{array} \right) A^{(n_1)}_{\nu_1} B^{(n_2)}_{\nu_2} \\
&= \sqrt{(n \mp \nu)(n \pm \nu + 1)} [A^{(n_1)} \times B^{(n_2)}]^{(n)}_{\nu \pm 1}.
\end{aligned}
$$

- For the justification of $*$, we consider

$$\Sigma = \sum_{m=-j}^{j} \sqrt{j(j+1) - m(m \pm 1)} f(m).$$

Using the substitution $n := m \pm 1$, we obtain

$$\Sigma = \sum_{n=-j\pm 1}^{j \pm 1} \sqrt{j(j+1) - (n \mp 1)n}\, f(n \mp 1).$$

We now consider the two cases separately.

1. For the upper sign, the argument of the square root satisfies

$$j(j+1) - (n-1)n = 0 \text{ for } \begin{cases} n = -j, \\ n = j+1. \end{cases}$$

Therefore, on the one hand we may add a zero by letting the sum already start at $n = -j$, on the other hand we omit a zero by letting the sum end at $n = j$.

2. In complete analogy, we obtain for the lower sign

$$j(j+1) - (n+1)n = 0 \text{ for } \begin{cases} n = -(j+1), \\ n = j. \end{cases}$$

In this case we omit a zero at the lower limit and add a zero at the upper limit.

For both cases we may thus also write

$$\sum_{n=-j}^{j} \cdots .$$

Renaming n with m, $n \to m$, we obtain

$$\Sigma = \sum_{m=-j}^{j} \sqrt{j(j+1) - m(m \mp 1)}\, f(m \mp 1).$$

- In $**$ we made use of the recursion relation for Clebsch-Gordan coefficients (see subsection 4.3.4 of Handout 11 and Exercise 6) with the substitutions $(j_1, j_2, j) \mapsto (n_1, n_2, n)$ and $\pm(m_1, m_2, m) \mapsto \mp(\nu_1, \nu_2, \nu)$.