Handout 11 (read by Jan. 15)

4.3.3 Algorithm for the determination of Clebsch-Gordan coefficients

Let us consider fixed j_1, j_2, and j with $|j_1 - j_2| \leq j \leq j_1 + j_2$. We describe a procedure for calculating all Clebsch-Gordan coefficients $\left(\begin{array}{cc|c} j_1 & j_2 & j \\ m_1 & m_2 & m \end{array}\right)$. We make use of the abbreviated form

$$|m_1; m_2\rangle := |j_1, m_1; j_2, m_2\rangle \quad \text{and} \quad |j, m\rangle := |(j_1, j_2)j, m\rangle.$$

The eigenvalue of $J_3 = J_3(1) + J_3(2)$ is additive, resulting in a vanishing Clebsch-Gordan coefficient unless $m = m_1 + m_2$.

1. We start with the coefficients for a maximal value of j, i.e., $j = j_1 + j_2$.

   - For the state with maximal j and maximal $m = j$ one has

     $$|j, j\rangle = |j_1; j_2\rangle \quad \Rightarrow \quad \left(\begin{array}{cc|c} j_1 & j_2 & j_1 + j_2 \\ j_1 & j_2 & j_1 + j_2 \end{array}\right) = 1.$$

     By choosing the value $+1$ rather than -1 we made use of the phase convention of Eq. (4.23).

   - We now apply the lowering operator $J_- = J_-(1) + J_-(2)$ to this state. In this context, we make use of

     $$J_-|j, j\rangle = \sqrt{(j + j)(j - j + 1)}|j, j - 1\rangle = \sqrt{2j}|j, j - 1\rangle$$

     to obtain

     $$J_-|j, j\rangle = \sqrt{2j}|j, j - 1\rangle = \sqrt{2j_1}|j_1 - 1; j_2\rangle + \sqrt{2j_2}|j_1; j_2 - 1\rangle.$$

     Projection, i.e., multiplying with the bras $\langle j_1 - 1, ; j_2|$ and $\langle j_1; j_2 - 1|$, results in the Clebsch-Gordan coefficients

     $$\begin{aligned} \left(\begin{array}{cc|c} j_1 & j_2 & j_1 + j_2 \\ j_1 - 1 & j_2 & j_1 + j_2 - 1 \end{array}\right) &= \langle j_1 - 1; j_2 | j_1 + j_2, j_1 + j_2 - 1\rangle \\ &= \sqrt{\frac{j_1}{j_1 + j_2}}, \\ \left(\begin{array}{cc|c} j_1 & j_2 & j_1 + j_2 \\ j_1 & j_2 - 1 & j_1 + j_2 - 1 \end{array}\right) &= \langle j_1; j_2 - 1 | j_1 + j_2, j_1 + j_2 - 1\rangle \\ &= \sqrt{\frac{j_2}{j_1 + j_2}}. \end{aligned}$$

   - $2(j_1 + j_2)$-fold application of the lowering operator generates all Clebsch-Gordan coefficients of the type

     $$\left(\begin{array}{cc|c} j_1 & j_2 & j_1 + j_2 \\ m_1 & m_2 & m \end{array}\right),$$

     where $m < j_1 + j_2$.

2. Next we consider the case $j = j_1 + j_2 - 1$.

   - We express the state with maximal m as a linear combination

$$|j, j\rangle = \alpha|j_1; j_2 - 1\rangle + \beta|j_1 - 1; j_2\rangle,$$

     where α and β are real, because Clebsch-Gordan coefficients are by definition real.
   - Determination of α and β:
     - The normalization of the state results in $\alpha^2 + \beta^2 = 1$.
     - Because of the Condon-Shortley convention we have $\alpha \geq 0$.
     - Apply the raising operator J_+:

$$\begin{aligned} 0 &= J_+|j, j\rangle \\ &= \alpha \underbrace{\sqrt{[j_2 - (j_2 - 1)](j_2 + j_2 - 1 + 1)}}_{= \sqrt{2j_2}} |j_1; j_2\rangle \\ &\quad + \beta\sqrt{2j_1}|j_1; j_2\rangle \\ &= \sqrt{2}\left(\alpha\sqrt{j_2} + \beta\sqrt{j_1}\right)|j_1; j_2\rangle. \end{aligned}$$

       Because $|j_1; j_2\rangle$ is not the zero vector, we conclude

$$0 = \sqrt{j_2}\alpha + \sqrt{j_1}\beta \Rightarrow \beta = -\sqrt{j_2/j_1}\alpha.$$

       Insertion into the normalization condition $\alpha^2(1 + j_2/j_1) = 1$ yields, in combination with the Condon-Shortley condition,

$$\begin{aligned} \alpha &= \left(\begin{array}{cc|c} j_1 & j_2 & j_1 + j_2 - 1 \\ j_1 & j_2 - 1 & j_1 + j_2 - 1 \end{array} \right) = \sqrt{\frac{j_1}{j_1 + j_2}}, \\ \beta &= \left(\begin{array}{cc|c} j_1 & j_2 & j_1 + j_2 - 1 \\ j_1 - 1 & j_2 & j_1 + j_2 - 1 \end{array} \right) = -\sqrt{\frac{j_2}{j_1 + j_2}}. \end{aligned}$$

   - $2(j_1 + j_2 - 1)$-fold application of the lowering operator generates all Clebsch-Gordan coefficients of the type

$$\left(\begin{array}{cc|c} j_1 & j_2 & j_1 + j_2 - 1 \\ m_1 & m_2 & m \end{array} \right)$$

     mit $m < j_1 + j_2 - 1$.

3. $j = j_1 + j_2 - 2$ (Exercise)

4. The procedure is repeated until $j = |j_1 - j_2|$.

4.3.4 Poperties of Clebsch-Gordan coefficients

- Selection rule

$$\left(\begin{array}{cc|c} j_1 & j_2 & j \\ m_1 & m_2 & m \end{array} \right) = 0,$$

  if one of the following conditions is satisfied: $m \neq m_1 + m_2$, $j > j_1 + j_2$ or $j < |j_1 - j_2|$.

- The Clebsch-Gordan coefficients are real. In combination with the Condon-Shortley condition of Eq. (4.23) they are uniquely fixed.

- The absolute value of a Clebsch-Gordan coefficient is always smaller than or equal to 1.

  (C is a real orthogonal $n \times n$ matrix, i.e., $CC^T = \mathbb{1}_{n\times n}$. Then applies

$$\begin{aligned} 1 = \sum_{j=1}^{n} C_{ij} C_{ji}^T &= \sum_{j=1}^{n} C_{ij}^2 \text{ for } i = 1, \ldots, n, \\ \Rightarrow \quad C_{ij}^2 &\leq 1 \text{ for } i, j = 1, \ldots, n, \\ \Rightarrow \quad |C_{ij}| &\leq 1 \text{ for } i, j = 1, \ldots, n. \end{aligned}$$

)

- Recursion relation (Exercise)

$$\begin{aligned} &\sqrt{(j \pm m)(j \mp m + 1)} \left(\begin{array}{cc|c} j_1 & j_2 & j \\ m_1 & m_2 & m \mp 1 \end{array} \right) \\ &= \sqrt{(j_1 \mp m_1)(j_1 \pm m_1 + 1)} \left(\begin{array}{cc|c} j_1 & j_2 & j \\ m_1 \pm 1 & m_2 & m \end{array} \right) \\ &\quad + \sqrt{(j_2 \mp m_2)(j_2 \pm m_2 + 1)} \left(\begin{array}{cc|c} j_1 & j_2 & j \\ m_1 & m_2 \pm 1 & m \end{array} \right). \end{aligned}$$

- Symmetry properties (see A. Lindner, Drehimpulse in der Quantenmechanik, Teubner, Stuttgart, 1984)

$$\begin{aligned} \left(\begin{array}{cc|c} j_1 & j_2 & j \\ m_1 & m_2 & m \end{array} \right) &= \left(\begin{array}{cc|c} j_2 & j_1 & j \\ -m_2 & -m_1 & -m \end{array} \right) \\ &= (-)^{j_1+j_2-j} \left(\begin{array}{cc|c} j_2 & j_1 & j \\ m_2 & m_1 & m \end{array} \right) \\ &= (-)^{j_1+j_2-j} \left(\begin{array}{cc|c} j_1 & j_2 & j \\ -m_1 & -m_2 & -m \end{array} \right). \end{aligned}$$

  In particular,

$$|(j_1, j_2) j, m\rangle = (-)^{j_1+j_2-j} |(j_2, j_1) j, m\rangle.$$