Name:

1	2	3	4	5	6	7	8	$\sum$

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Symmetries in Physics (WS 2018/2019)
Exercise 6

1. **[2]** Verify $\sum_{j=|j_1-j_2|}^{j_1+j_2}(2j+1)=(2j_1+1)(2j_2+1)$ for $j_1, j_2 \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$.

2. **[2]** Define $B=\varphi\mathcal{T}_3$ with $0\leq\varphi<2\pi$ and

$$\mathcal{T}_3=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Verify explicitly

$$R_3(\varphi):=\exp(B)=\begin{pmatrix}\cos(\varphi) & -\sin(\varphi) & 0\\ \sin(\varphi) & \cos(\varphi) & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Remark: The rotation matrices $R_1(\varphi)$ and $R_2(\varphi)$ are obtained analogously using $B=\varphi\mathcal{T}_1$ and $B=\varphi\mathcal{T}_2$, respectively.

3. **[2]** Consider the decomposition

$$|j_1,m_1;j_2,m_2\rangle=\sum_{j,m}\left(\begin{array}{cc|c} j_1 & j_2 & j\\ m_1 & m_2 & m\end{array}\right)|(j_1,j_2)j,m\rangle.$$

Using

$$\sum_{m_1,m_2}\left(\begin{array}{cc|c} j_1 & j_2 & j\\ m_1 & m_2 & m\end{array}\right)\left(\begin{array}{cc|c} j_1 & j_2 & j'\\ m_1 & m_2 & m'\end{array}\right)=\delta_{jj'}\delta_{mm'},$$

derive the relation

$$|(j_1,j_2)j,m\rangle=\sum_{m_1,m_2}\left(\begin{array}{cc|c} j_1 & j_2 & j\\ m_1 & m_2 & m\end{array}\right)|j_1,m_1;j_2,m_2\rangle.$$

4. **[2]** Using $\langle(j_1,j_2)j,m|J_\pm|j_1,m_1;j_2,m_2\rangle$, $J_\pm=J_1\pm iJ_2$, derive the recursion relation

$$\begin{aligned}&\sqrt{(j\pm m)(j\mp m+1)}\left(\begin{array}{cc|c} j_1 & j_2 & j\\ m_1 & m_2 & m\mp 1\end{array}\right)\\ =\;&\sqrt{(j_1\mp m_1)(j_1\pm m_1+1)}\left(\begin{array}{cc|c} j_1 & j_2 & j\\ m_1\pm 1 & m_2 & m\end{array}\right)\\ &+\sqrt{(j_2\mp m_2)(j_2\pm m_2+1)}\left(\begin{array}{cc|c} j_1 & j_2 & j\\ m_1 & m_2\pm 1 & m\end{array}\right).\end{aligned}$$

Hint: $\langle j,m|J_\pm=(J_\mp|j,m\rangle)^\dagger$.

5. **[5]** Consider j_1, j_2, and j with $|j_1 - j_2| \leq j \leq j_1 + j_2$. We make use of the abbreviated form

$$|m_1; m_2\rangle := |j_1, m_1; j_2, m_2\rangle,$$
$$|j, m\rangle := |(j_1, j_2)j, m\rangle.$$

Determine in step 3 of subsection 4.3.3 (Handout 11) the coefficients α, β, and γ of the decomposition

$$|j_1 + j_2 - 2, j_1 + j_2 - 2\rangle = \alpha|j_1; j_2 - 2\rangle + \beta|j_1 - 1; j_2 - 1\rangle + \gamma|j_1 - 2; j_2\rangle.$$

6. **[5]** Using the ladder operators, determine in analogy to the lecture the Clebsch-Gordan coefficients

$$\left(\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{3}{2} \\ -1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right), \quad \left(\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right), \quad \left(\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{3}{2} \\ -1 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right),$$
$$\left(\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right), \quad \left(\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right), \quad \left(\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right), \quad \left(\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right).$$

Hint: If applicable, you may also use the symmetry properties of subsection 4.3.4 (Handout 11).

7. Consider pion-nucleon scattering.

   (a) **[1]** Identify the eight independent physical reactions which are consistent with charge conservation.
   Hint: Because of the time-reversal invariance of the strong interaction, the reactions $\pi^0 p \to \pi^+ n$ and $\pi^+ n \to \pi^0 p$ do not count as independent reactions.

   (b) **[4]** The invariance of the Hamilton operator of the strong interaction with respect to SU(2) (isospin symmetry) results in the invariance of the S matrix under SU(2). Using

   $$\langle I', M_I'|T|I, M_I\rangle = T_I \delta_{I'I} \delta_{M_I' M_I},$$

   express the eight processes in terms of the isospin amplitudes $T_{\frac{1}{2}}$ and $T_{\frac{3}{2}}$.

   (c) **[2]** Let

   $$T_A := T(\pi^+ n \to \pi^+ n),$$
   $$T_B := T(\pi^+ n \to \pi^0 p).$$

   Determine $|T_A|^2 + |T_B|^2$ using the isospin amplitudes. Note that T_A and T_B are complex numbers.

   (d) **[1]** In the Δ-resonance region, we may neglect $T_{\frac{1}{2}}$ in comparison to $T_{\frac{3}{2}}$. What do you obtain for the ratio $\sigma(\pi^+ p)/\sigma(\pi^- p)$ of the elastic cross sections?

8. We consider the strong decay of a nucleon resonance N^* with isospin $I = \frac{1}{2}$ into a πN final state.

   (a) **[1]** How many independent isospin amplitudes are required to describe the decay?

   (b) **[2]** What is the relation between the amplitudes of the decays $N^{*+} \to \pi^+ n$ and $N^{*+} \to \pi^0 p$?

   (c) **[1]** The rate $1/\tau$ of a decay is proportional to the absolute value square of the matrix element. What is the ratio of the rates of the two decays?