Theoretical Elementary Particle Physics
Exercise 5

7 January 2019

1 Running mass and anomalous dimension in QCD (50 points)

We define a running quark mass $m(\mu)$ through the equation

$$\mu \frac{d}{d\mu} m(\mu) = \gamma_m m(\mu)\,, \tag{1}$$

where γ_m is the so-called anomalous dimension of the mass.

(a) **(10 points)** Show first that γ_m is equivalently given through the renormalization group equation

$$\gamma_m = -\frac{1}{Z_m(\mu)} \mu \frac{d}{d\mu} Z_m(\mu)\,, \tag{2}$$

where Z_m is the mass renormalization constant.

(b) **(15 points)** The anomalous dimension of the mass can be expanded in a perturbation series in the strong coupling constant as:

$$\gamma_m = \gamma_m^0 \frac{g^2(\mu)}{(4\pi)^2} + \mathcal{O}(g^4)\,. \tag{3}$$

Use the known expression for $Z_m = 1 - \frac{g^2}{(4\pi)^2}\frac{4}{\epsilon}$ to derive the value of γ_m^0. **Hint:** formula

$$\beta(g) = -\epsilon g \left[1 + \frac{1}{\epsilon}\frac{\beta_0}{(4\pi)^2} g^2 + \mathcal{O}(g^4)\right] \tag{4}$$

is useful and the expression for β_0 is shown in Eq. 7.

(c) **(25 points)** By inserting Eq. 3 into Eq. 1, derive the one-loop expression of the running quark mass:

$$m(Q^2) = \frac{\hat{m}}{(\ln(Q^2/\Lambda^2_{\text{QCD}}))^{d_m}}\,, \tag{5}$$

with $d_m = 4/(11 - 2/3 N_f)$ and where $\hat{m}$ is an integration constant (the mass analog of Λ_{QCD}) that equals $\hat{m} = m(Q^2 = e\Lambda^2_{\text{QCD}})$ where e stands for Euler's number.

2 Running coupling constant in QCD in two loop order (50 points)

In QCD, the β-function in two loop order is of the form

$$\beta(g) = -\frac{\beta_0}{(4\pi)^2} g^3 - \frac{\beta_1}{(4\pi)^4} g^5 + \cdot\,, \tag{6}$$

where

$$\beta_0 = 11 - \frac{2}{3} N_f\,, \tag{7}$$

and

$$\beta_1 = 102 - \frac{38}{3} N_f\,. \tag{8}$$

Show that the running coupling constant can be written as:

$$\alpha_s(Q^2) \equiv \frac{g^2(Q^2)}{4\pi} \approx \frac{4\pi}{\beta_0 \ln\left(Q^2/\Lambda^2_{\rm QCD}\right)} \left[1 - \frac{\beta_1}{\beta_0^2} \frac{\ln\ln\left(Q^2/\Lambda^2_{\rm QCD}\right)}{\ln\left(Q^2/\Lambda^2_{\rm QCD}\right)}\right]\,. \tag{9}$$

Hints: the strategy in generally similar to the one where only 1-loop contribution is considered. Integrate between two mass scales and separate the dependence on each scale which implies that both LHS and RHS are equal to a constant that is chosen to be $\frac{-\beta_0}{4\pi} \ln\left(\Lambda^2_{\rm QCD}\right)$.

In general, we are interested in $Q^2 \gg \Lambda^2_{\rm QCD}$ because perturbative expansion is valid in that case. Therefore, the following approximations can be applied:

$$\ln\left[\frac{\beta_0^2}{4\pi} \ln\left(\frac{Q^2}{\Lambda^2_{\rm QCD}}\right) + \frac{\beta_1}{4\pi}\right] \approx \ln\ln\left[\frac{Q^2}{\Lambda^2_{\rm QCD}}\right] \tag{10}$$

$$\left[1 + \frac{\beta_1}{\beta_0^2} \frac{\ln\ln\left(Q^2/\Lambda^2_{\rm QCD}\right)}{\ln\left(Q^2/\Lambda^2_{\rm QCD}\right)}\right]^{-1} \approx \left[1 - \frac{\beta_1}{\beta_0^2} \frac{\ln\ln\left(Q^2/\Lambda^2_{\rm QCD}\right)}{\ln\left(Q^2/\Lambda^2_{\rm QCD}\right)}\right] \tag{11}$$