Quantum chromodynamics in the non-perturb. regime: implications of unitarity, dispersion relations

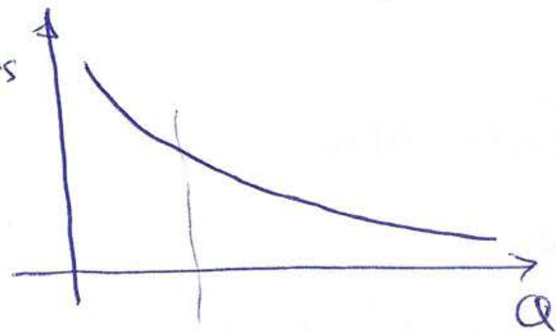
$$\mathcal{L}_{QCD} = \bar{q}\,(i\gamma^\mu \partial_\mu - m)\,q - g\,\bar{q}\,\gamma^\mu \frac{\lambda_a}{2}\,q\,A_\mu^a - \frac{1}{4} F^{a,\mu\nu} F_{\mu\nu}^a + \mathcal{L}_{GHOST} + \mathcal{L}_{GAUGE\ FIXING}$$

$\downarrow$ RENORMALIZATION

$$\mathcal{L}_{BARE} = \underbrace{\mathcal{L}_{RENORM}}_{\text{physical}} + \mathcal{L}_{COUNTER\ TERMS}$$

$$\alpha_s(Q^2) = \frac{g^2(Q^2)}{4\pi} = \frac{4\pi}{\beta_0 \ln\left(Q^2/\Lambda_{QCD}^2\right)}, \quad \beta_0 = 11 - \frac{2}{3} N_f > 0$$

$\Lambda_{QCD} \simeq 0.2$ GeV
FUNDAM. PAR. OF QCD
(MASS SCALE)



Q (CHAR. MOMENTUM, ENERGY SCALE)

1) AT LARGE SCALES (α_s IS SMALL) $\Rightarrow$ PERT. THEORY.
2) AT SMALL SCALES (α_s IS LARGE) $\Rightarrow$ NON PERT. REGION

QUARKS AND GLUONS ARANGE THEMSELVES INTO COLOURLESS **HADRONS**: $\underbrace{\text{MESONS, BARYONS, ..}}_{\text{NEW D.O.F.}}$

POSSIBLE WAY OUT:

1) EFFECTIVE FIELD THEORIES
   (BASED ON SYMMETRIES OF QCD)
2) LATTICE QCD
3) S-MATRIX CONSTRAINTS (sometimes called "DISPERSION THEORY")

S-matrix CONSTRAINTS

1) CROSSING SYMMETRY (antiparticle is not a different particle)
2) UNITARITY (sum of propabilities of all possible outcomes equal to one)
3) CAUSALITY (future cannot change the past)

THESE CONSRAINTS + EXP. DATA $\Rightarrow$ ONE CAN MAKE A PREDICTION.

CROSSING SYMMETRY

$S = 1 \underline{\pm} iT$ ⚠ NOTE CONVENTION

initial state $t = -\infty$ $|i\rangle$
final state $t = +\infty$ $|f\rangle$

$$S_{fi} = \delta_{fi} + (2\pi)^4 \delta^4(P_f - P_i) \underbrace{iM_{fi}}_{\text{SCATT. AML.}}$$

EXP. OBSERVABLE: CROSS SECTION

$$d\sigma_{2\to 2} = \frac{1}{2E_1 2E_2 |v_1 - v_2|} \underbrace{\left(\prod_{f=1}^{2} \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) (2\pi)^4 \delta(P_f - P_i)}_{\text{2 body phase space}} \times$$

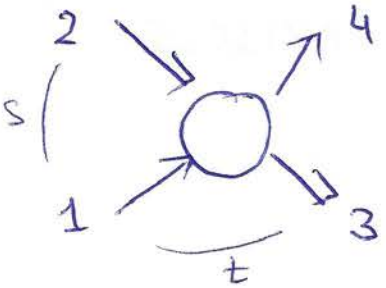
$$\times |M_{fi}|^2$$

PARTICLES WITH SPIN

$$|M_{fi}|^2 \to \overline{|M_{fi}|^2} = \frac{1}{(2S_1+1)(2S_2+1)} \sum_{\substack{s_1 s_2 \\ s_1' s_2'}} |M_{s_1 s_2 \ldots}|^2$$

AVERAGE OVER INITIAL POLARIZ.
SUM OVER FINAL POLARIZ.

SPINLESS PARTICLES
(FOR SIMPLICITY)



$M(s,t,u)$ SCATT. AMPL IS A FUNC. OF INVARIANTS (DUE TO LORENZ INVARIANCE)

DEFINE:

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$
$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2$$
$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2$$

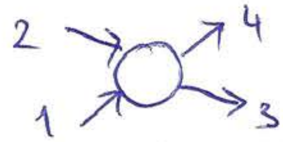
$$p_1 + p_2 = p_3 + p_4 \ , \quad p_i^2 = m_i^2$$

$$\Rightarrow s + t + u = m_1^2 + m_2^2 + 2p_1p_2 + p_1^2 + m_3^2 - 2p_1p_3 + p_1^2 + m_4^2 - 2p_1p_4 = \sum_{i=1}^{4} m_i^2 + 2p_1\underbrace{(\cancel{p_1 + p_2 - p_3 - p_4})}_{0}$$

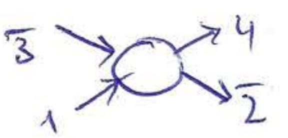
$$\boxed{s + t + u = \sum_{i=1}^{4} m_i^2}$$

GENERAL CASE: $(3n - 10)$ INDEPENDENT VARIABLES FOR n-POINT FUNCTION

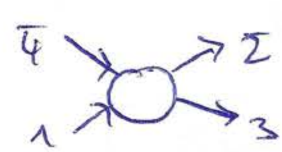
$M(s,t)$ describes:



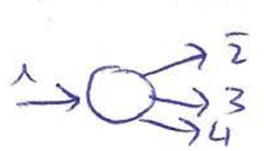
$1+2 \to 3+4$, s-channel



$1+\bar{3} \to \bar{2}+4$, t-channl



$1+\bar{4} \to \bar{2}+3$, u-channel



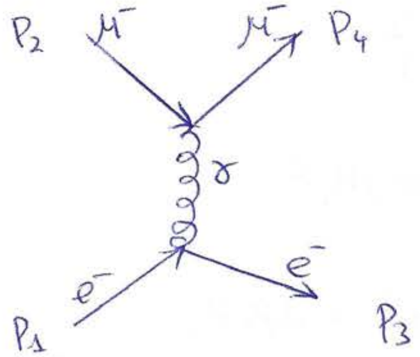
$1 \to \bar{2}+3+4$, DECAY (IF $m_1 > m_2 + m_3 + m_4$)

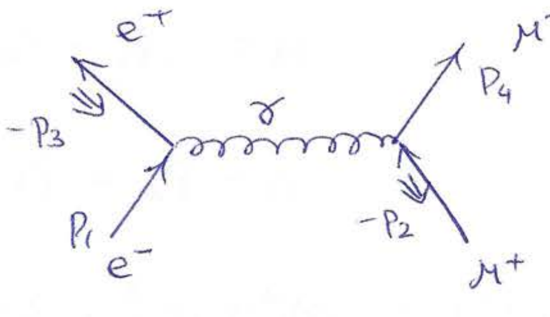
IN REL. QUANTUM FIELD THEORY:

AN INCOMING PARTICLE WITH MOM $+P$, CORRESPOND TO OUTGOING ANTIPARTICLE WITH MOM $-P$.

EXAMPLE (from QED)

$e^-\mu^- \to e^-\mu^-$ $\qquad\qquad$ $e^-e^+ \to \mu^-\mu^+$





$$s = (P_1+P_2)^2 = (P_3+P_4)^2$$
$$t = (P_4-P_2)^2 = (P_3-P_1)^2$$
$$u = (P_4-P_1)^2 = (P_3-P_2)^2$$

$$\tilde{s} = (P_1+(-P_3))^2 = (P_4+(-P_2))^2$$
$$\tilde{t} = (P_4-(-P_3))^2$$
$$\tilde{u} = (P_4-P_1)^2$$

$$s \to \tilde{t}$$
$$t \to \tilde{s}$$
$$u \to \tilde{u}$$

$$\overline{|M|^2}_{e\mu\to e\mu} = 2e^4\left(\frac{s^2+u^2}{t^2}\right) \qquad \text{s-channel}$$

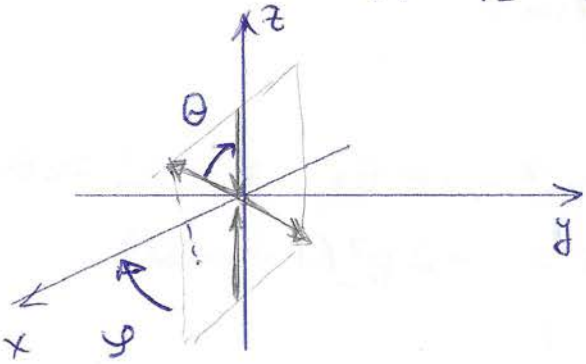
$$\overline{|M|^2}_{ee\to\mu\mu} = 2e^4\left(\frac{\tilde{t}^2+\tilde{u}^2}{\tilde{s}^2}\right) \qquad \text{t-channel}$$

ONE CAN GET THE FORMULA FOR t- AND u-channels FROM THE s-CHANNEL BY APPLYING APPROPRIATE VARIABLE EXCHANGE

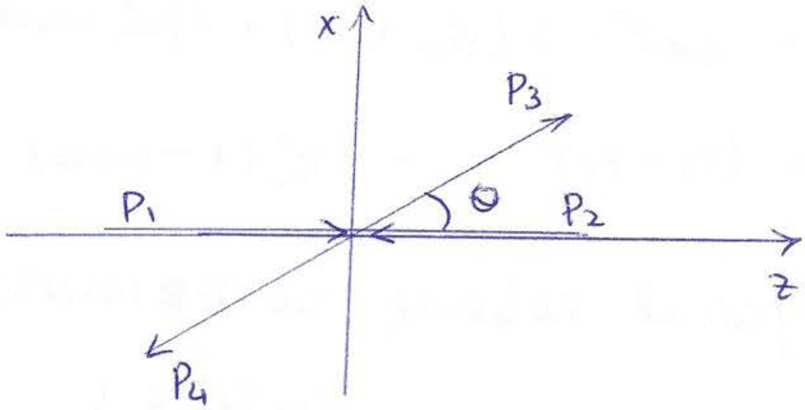
PHYSICAL REGIONS FOR s, t AND u-channels
ARE DEFFERENT ?!

LET'S CONSIDER CENTER OF MASS FRAME (C.M).

$$\vec{P}_1 + \vec{P}_2 = \vec{P}_3 + \vec{P}_4$$



$\varphi = 0$: ONE CAN CHOOSE
XZ PLANE



$$|\vec{P}_1| = |\vec{P}_2| = p_{cm}$$
$$|\vec{P}_3| = |\vec{P}_4| = p'_{cm}$$

$$P_1 = \{E_1, 0, 0, p_{cm}\}$$
$$P_2 = \{E_2, 0, 0, -p_{cm}\}$$
$$P_3 = \{E_3, p'_{cm}\sin\theta, 0, p'_{cm}\cos\theta\}$$
$$P_4 = \{E_4, -p'_{cm}\sin\theta, 0, -p'_{cm}\cos\theta\}$$

$$\begin{cases} E_1^2 = p_{cm}^2 + m_1^2 \\ E_2^2 = p_{cm}^2 + m_2^2 \end{cases} \qquad \begin{aligned} s &= (p_1+p_2)^2 = (E_1+E_2)^2 \\ \sqrt{s} &= E_1 + E_2 \end{aligned}$$

$$\Downarrow$$

$$\begin{cases} p_{cm} = \dfrac{\lambda^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}} \\[2ex] p'_{cm} = \dfrac{\lambda^{1/2}(s, m_3^2, m_4^2)}{2\sqrt{s}} \end{cases}, \qquad \lambda \equiv \left(s-(m_1-m_2)^2\right)\times\left(s-(m_1+m_2)^2\right)$$

källen function

PHYSICAL REGIONS FOR $m_1 = m_2 = m_3 = m_4 = m$

$p'_{cm} = p_{cm}$

$E_1 = E_2 = E_3 = E_4 = \sqrt{p_{cm}^2 + m^2}$

$$s = (E_1 + E_2)^2 = 4(\underbrace{p_{cm}^2}_{\geq 0} + m^2) \geq 4m^2$$

$$t = (p_3 - p_1)^2 = p_3^2 + p_1^2 - 2p_1 p_3 = 2m^2 - 2E_1E_3 + 2p_{cm}p'_{cm}\cos\theta$$
$$= \cancel{2m^2} - 2(p_{cm}^2 + \cancel{m^2}) + 2p_{cm}^2\cos\theta = -2p_{cm}^2(1-\cos\theta)$$

$$u = (p_3 - p_2)^2 = -2p_{cm}^2(1-\cos\theta)$$

Physical REGION DETERMINED AS

$$|\cos\theta| \leq 1$$

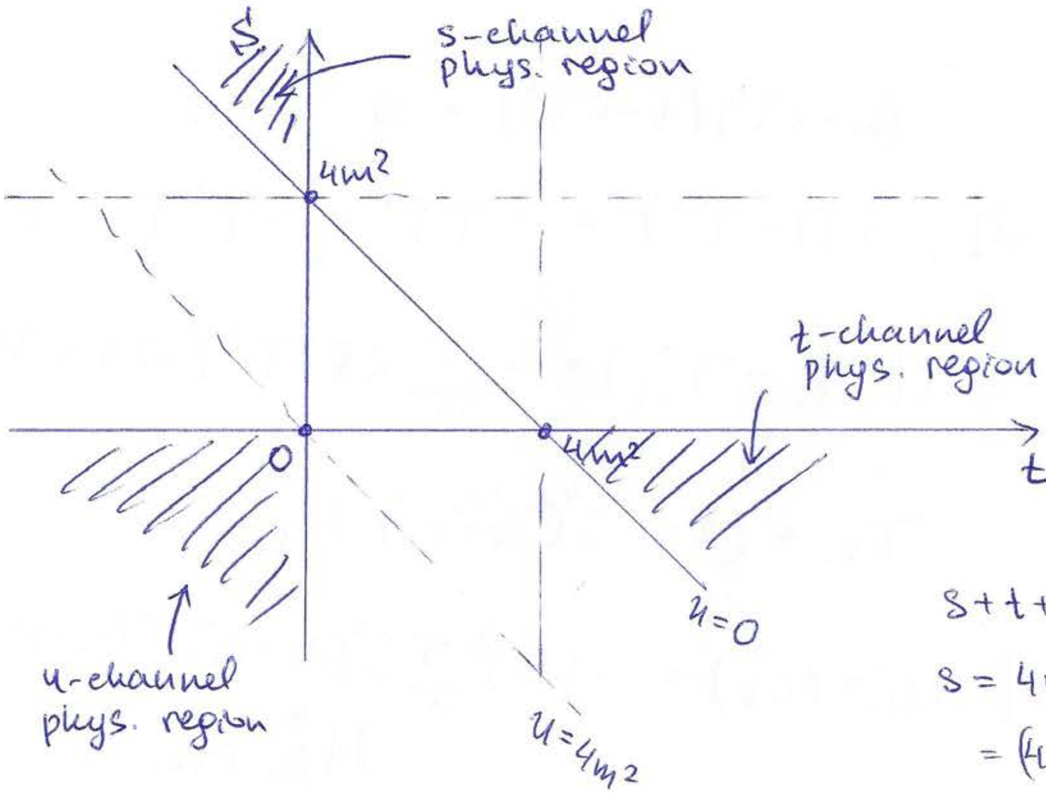
$\Rightarrow$

CROSSING	$s \geq 4m^2$,	$t, u \leq 0$	s-channel
	$t \geq 4m^2$,	$s, u \leq 0$	t-channel
	$u \geq 4m^2$,	$s, t \leq 0$	u-channel
			$m_1 = m_2 = m_3 = m_4 = m$

FOR ARBITRARY MASSES, THE BORDER OF PHYS. REGION IS MORE COMPLICATED $t(s)$, $s \geq 4m^2$

$\uparrow$ curve from $|\cos\theta| \leq 1$ condition.

GRAPHYCAL REPRESENTATION

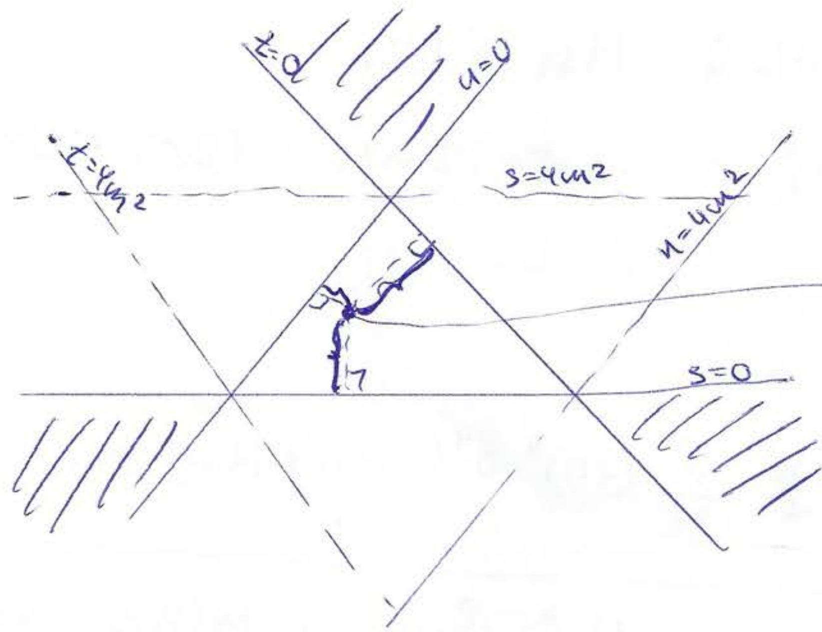


$$s+t+u = 4m^2$$

$$s = 4m^2 - t - u =$$

$$= (4m^2 - u) - t$$

ANOTHER REPRESENTATION (MORE COMMON)



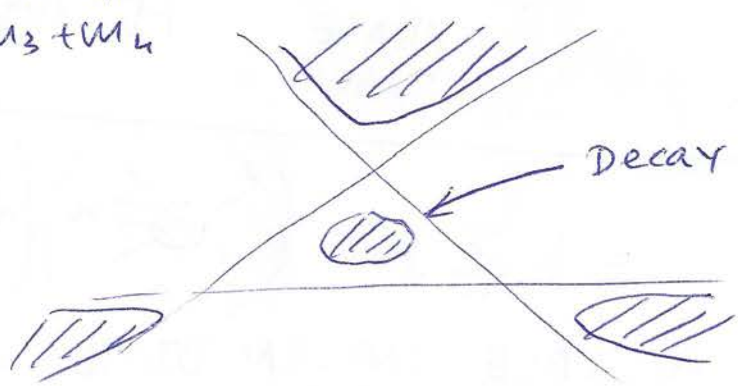
$$s+t+u = 4m^2$$

$$= \sum m_i^2$$

SUM OF PERPEND. DISTANCES TO EACH SIDE IS EQUAL TO THE HEIGHT OF TRIANGLE:

$$\sum m_i^2$$

[$m_1 \neq m_2 \neq m_3 \neq m_4$
& $m_1 > m_2 + m_3 + m_4$



Unitarity

$SS^\dagger = S^\dagger S = 1$

$SS^\dagger = 1\!\!1$

$S = 1 + iT$

$(1 + iT)(1 - iT^\dagger) = 1\!\!1$

$\langle f | \quad i(T - T^\dagger) = -TT^\dagger \equiv -T^\dagger T \quad | i \rangle$

$$i(T_{fi} - T^*_{if}) = -\sum_n \langle f | T^\dagger | n \rangle \langle n | T | i \rangle$$

$$T_{fi} \equiv (2\pi)^4 \delta^4(P_f - P_i) M_{fi}$$

$$i(2\pi)^4 \delta^4(P_f - P_i)\left(M_{fi} - M^*_{if} \right) = -(2\pi)^8 \sum_n \delta^4(P_f - P_n)\delta^4(P_n - P_i) M^*_{nf} M_{ni}$$

$$M_{fi} - M^*_{if} = i(2\pi)^4 \sum_n \delta^4(P_n - P_i) M^*_{nf} M_{ni}$$

TIME REVERSAL: $M_{fi} = M_{if}$

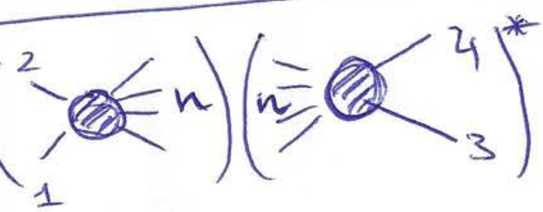
$$M_{fi} - M^*_{fi} = \cancel{\mathrm{Re}M} + i\,\mathrm{Im}M - (\cancel{\mathrm{Re}M} - i\,\mathrm{Im}M) = 2i\,\mathrm{Im}\,M_{fi}$$

$$\mathrm{Im}\,M_{fi} = \frac{1}{2} \sum_n (2\pi)^4 \delta^4(P_n - P_i) M^*_{nf} M_{ni}$$

$$\boxed{\mathrm{Im}\,M(p_1p_2 \to p_3p_4) = \frac{1}{2} \sum_n \underbrace{\int d\Phi_n}_{\text{PHASE SPACE}} \cdot M(p_1p_2 \to \{n\}) \cdot M^*(\{n\} \to p_3p_4)}$$

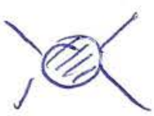
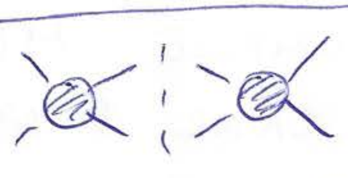
FORWARD SCATT.
$p_1 = p_3,\ p_2 = p_4$: OPTICAL THEOREM

$$= \frac{1}{2} \sum_n \int d\Phi_n$$



SUM OVER ALL POSSIBLE INTERMEDIATE STATES

ELASTIC UNITARITY

Im  $= \frac{1}{2}\int d\Phi_2$ 

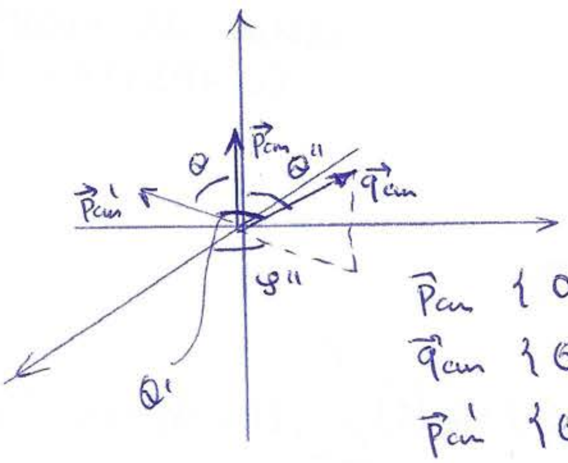
$$d\Phi_2 = \frac{d^3q_1}{(2\pi)^3 2E_{q_1}}\frac{d^3q_2}{(2\pi)^3 2E_{q_2}}(2\pi)^4\,\delta^4(p_1+p_2-q_1-q_2)$$

$$= \begin{Bmatrix}\text{C.M.}\\ \text{FRAME}\end{Bmatrix}\;\frac{dq_{cm}\cdot q_{cm}^2\,d\Omega''}{(2\pi)^3\,2E_{q_1}\,2E_{q_2}}(2\pi)\,\delta(\sqrt{s}-E_{q_1}-E_{q_2}) \;=$$

$$E_{q_1} = \sqrt{q_{cm}^2+m_1^2}$$

$$E_{q_2} = \sqrt{q_{cm}^2+m_2^2}$$

$$\delta(f(x)) = \frac{\delta(x-x_0)}{|f'(x)|_{x=x_0}}$$



$\vec{p}_{cm}\ \{0,0\}$ ALONG OZ

$\vec{q}_{cm}\ \{\Theta'',\varphi''\}$

$\vec{p}\,'_{cm}\ \{\Theta,\varphi=0\}$

$$\delta\left(\sqrt{q_{cm}^2+m_1^2}+\sqrt{q_{cm}^2+m_2^2}-\sqrt{s}\right) = \frac{\delta(q_{cm}-q_{cm}^*)}{\left|\frac{q_{cm}}{E_{q_1}}+\frac{q_{cm}}{E_{q_2}}\right|} =$$

$$q_{cm}^* = \frac{\lambda^{1/2}(s,m_1^2,m_2^2)}{2\sqrt{s}} \qquad = \frac{\delta(q_{cm}-q_{cm}^*)\,E_{q_1}E_{q_2}}{q_{cm}(E_{q_1}+E_{q_2})}$$

$$= \frac{q_{cm}^*}{16\pi^2}\frac{d\Omega''}{\sqrt{s}}$$

$$\boxed{\begin{aligned} d\Phi_2 &= d\Omega''\,\frac{\beta(s)}{32\pi^2}\\ \beta(s) &= \frac{2q_{cm}}{\sqrt{s}}\end{aligned}}$$

$$\text{Im}[M(s,\cos\Theta)] = \frac{1}{2}\int d\Phi_2\cdot$$
$$\cdot[M(s,\cos\Theta'',\varphi'')][M^*(s,\cos\Theta')]$$

PARTIAL WAVE AMPLITUDES

USEFUL TO DECOMPOSE $M(s,t) \equiv M(s,\cos\theta)$ IN TERMS OF RATIONAL FUNCTIONS

$$\underbrace{M(s,\cos\theta)}_{\text{FULL AMPL.}} = \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(\cos\theta) \underbrace{M_\ell(s)}_{\text{P.W. AMPLITUDE}}$$

UNITARITY FOR P.W. AMPLITUDES

NOTE: FOR PARTICLES WITH SPINS, EXPANSION IS MORE COMPLICATED.

$$\mathrm{Im}\left(\sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(\cos\theta)\, M_\ell(s) \right) =$$

$$= \frac{1}{2} \int d\Omega'' \frac{\beta(s)}{32\pi^2} \sum_{\ell''=0}^{\infty} (2\ell''+1)\, P_{\ell''}(\cos\theta'')\, M_{\ell''}(s) \sum_{\ell'=0} (2\ell'+1)\, P_{\ell'}(\cos\theta')\, M^*_{\ell'}(s)$$

ADDITION THEOREM:

$$\cos\theta' = \sin\theta \sin\theta'' \cos\varphi'' + \cos\theta \cos\theta''$$

$$\int d\Omega''\, P_{\ell''}(\cos\theta'')\, P_{\ell'}(\cos\theta') = \delta_{\ell'\ell''} \frac{4\pi}{2\ell'+1} P_{\ell'}(\cos\theta)$$

$$\Rightarrow \boxed{\mathrm{Im}\, M_\ell(s) = \underbrace{\frac{1}{16\pi} \beta(s)}_{\equiv \rho(s)} |M_\ell|^2}$$

IF PARTICLES IDENTICAL: $\times \frac{1}{2}$.

CAUSALITY & ANALYTICITY

UNITARITY: CONSTRAINT IMAGINARY PART OF THE AMPLITUDE.

HOWEVER, $M(s,t)$ IS REAL UNLESS DENOMINATORS

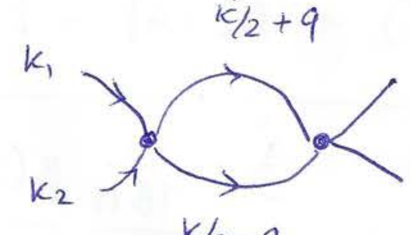
$$\frac{1}{k^2 - m^2 + i\varepsilon}$$

VANISH, SO THAT $i\varepsilon$ PRESCRIPTION FOR TREATING POLES BECOMES RELEVANT.

THE SIGN OF $\underline{+i\varepsilon}$ IS DICTATED BY CAUSALITY (SEE QFT I).

$$i\Delta(x-y) = \langle 0 | T(\varphi(x)\varphi^{+}(y)) | 0 \rangle =$$
$$= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{1}{k^2 - m^2 + i\varepsilon}$$

LET'S CONSIDER EXAMPLE



φ^4 theory

$$iM(s) = \underbrace{\frac{\lambda^2}{2}}_{\text{SYM. FACTOR}} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k/2+q)^2 - m^2 + i\varepsilon} \cdot \frac{1}{(k/2-q)^2 - m^2 + i\varepsilon}$$

$$= \frac{\lambda^2}{2} \cdot \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx \frac{1}{(\ell^2 - \Delta + i\epsilon)^2}$$

$$= \frac{i\lambda^2}{2\cdot(4\pi)^2} \cdot \left(\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi\mu^2) - \int_0^1 dx \ln(m^2 - x(1-x)s - i\varepsilon) \right)$$

$\varepsilon \equiv 2 - d/2$

$\lambda \to \lambda\mu^{\varepsilon}$

THE DIVERGENT CONTRIBUTION CAN BE SEPERATED AS

$$\overline{M}(s) = M(s) - M(0) =$$

$$= -\frac{1}{2}\cdot\frac{\lambda^2}{16\pi^2}\int_0^1 dx \ln\left(1 - \frac{x(1-x)s}{m^2} - i\varepsilon\right)$$

$$= \frac{\lambda^2}{2}\cdot\frac{1}{16\pi^2}\left(2 + \beta \ln\left(\frac{\beta-1}{\beta+1}\right)\right)$$

$$\beta = \sqrt{1 - \frac{4m^2}{s}} \equiv \frac{2p_{cm}}{\sqrt{s}}$$

PROPERTIES OF $\overline{M}(s)$

$$\boxed{\overline{M}(s) \quad \text{REAL FOR} \quad s < 4m^2}$$

$1/2 - 1/2\beta < x < 1/2 + 1/2\beta$

SINCE $\ln(A - i\varepsilon) = \ln|A| - i\pi\,\Theta(A<0)$

$$\boxed{\operatorname{Im}\overline{M}(s) = \frac{\lambda^2}{2}\cdot\frac{1}{16\pi}\,\beta(s)\,\Theta(s>4m^2)}$$

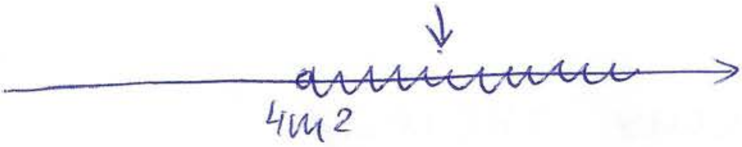
$$\operatorname{Im}\overline{M} = -\frac{\lambda^2}{2\cdot 16\pi^2}\cdot\int_{1/2-\frac{1}{2}\beta}^{1/2+1/2\beta} dx\cdot(-\pi) = \frac{\lambda^2}{2}\cdot\frac{1}{16\pi}\cdot\beta$$

CONSEQUENCES

IF WE IGNORE $i\varepsilon$ PRESCRIPTION IN THE PROPAGATOR, THEN

$$\operatorname{Im}\overline{M}(s \pm i\varepsilon) = \pm\frac{\lambda^2}{2}\,\frac{1}{16\pi}\,\beta(s)\,\Theta(s>4m^2)$$

$\Rightarrow$ THERE IS A BRANCH CUT SINGULARITY RELATED TO UNITARITY.



BUT $i\varepsilon$ PRESCRIPTION (CAUSALITY) DICTATES THAT THE PHYSICAL REGION IS DETERMINED AS THE LIMIT

$$M^{PHYS}(s) \equiv \lim_{\epsilon \to +0} M(s+i\varepsilon)$$

IDEA: BASED ON **PERTURBATION THEORY** (FEYNMAN GRAPHS) ANALYTICALLY CONTINUE THE AMPLITUDE FROM THE PHYSICAL REGION TO THE COMPLEX PLANE.

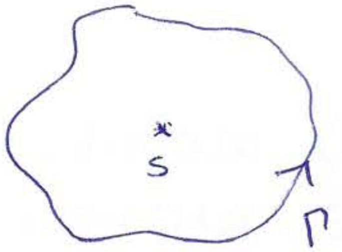
SINCE $M(s)$ is REAL BELOW THE CUT, SCHWARZ' REFLECTION PRINCIPLE MUST HOLD

$$M^*(s+i\varepsilon) = M(s-i\varepsilon)$$ ← (CALLED HERMITEAN ANALYTIC FUNC.)

ALLOWS TO RELATE $Im(M)$ TO DISCONTINUITY:

$$\begin{aligned} \text{Disc}\, M(s) &\equiv M(s+i\varepsilon) - M(s-i\varepsilon) = \\ &= M(s+i\varepsilon) - M^*(s+i\varepsilon) = \\ &= 2i\, M(s+i\varepsilon) \end{aligned}$$

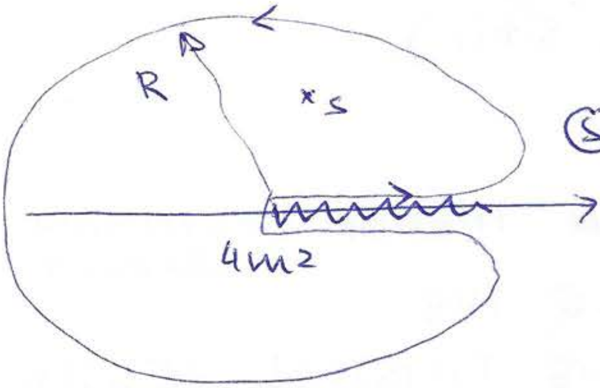
COMPLEX ANALYSIS



CAUCHY THEOREM
(FOR ANALYTIC FUNC)

$$M(s) = \frac{1}{2\pi i} \oint_{\Gamma} ds' \, \frac{M(s')}{s'-s}$$

INTEGRATION : ANTICLOCKWISE



(S)
$$M(s) = \frac{1}{2\pi i} \int_{4m^2}^{\infty} ds' \, \frac{M(s'+i\epsilon)}{s'-s} + \underbrace{\cancel{\frac{1}{2\pi i} \oint_{|s'|=R} ds' \, \frac{M(s')}{s'-s}}}_{0} + \frac{1}{2\pi i} \int_{\infty}^{4m^2} ds' \, \frac{M(s'-i\varepsilon)}{s'-s}$$

ASSUME THAT
$M(s \to \infty) \to 0$
ON THE LARGE
SEMI-CIRCLE

$$M(s) = \frac{1}{2\pi i} \int_{4m^2}^{\infty} ds' \, \frac{M(s'+i\varepsilon) - M(s'-i\varepsilon)}{s'-s} \equiv \frac{1}{2\pi i} \int_{4m^2}^{\infty} ds' \, \frac{\text{Disc}\, M(s')}{s'-s} \Rightarrow$$

$$\boxed{M(s) = \frac{1}{\pi} \int_{4m^2}^{\infty} ds' \, \frac{\text{Im}\, M(s'+i\varepsilon)}{s'-s}}$$

UNSUBTRACTED
DISPERSION
RELATION

NOTATIONS: $+i\varepsilon$ is omitted for simplicity in the argument whenever function has to be evaluated on the upper rim of the cut, otherwise (i.e. $-i\varepsilon$) we will write explicitly.

CHECK UNITARITY

$$\overset{\text{phys}}{M(s)} = \lim_{\epsilon \to +0} M(s+i\varepsilon) = \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \, \frac{\operatorname{Im} M(s')}{s'-s-i\epsilon} \; (=)$$

Sokhotski–Plemelj theorem

$$\int dx \, \frac{f(x)}{x \pm i\varepsilon} = \text{p.v.} \int dx \frac{f(x)}{x} \mp i\pi f(0)$$

$$(=) \; \underbrace{\frac{1}{\pi} \text{P.V.} \int_{4m^2}^{\infty} ds' \, \frac{\operatorname{Im} M(s')}{s'-s}}_{\operatorname{Re} M(s)} + i \operatorname{Im} M(s)$$

ASSYMPTOTICS

WE ASSUMED $M(s) \underset{s\to\infty}{\to} 0$, SO INTEGRAL CONVERGES AND EQUAL 0 ON THE LARGE SEMI-CIRCLE. OTHERWISE ONE CAN WRITE A DISPERSION RELATION FOR

$$\frac{M(s) - M(s_0)}{s - s_0}$$

$$M(s) = \underbrace{M(s_0)}_{} + \frac{s-s_0}{\pi} \int_{4m^2}^{\infty} \underbrace{\frac{ds'}{s'-s_0}}_{} \, \frac{\operatorname{Im} M(s')}{s'-s}$$

SO CALLED <u>SUBTRACTION</u> CONSTANT

IMPROVED CONVERGENCE (ADDIT. POWER OF s' IN DENOM)

<u>IMPORTANT:</u> $s_0 < 4m^2$, SO $M(s_0)$ IS REAL.

DISPERSION REPRESENTATION FOR A SCALAR LOOP FUNCTION (ALSO CALLED SPECTRAL REPRESENTATION)

$$J(s) \equiv \frac{M(s) \times 2}{\lambda^2}; \qquad \boxed{\bar{J}(s) = J(s) - J(0) = = \frac{1}{16\pi^2}\left(2 + \beta \ln \frac{\beta-1}{\beta+1}\right)}$$

$\lambda = 1$

$$\mathrm{Im} J(s) = \mathrm{Im} \bar{J}(s) = \frac{1}{16\pi} \beta(s)\, \Theta(s > 4m^2)$$

$$\Rightarrow \quad \left.\begin{array}{c}\mathrm{Im} J(s) \to \text{const} \\ s \to \infty\end{array}\right\| \Rightarrow \text{NEED ONE SUBTRACTION}$$

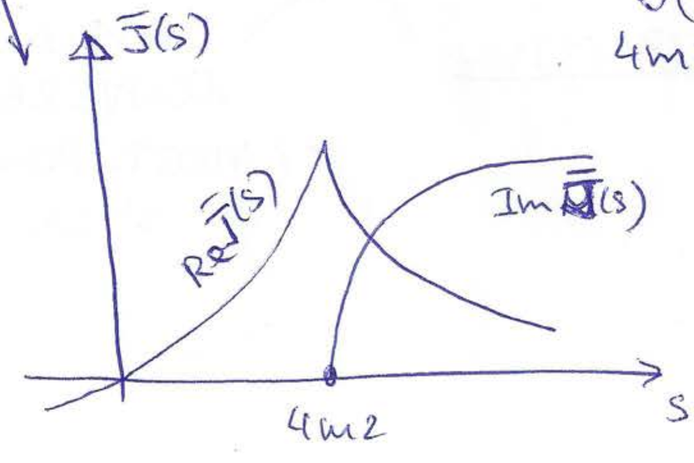
CHOOSE $s_0 = 0$

$$J(s) = J(0) + \frac{s}{\pi} \int_{4m^2}^{\infty} \frac{ds'}{s'} \frac{\mathrm{Im} J(s')}{s' - s} \Rightarrow$$

$$\boxed{\bar{J}(s) = J(s) - J(0) = \frac{s}{\pi} \int_{4m^2}^{\infty} \frac{ds'}{s'} \cdot \frac{1/16\pi\, \beta(s')}{s' - s}}$$

SUBTRACTIONS: GENERAL, n-subtracted dispersion relation

$$M(s) = \sum_{i=0}^{n-1} \frac{1}{i!} F^{(i)}(s_0)(s - s_0)^i + \frac{(s-s_0)^n}{\pi} \cdot \int_{4m^2}^{\infty} \frac{ds'}{(s'-s_0)^n} \frac{\mathrm{Im} M(s')}{s' - s}$$

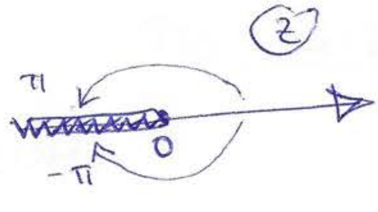


COMPLEX ALGEBRA

$$\ln(\underbrace{A}_{A=|A|e^{i\varphi}} - i\varepsilon)$$

CONVENTION FOR LOG:

$$-\pi \leq \varphi < \pi$$



1) $A > 0$

$$\ln(A - i\varepsilon) = \ln(A)$$

2) $A < 0$

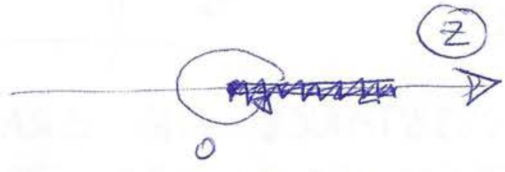
$$\ln(A \underbrace{- i\varepsilon}) = \ln(|A| \cdot e^{-i\pi}) = \ln(|A|) - i\pi\, \Theta(A<0)$$

means $|A|$ reached
in the III^{rd} quadrant
from below

$$\ln(A + i\varepsilon) = \ln(|A|\, e^{i\pi}) = \ln(|A|) + i\pi\, \Theta(A<0)$$

ANOTHER CHOICE FOR LOG:

$$0 \leq \varphi < 2\pi$$



$$\log(|A|) +$$

$$\log(|A|) + i2\pi\, \Theta(A>0)$$

ANALYTIC FUNCTIONS FORM A VERY STRINGENT CLASS OF FUNCTIONS. $\Rightarrow$

IF PHYSICAL OBSERVABLES COMES AS A LIMIT FROM COMPLEX ANALYTIC FUNCTIONS $\rightarrow$ IT IS A <u>CONSTRAINT</u>.

$f = u(x,y) + iv(x,y)$

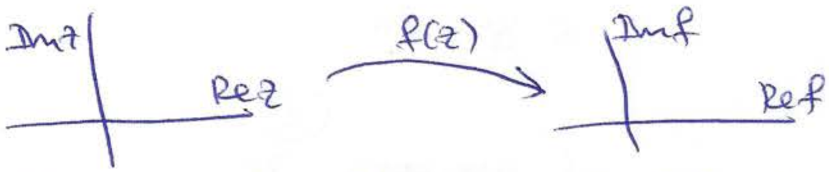
$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

SMOOTH, HOLOMORPHIC, ANALYTIC = SYNONYMS (i.e. function is differenciable)

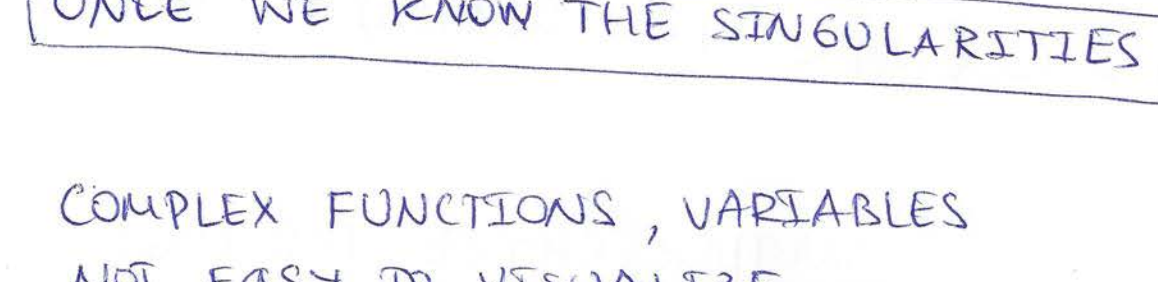
IF THE FUNCTION IS COMPLETELY SMOOTH $\Rightarrow$ THINGS ARE BOARING, IT IS JUST A POLYNOMIAL. BUT THERE ARE SINGULARITIES $\Rightarrow$ CUTS, POLES.

$\boxed{\text{ONCE WE KNOW THE SINGULARITIES} \rightarrow \text{we know THE FUNCTION.}}$

COMPLEX FUNCTIONS, VARIABLES NOT EASY TO VISUALIZE



DUE TO EXSISTANCE OF BRANCHING CUTS, IT IS NOT STREIGT FORWARD TO DO ONE-TO-ONE MAPPING: $\sqrt{-25} \rightarrow 5I$ OR $-5I$, OR



$e^{z \pm 2\pi i} = e^z$ $\left(e^z = e^{\mathrm{Re}\, z}(\cos[\mathrm{Im}\, z] + i \sin[\mathrm{Im}\, z]) \right)$

periodic function

But INVERSE IS MORE COMPLICATED.

```
Exit[];
```

In[181]:=

```
m = 1;
ϵ = 0.00001;
```

$$\beta[\text{s\_}] := \sqrt{1 - \frac{4\,\text{m}^2}{\text{s}}}\;;$$

$$\text{Jloop1}[\text{s\_}] := -\frac{1}{16\,\pi^2}\,\text{NIntegrate}\left[\text{Log}\left[1 - \frac{\text{s}}{\text{m}^2}\,\text{x}\,(1-\text{x}) - \text{I}\,\epsilon\right],\ \{\text{x},\,0,\,1\}\right];$$

$$\text{Jloop2}[\text{s\_}] := \frac{1}{16\,\pi^2}\left(2 + \beta[\text{s}]\ \text{Log}\left[\frac{\beta[\text{s}] - 1}{\beta[\text{s}] + 1}\right]\right);$$

$$\text{Jdisp}[\text{s\_}] :=$$

$$\text{NIntegrate}\left[\frac{\frac{1}{16\,\pi}\,\beta[\text{sb}]}{\text{sb} - \text{s} - \text{I}\,\epsilon}\,\frac{\text{s}}{\text{sb}}\,\frac{1}{\pi},\ \{\text{sb},\,4\,\text{m}^\wedge 2,\,\text{Infinity}\},\ \text{MaxRecursion} \to 20\right];$$

```
Plot[{Re[Jloop1[s]], Im[Jloop1[s]], Re[Jloop2[s]],
  Re[Jdisp[s]], Im[Jloop2[s]], Im[Jdisp[s]]}, {s, -1, 8},
 PlotPoints → 150, MaxRecursion → 0, FrameLabel → {"s", "J(s)"},
 PlotStyle → {Blue, Red, {Darker[Blue], Dashed}, {Lighter[Blue], Dotted},
   {Dashed, Darker[Red]}, {Lighter[Red], Dotted}},
 PlotLegends → Placed[{"Re(J(s))", "Im(J(s))"}, {Left, Top}]]
```

Out[187]=