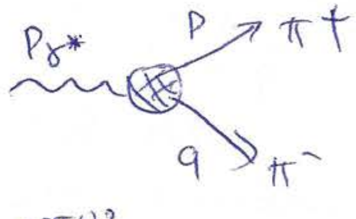
APPLICATION: PION VECTOR FORM FACTOR

CONSIDER THE PROCESS $\gamma^* \to \pi\pi$.

IT IS AN IMPORTANT BULDING BLOCK FOR REACTIONS $ee \to \pi\pi$, $\tau \to \pi\pi\nu$, ...



FORM FACTOR DEFINED AS MATRIX ELEMENT

TOT SPIN:
π: $J=0$
$m \simeq 0.14$ GeV

$$\boxed{\langle \pi(p)\pi(q) | \underbrace{\mathring{J}_\mu}_{\text{EM CURRENT}} | 0 \rangle = (p-q)_\mu F_\pi(s)}$$

GENERAL FORM

$$\langle \pi\pi | \mathring{J}_\mu | 0 \rangle = (\alpha_1 p_\mu + \alpha_2 q_\mu) F_\pi(s)$$

GAUGE INVAR.

$$P_\gamma^\mu \langle \pi\pi | \mathring{J}_\mu | 0 \rangle = 0 = (p+q)^\mu (\alpha_1 p_\mu + \alpha_2 q_\mu) F_\pi(s)$$

$$\alpha_1 p^2 + \alpha_2 q^2 + (\alpha + \alpha_2)\, p\cdot q = 0$$

$$(\alpha_1 + \alpha_2)(m^2 + (s - m^2)) = 0$$

$P_{\gamma^*} = p + q$
$P_{\gamma^*}^2 = s = p^2 + 2pq + q^2$
$pq = s - m^2$

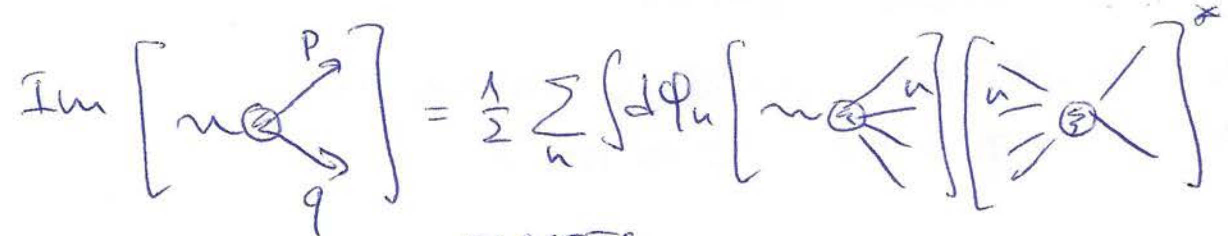
$$\Rightarrow \alpha_1 = -\alpha_2$$

$$\langle \pi\pi | J_\mu | 0 \rangle = (p-q)_\mu F_\pi(s)$$

CHARGE CONSERVATION: $F_\pi(s=0) = 1$

FOR ZERO MOMENTUM TRANSFER NOTHING HAPPENS $\Rightarrow$ FORM FACTOR $= 1$.

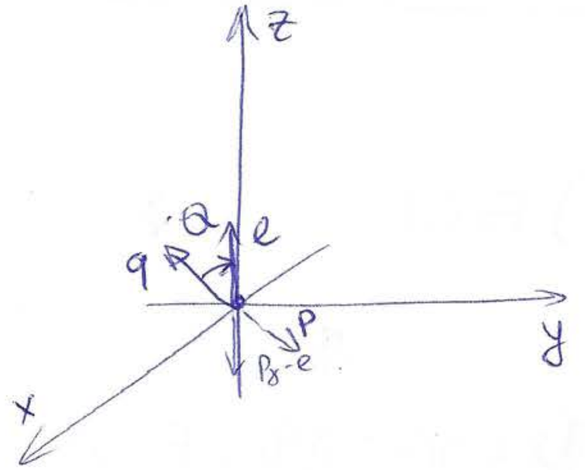
Unitarity relation



$=$ ELASTIC

$\times \underbrace{\frac{1}{2}}$ SYM. FACTOR (IDENTICAL PIONS)

$$(p-q)_\mu \, \text{Im}\, F_\pi(s) = \frac{1}{4}\int d\Omega \, \frac{\beta(s)}{32\pi^2}\,(P_\gamma - 2l)_\mu \, F_\pi(s)\, T^*_{\pi\pi}(s,\cos\Theta)$$



$$\vec{P}_\gamma = 0 = \vec{p} + \vec{q}$$

$$\Rightarrow l^0 = p^0 = q^0 = \frac{\sqrt{s}}{2}$$

$$|\vec{p}| = |\vec{q}| = |\vec{l}| = \frac{\sqrt{s}}{2}\beta(s)$$

$$\beta(s) = \sqrt{1 - \frac{4m^2}{s}}$$

$$pl = \frac{s}{4}(1 + \beta^2 z) = p^0 l^0 - \vec{p}\vec{l}$$

$$ql = \frac{s}{4}(1 - \beta^2 z) = q^0 l^0 - \vec{q}\vec{l}$$

GENERAL FORM:

$$\int d\Omega \, ((p+q) - 2l)_\mu T^*_{\pi\pi} = L_1 (p+q)_\mu + L_2 (p-q)_\mu$$

CONTRACT $(p+q)^\mu$, $(p-q)^\mu$

$$\int d\Omega \, (\underbrace{(p+q)^2}_{s} - \underbrace{2l(p+q)}_{2\cdot s/2}) T^*_{\pi\pi} = L_1 (p+q)^2 + L_2 \overbrace{(p+q)(p-q)}^{p^2-q^2=0}$$

$$L_1 = 0$$

$$\int d\Omega \, (\underbrace{\cancel{(p+q)}(p-q)}_{0} - 2l(p-q)) T^* = L_1 \overbrace{\cancel{(p+q)}(p-q)}^{0} + L_2 (p-q)^2$$

$$\int d\Omega \, (-\underbrace{s\beta^2}_{s-4m^2} z) T^*_{\pi\pi} = (\underbrace{-2pq + 2m^2}_{4m^2 - s}) L_2 \qquad pq = \frac{s}{4} + \left(\frac{\sqrt{s}\beta}{2}\right)^2_{\cos\pi = -1}$$

$$L_2 = + \int d\Omega \; T^*_{\pi\pi}(s,z) \cdot z$$

AND

$$\int d\Omega \, \left((p+q) - 2\ell \right)_\mu T^*_{\pi\pi}(s,z) = 2\pi \int_{-1}^{1} dz \; z \cdot T^*_{\pi\pi}(s,z)\,(p-q)_\mu$$

$T_{\pi\pi}$ is the $\pi\pi \to \pi\pi$ SCATTERING AMPLITUDE

$$T_{\pi\pi}(s,z) = \sum_{=0}^{\infty} (2\ell+1)\, t_J(s)\, P_J(z)$$

$$\int_{-1}^{1} dz \; P_J(z)\, P_{J'}(z) = \frac{2\delta_{JJ'}}{2J+1}$$

$$\Rightarrow \int_{-1}^{1} dz \cdot P_J(z) \cdot \underbrace{z}_{P_1(z)} = \frac{2\delta_{J1}}{(2J+1)} \leftarrow$$

ONLY $J=1$ SURVIVE ⚠

PHOTON HAS SPIN 1 $\Rightarrow$ OK!

$$\mathrm{Im}\, F_\pi(s) = \cancel{\frac{1}{4}} \cdot \frac{\beta(s)}{32\pi^{\cancel{2}}} \cdot \cancel{2\pi} \cdot \cancel{2} \,.\, F_\pi(s)\, t^*_1(s)\; \theta(s > 4m^2)$$

$$\boxed{\mathrm{Im}\, F_\pi(s) = \underbrace{\frac{\beta(s)}{32\pi}}_{\rho(s)} F_\pi(s)\, t^*_1(s)\; \theta(s > 4m^2)}$$

$$t_{J=1}(s) = |t_{J=1}(s)|\, e^{i\delta_{J=1}(s)} \,, \quad \delta_{J=1}(s) \text{ - PHASE SHIFT}$$

$$= \frac{\sin\delta_{J=1}\, e^{i\delta_{J=1}}}{\rho} \leftarrow$$

TASK, DERIVE FROM P.W. UNITARITY $\mathrm{Im}\, t_\ell = \rho\, |t_\ell|^2$

$$\rho(s) \cdot t^*_{J=1} = \sin\delta \cdot e^{-i\delta} \qquad \Big\| \qquad \cancel{|F|} \sin\varphi = \cancel{|F|}\, e^{i\varphi} \sin\delta\, e^{i\delta}$$

$$F_\pi = |F_\pi|\, e^{i\varphi} \qquad \Big\| \qquad \varphi = \delta$$

$$\Rightarrow \quad F_\pi(s) = |F_\pi(s)| e^{i\delta(s)} \quad ; \quad \varphi = \delta$$

WATSON THEOREM: THE PHASE OF THE FORM FACTOR IS DETERMINED BY THE TWO-PARTICLE PHASE SHIFT.

MUSKHELISHVILI-OMNÈS PROBLEM

WE WANT TO FIND THE MOST GENERAL REPRESENTATION FOR A FUNCTION $F_\pi(s)$, WICH HAS A PHASE

$$\mathrm{Arg}\,(F_\pi(s)) = \delta(s)$$

UNITARITY CUT $s > 4m^2$

$$\mathrm{Im}\, F_\pi = \rho\, F_\pi\, t^* = F_\pi \cdot \sin\delta\, e^{-i\delta}$$

LOOK FOR A SOLUTION IN THE FORM

$$F_\pi(s) = P(s) \cdot \Omega(s) \rightsquigarrow \mathrm{Im}\,\Omega = \Omega \sin\delta\, e^{-i\delta}$$

$$\frac{1}{2i}\Big(\Omega(s+i\varepsilon) - \Omega(s-i\varepsilon)\Big) = \Omega(s+i\varepsilon)\sin\delta\, e^{-i\delta}$$

$$\Omega(s+i\varepsilon)\underbrace{\Big[\frac{1}{2i} - \underbrace{\sin\delta}_{\frac{1}{2i}(e^{i\delta}-e^{-i\delta})}\, e^{-i\delta}\Big]}_{\frac{1}{2i}e^{-2i\delta}} = \Omega(s-i\varepsilon)\cdot\frac{1}{2i}$$

$$\Omega(s+i\varepsilon)e^{-2i\delta} = \Omega(s-i\varepsilon)$$

$$\ln\Omega(s+i\varepsilon) - 2i\delta = \ln\Omega(s+i\varepsilon)$$

$$\mathrm{Disc}\big(\ln\Omega(s)\big) = 2i\delta$$

DISPERSION RELATION

$$\ln \Omega(s) = \frac{1}{2i}\int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{\mathrm{Disc}(\ln \Omega(s'))}{s'-s} = \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{\delta(s')}{s'-s}$$

$$\Omega(s) = \exp\left(\int_{4m^2}^{\infty} \frac{ds'}{\pi}\frac{\delta(s')}{s'-s}\right) = \exp\left(a + \frac{s}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'}\frac{\delta(s')}{s'-s}\right)$$

NORMALIZATION

$$\begin{cases} \Omega(s=0) = 1 \Rightarrow a = 0 \\ \Omega(s) = \exp\left(\frac{s}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'}\frac{\delta(s')}{s'-s}\right) \end{cases}$$

Omnès function

ASYMPTOTIC BEHAVIOR

IN PHYSICAL REGION

$$\Omega(s+\underline{\underline{i\varepsilon}}) = \exp\left(\frac{s}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'}\frac{\delta(s')}{s'-s-\underline{\underline{i\varepsilon}}}\right) =$$

$$= \exp\left(\frac{s}{\pi}\,\mathrm{P.V.}\int_{4m^2}^{\infty}\frac{ds'}{s'}\frac{\delta(s')}{s'-s} + \frac{\cancel{s}}{\cancel{\pi}}\cdot\frac{i\cancel{\pi}\,\delta(s)}{\cancel{s}}\right)$$

$$= \exp(i\delta(s))\,\exp\left(\frac{s}{\pi}\,\mathrm{P.}\!\!\int_{4m^2}^{\infty}\frac{ds'}{s'}\frac{\delta(s')}{s'-s}\right)$$

TRICK FOR P.V. INTEGRALS

$$\mathrm{P.V.}\int_{4m^2}^{\infty}\frac{ds'}{s'}\frac{\delta(s')-\delta(s)+\delta(s)}{s'-s} = \underbrace{\int_{4m^2}^{\infty}\frac{ds'}{s'}\frac{\delta(s')-\delta(s)}{s'-s}}_{\text{SMOOTH}} + \delta(s)\underbrace{\mathrm{P.}\!\!\int_{4m^2}^{\infty}\frac{ds'}{s'}\frac{1}{s'-s}}_{\frac{1}{s}\ln\left(\frac{4m^2}{s-4m^2}\right)}$$

$$\Rightarrow \Omega(s\to\infty) \simeq \exp(i\delta)\,\exp(\text{SMOOTH})\,\exp\left(\frac{s}{\pi}\cdot\frac{\delta(s)}{s}\ln\left(\frac{4m^2}{s-4m^2}\right)\right)$$

$$\simeq \frac{1}{s^{\delta(\infty)/\pi}}$$

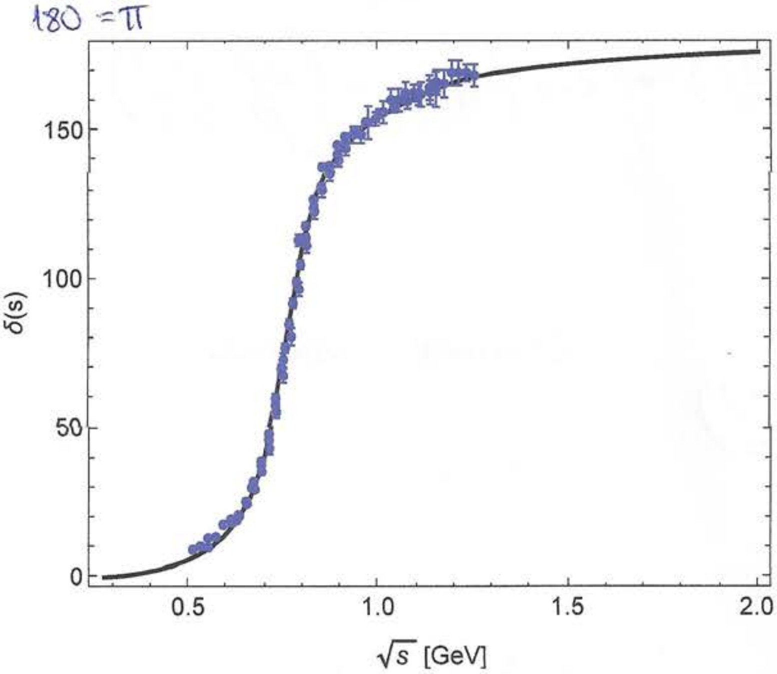
FROM PERT. QCD

$F_\pi(s\to\infty) \sim \frac{1}{s}$; WE CAN ASSUME $\delta(\infty)\to\pi$

$P(s) = 1$

$$\Rightarrow \boxed{F_\pi(s) = \Omega(s)}$$

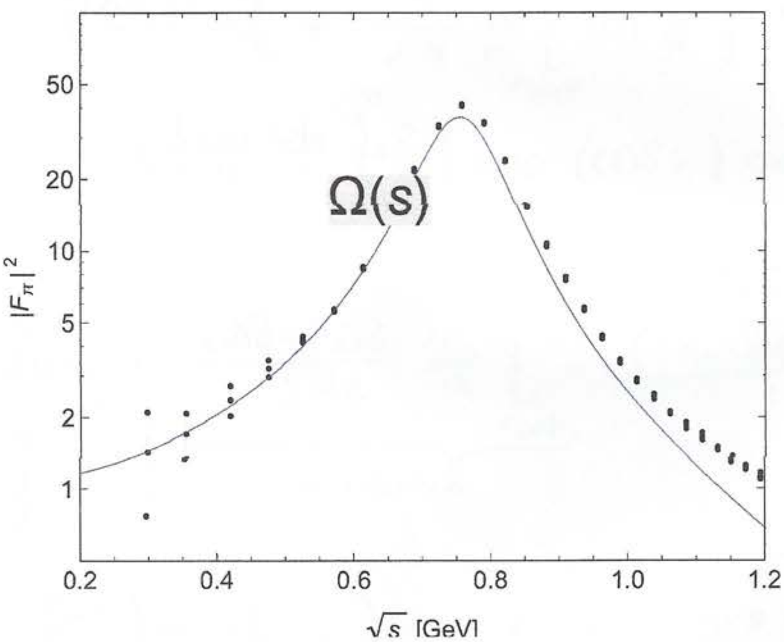
$\pi\pi$ phase shift (J=1)



$\delta(s \to \infty) = \pi$
REASONABLE
ASSUMPTION

$\Downarrow$ PREDICTION

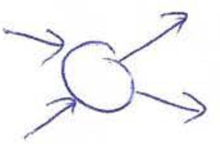
Pion vector transition form factor



FIXED t DISPERSION RELATIONS

SO FAR WE HAVE ONLY CONSIDERED THE DISP. REL. FOR A FUNCTION OF A SINGLE VARIABLE.

IN GENERAL



$\mathcal{M}(s, t)$

CONSIDER SIMPLE CASE: $m_i = m$ (EQUAL MASSES).

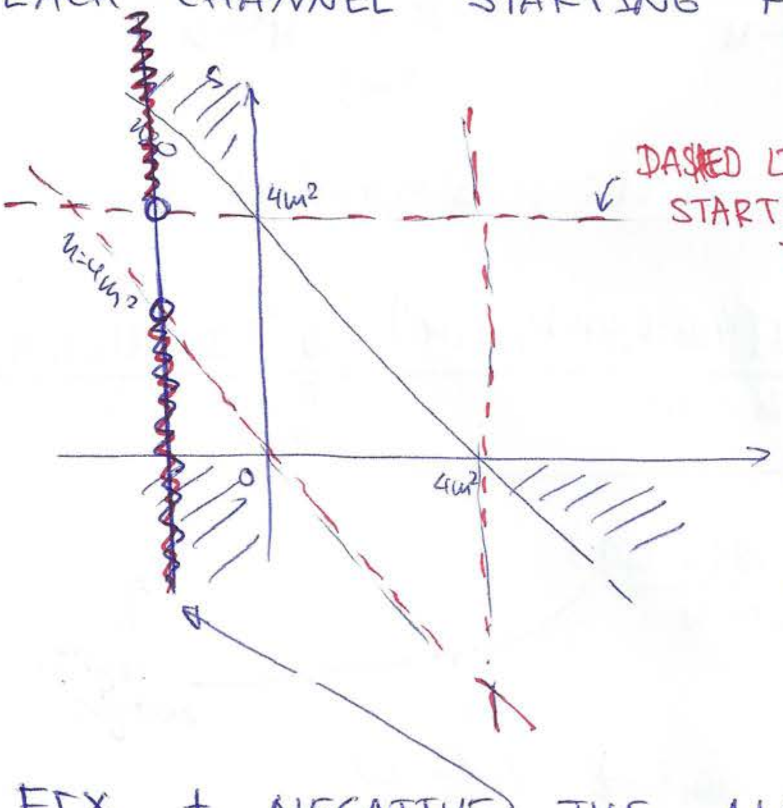
WE KNOW THAT PHYSICAL REGION IS

$s \geq 4m^2, \quad t, u \leq 0$ s-channel

$t \geq 4m^2, \quad s, u \leq 0$ t-channel

$u \geq 4m^2, \quad s, t \leq 0$ u-channel

AND UNITARITY REQUIRES TO HAVE A CUT IN EACH CHANNEL STARTING FROM $4m^2$.



IN EACH CHANNEL SCHWARZ REFLECTION PRINCIPLE HOLDS

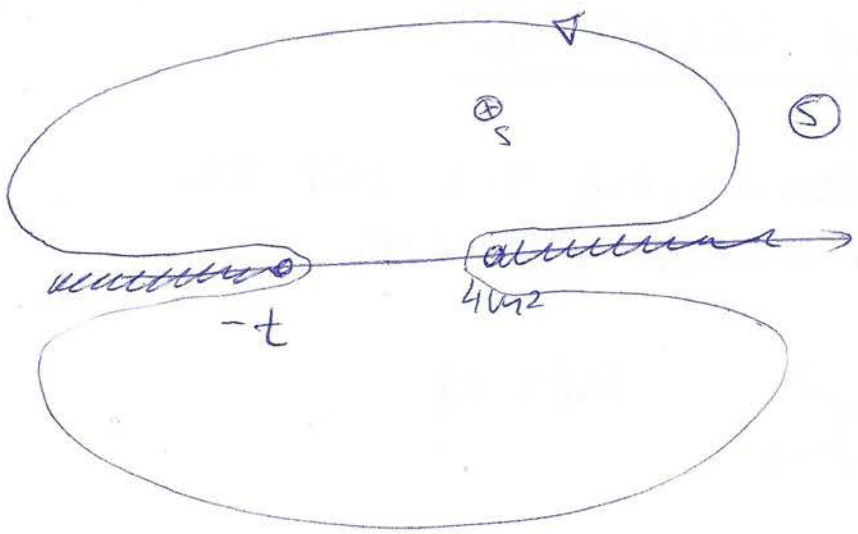
$$\frac{\mathcal{M}(s+i\varepsilon, t, u) - \mathcal{M}(s-i\varepsilon, t, u)}{2i} =$$

$$= \operatorname{Im} \mathcal{M}(s+i\varepsilon, t, u)$$

FIX t NEGATIVE, THE $\mathcal{M}(s, \underset{\uparrow}{t})$ IS A FUNCTION OF ONE VARIABLE, WHICH FIXED
HAS TWO CUTS:

$$s = [4m^2, \infty) \qquad 4m^2$$

$$s = (-\infty, 4m^2 - t - \overset{\shortparallel}{u} = -t]$$



EVALUATED ON THE UPPER RIM OF THE CUT

FIXED t- DISPERSION RELATION

$$M(s,t,u) = \underbrace{\frac{1}{\pi}\int_{-\infty}^{-t} \frac{\operatorname{Im} M(s',t,u)}{s'-s}\, ds'}_{} + \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{\operatorname{Im} M(s',t,u)}{s'-s}\, ds'$$

SIMILAR ONE CAN WRITE INTEGRALS IN TERMS OF u.

$$M(s,t,u) = \frac{1}{\pi}\int_{-\infty}^{-t} \frac{\operatorname{Im} M(s,t,u')}{u'-u}\, du' + \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{\operatorname{Im} M(s,t,u')}{u'-u}\, ds'$$

REWRITE AS AN INTEGRAL OVER u: $\quad s' = 4m^2 - t - u'$

$$M(s,t,u) = \underbrace{\frac{1}{\pi}\int_{\infty}^{4m^2} -du' \, \frac{\operatorname{Im} M(s,t,u')}{u-u'}}_{-\frac{1}{\pi}\int_{4m^2}^{\infty} du' \, \frac{\operatorname{Im} M(s,t,u')}{u'-u}} + \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{\operatorname{Im} M(s',t,u)}{s'-s}\, ds'$$

$$s' + i\epsilon = (4m^2 - t - u') + i\epsilon = 4m^2 - t - \underline{\underline{(u' - i\epsilon)}}$$

$$\operatorname{Im} M(s'+i\epsilon, t, u) = \operatorname{Im} M(s,t,u'-i\epsilon) = -\operatorname{Im} M(s,t,u'+i\epsilon)$$

SCHWARZ REFLECTION PRINCIPLE

$$\boxed{M(s,t,u) = \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \frac{\mathrm{Im}\, M(s',t,u)}{s'-s} + \frac{1}{\pi}\int_{4m^2}^{\infty} du' \frac{\mathrm{Im}(s,t,u')}{u'-u} \qquad (1)}$$

FIXED-t DISP. REL.

WHERE IN EACH INTEGRAL THE IMAGINARY PART MUST BE EVALUATED ON THE UPPER RIM OF THE CUT.

IF THERE ARE ISOLATED POLES, THEN

$$M(s,t,u) = \dots + \frac{\mathrm{Res}(M(s,t,u), s=M^2)}{s-M^2} + \dots \quad t,u$$

IT IS NOT COMPULSORY TO KEEP t-fixed. MANDESTAM ARGUED (NO FORMAL PROOF) THAT ONE CAN WRITE DOUBLE DISPERSION RELATION

$$\boxed{\begin{aligned} M(s,t,u) = {} & \frac{1}{\pi^2}\int_{4m^2}^{\infty} ds' \int_{4m^2}^{\infty} dt' \frac{\rho_{st}(s',t')}{(s'-s)(t'-t)} + \frac{1}{\pi^2}\int_{4m^2}^{\infty} dt' \int_{4m^2}^{\infty} du' \frac{\rho_{tu}(t',u')}{(t'-t)(u'-u)} \\ & + \frac{1}{\pi^2}\int_{4m^2}^{\infty} ds' \int_{4m^2}^{\infty} du' \frac{\rho_{su}(s',u')}{(s'-s)(u'-u)} + \text{poles} \qquad (2) \end{aligned}}$$

WHERE $\rho_{st}, \rho_{tu}, \rho_{su}$ ARE REAL SPECTRAL FUNCTIONS.

HOWEVER, ONE CAN DERIVE (1) FROM (2):

$$\frac{1}{(s'-s)(u'-u)} = \left(\frac{1}{s'-s} + \frac{1}{u'-u}\right) \Big/ \left(\frac{1}{s'-s+u'-u}\right)$$

$$(s'-s) + (u'-u) = s' - (4m^2 - t - u') = u' - (4m^2 - t - s')$$

THEN

$$\begin{aligned} M(s,t,u) = {} & \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \frac{1}{s'-s}\left(\frac{1}{\pi}\int_{4m^2}^{\infty} dt' \frac{\rho_{st}}{t'-t} + \frac{1}{\pi}\int_{4m^2} du' \frac{\rho_{su}}{u'-(4m^2-t-s')}\right) \\ & + \frac{1}{\pi}\int_{4m^2}^{\infty} du' \frac{1}{u'-u}\left(\frac{1}{\pi}\int_{4m^2}^{\infty} dt' \frac{\rho_{tu}}{t'-t} + \frac{1}{\pi}\int_{4m^2} ds' \frac{\rho_{su}}{s'-(4m^2-t-u')}\right) \end{aligned}$$

USING SOKHOTSKI - PLEMELIJ FORMULA

$$\frac{1}{x \pm i\epsilon} = \mp i\pi\delta(x) + P \cdot \frac{1}{x}$$

WE CAN EVALUATE $\mathrm{Im}[\,]$ of $M(s,t,u)$ AT $s+i\epsilon$

$$\mathrm{Im}_s A(s+i\epsilon, t, u) = \frac{1}{\pi}\int dt' \frac{\rho_{st}(s,t')}{t'-t} + \frac{1}{\pi}\int du' \frac{\rho_{su}(s,u')}{u'-u}$$

$$\mathrm{Im}_u A(s,t,u+i\epsilon) = \frac{1}{\pi}\int dt' \frac{\rho_{tu}}{t'-t} + \frac{1}{\pi}\int ds' \frac{\rho_{su}}{s'-s}$$

$\Rightarrow$ LEADS TO FIXED-t DISPERSION RELATION

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Book List

June 12, 2017

1. **V. Gribov**, “Strong Interactions of Hadrons at High Energies”
2. **V. Gribov**, “The Theory of Complex Angular Monemta”
3. **P. D. B. Collins**, “Regge Theory and High Energy Physics”
4. **E. Byckling and K. Kajantie** “Particle Kinematics”
5. **R. Hagedorn**, “Relativistic Kinematics”
6. **M. Perl**, “High Energy Hadron Physics”
7. **Martin and Spearman**, “Elementary Particle Physics”
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9. **N. M. Queen and G, Violini**, “Dispersion Theory in High Energy Physics”
10. **R. G. Newton**, “The Complex j-Plane”
11. **A. O. Barut**, “The Theory Scattering Matrix”
12. **E. Leader**, “Spin in Particle Physics”
13. **O. Nachtmann**, “Elementary Particle Physics”
14. **H. M. Pilkuhn**, “Relativistic Particle Physics”
15. **J. Werle**, “Relativistic Theory of Reactions”
16. **S. Donnachie et al**, “Pomeron Physics and QCD”
17. **J. D. Bjorken and S. D. Drell**, “ Relativistic Quantum Fields”
18. **M. E. Peskin and D. V. Schroeder**, “Quantum Field Theory”
19. **S. Weinberg**, "Th Quantum Theory of Fields I”
20. **S. Weinberg**, "Th Quantum Theory of Fields II”