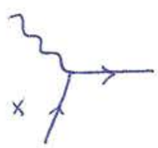
SCALING VIOLATIONS IN DIS, EVOLUTION OF PARTON DISTRIBUTIONS

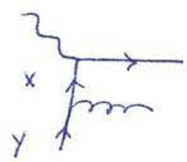
* INTRODUCTION

→ DIS IN LO IN α_s



$\Longrightarrow \quad q(x)$

→ DIS IN NLO IN α_s



$\Rightarrow \quad q(x, Q^2)$

QUARK DISTRIBUTION GETS SCALE DEPENDENCE

Q^2 DEPENDENCE OF PARTON DISTRIBUTION GOVERNED BY EVOLUTION EQUATION

2 TYPES OF EVOLUTION EQUATIONS :

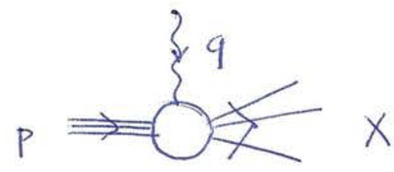
→ FLAVOR NON-SINGLET

$$q_v = q - \bar{q}$$

→ FLAVOR SINGLET ($SU(3)_F$ SINGLET)

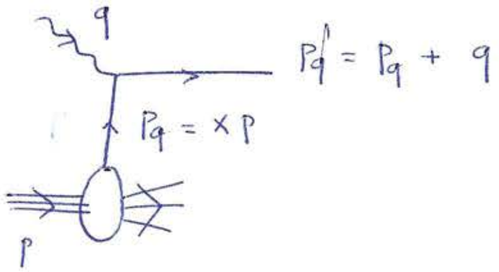
$$\left.\begin{array}{l} \nearrow \frac{1}{3}(u + d + s) \\ \searrow g \quad (\text{GLUON}) \end{array}\right\} \quad \text{QUARK SINGLET \& GLUON MIX.}$$

* DIS TENSOR : GENERAL



$$W^{\mu\nu} = \frac{1}{2\pi} \sum_X (2\pi)^4 \delta^4(p + q - p_X) < p | J^{\mu +}(0) | X > < X | J^{\nu}(0) | p >$$

* DIS TENSOR : PARTON MODEL



$$W^{\mu\nu}_{\text{PARTON}} = \sum_q e_q^2 \int_0^1 \frac{dx}{x} \, q(x) \, w^{\mu\nu}_{q \to q}$$

INITIAL

TENSOR FOR SCATTERING OFF QUARK

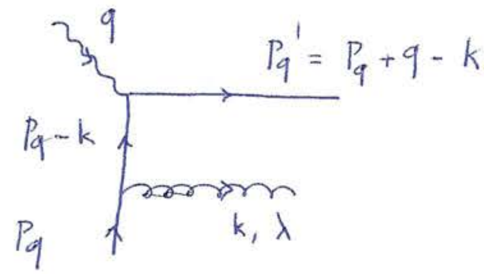
$$\frac{1}{2E_q} = \frac{1}{x(2E)} \qquad \left(E = \sqrt{\vec{p}^2 + M^2} \right)$$

UNPOL SCATTERING

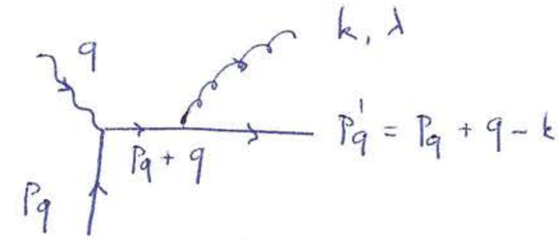
$$w^{\mu\nu}_{q \to q} = \frac{1}{2\pi} \frac{1}{2} \sum_{s, s'} \int \frac{d^3 \vec{p}_q^{\,\prime}}{(2\pi)^3 2E_q'} (2\pi)^4 \delta^4(p_q + q - p_q') \, J^{\mu *}_{q \to q} \cdot J^{\nu}_{q \to q}$$

$$J^{\nu}_{q \to q} = \bar{u}(p_q', s') \, \gamma^{\nu} \, u(p_q, s)$$

* DIS TENSOR : ONE - GLUON EMISSION



(a)



(b)

- $$W^{\mu\nu}_{q\to qg} = \frac{1}{2\pi} \underbrace{\frac{1}{2}\sum_{ss'}}_{\text{UNPOL.}} \sum_\lambda \int \frac{d^3\bar{k}}{(2\pi)^3 2|\bar{k}|} \int \frac{d^3\bar{P}_q'}{(2\pi)^3 2E_q'} (2\pi)^4 \delta^4(P_q + q - P_q' - k') \cdot J^{\mu *}_{q\to qg} J^{\nu}_{q\to qg}$$

- $$J^{\nu}_{q\to qg(a)} = \bar{U}(P_q + q - k, s')\, \gamma^\nu \frac{i(\not{P}_q - \not{k})}{(P_q - k)^2} \left[-i g \gamma^\beta \frac{\lambda_a}{2}\right] U(P_q, s) \cdot \varepsilon^*_\beta(k, \lambda)$$

- $$J^{\nu}_{q\to qg(b)} = \bar{U}(P_q + q - k, s') \left[-i g \gamma^\beta \frac{\lambda_a}{2}\right] \frac{i(\not{P}_q + \not{q})}{(P_q + q)^2} \gamma^\nu U(P_q, s) \cdot \varepsilon^*_\beta(k, \lambda)$$

↳ $$\int \frac{d^3\bar{k}}{(2\pi)^3 2|\bar{k}|} \cdots = \int \frac{d^4 k}{(2\pi)^3} \delta_+(k^2) \cdots \qquad \delta_+(k^2) \equiv \delta(k^2)\,\Theta(k^0)$$

↳ $$\int \frac{d^3\bar{P}_q'}{(2\pi)^3 2E_q'} \cdots = \int \frac{d^4 P_q'}{(2\pi)^3} \delta_+(P_q'^2 - \underset{0}{m_q^2}) \cdots$$

↳ COLOR FACTOR SU(3)

$$C_F = \underset{\substack{\uparrow \\ \text{AVERAGE} \\ \text{OVER 3} \\ \text{INITIAL QUARK COLORS}}}{\frac{1}{3}} \sum_{i=1,2,3} \underset{\substack{\uparrow \\ \text{SUM OVER 8 GLUONS}}}{\sum_{a=1}^{8}} \left(\frac{\lambda_a}{2} \cdot \frac{\lambda_a}{2} \right) = \frac{1}{3} \cdot 4 = \frac{4}{3}$$

↳ $$w^{\mu\nu}_{q \to qg\,(a)} = \frac{1}{2} \cdot \frac{1}{(2\pi)^3} \cdot g^2 \cdot C_F$$

$$\cdot \int \frac{d^3 \bar{k}}{2|\bar{k}|} \, \delta_+ \left((P_q + q - k)^2 \right) \left[\sum_\lambda \varepsilon_\alpha(k,\lambda) \, \varepsilon^*_\beta(k,\lambda) \right]$$

$$\cdot \frac{1}{(P_q - k)^4} \cdot \mathrm{Tr} \left\{ \gamma^\alpha (\not{P}_q - \not{k}) \gamma^\mu (\not{P}_q + \not{q} - \not{k}) \gamma^\nu (\not{P}_q - \not{k}) \gamma^\beta \not{P}_q \right\}$$

↳ $$w^{\mu\nu}_{q \to qg\,(b)} = \frac{1}{2} \cdot \frac{1}{(2\pi)^3} \cdot g^2 \cdot C_F$$

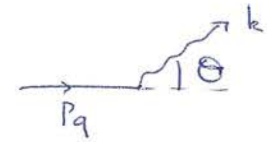
$$\cdot \int \frac{d^3 \bar{k}}{2|\bar{k}|} \, \delta_+ \left((P_q + q - k)^2 \right) \left[\sum_\lambda \varepsilon_\alpha(k,\lambda) \, \varepsilon^*_\beta(k,\lambda) \right]$$

$$\cdot \frac{1}{(P_q + q)^4} \cdot \mathrm{Tr} \left\{ \gamma^\mu (\not{P}_q + \not{q}) \gamma^\alpha (\not{P}_q + \not{q} - \not{k}) \gamma^\beta (\not{P}_q + \not{q}) \gamma^\nu \not{P}_q \right\}$$

↳ DIAGRAM (a) HAS SINGULARITY

WHEN $(P_q - k)^2 = 0$

$$(P_q - k)^2 = P_q^2 - 2|\vec{k}|\left(P_q^0 - |\vec{P}_q|\cos\theta\right)$$



$$\simeq - 2|\vec{k}|\,|\vec{P}_q|\,(1 - \cos\theta)$$

FOR MASSLESS QUARKS

$\parallel$ $\cos\theta = 1 \iff$ COLLINEAR DIVERGENCE

DIV. WHEN GLUON IS EMITTED IN DIR. OF QUARK

- REGULARIZE BY PUTTING QUARK OFF-SHELL

  i.e. $P_q^2 = -\mu^2$

- DIVERGENT TERM IN $\mu \to 0$ LIMIT

  ↳ LOGARITHMIC CORRECTION $\ln\left(Q^2/\mu^2\right)$

↳ DIAGRAM (b) HAS <u>NO</u> SINGULARITY OVER GLUON INTEGRATION RANGE

$(P_q + q)^2$ IS INDEPENDENT OF GLUON MOMENTUM k

↳ CHOOSE AXIAL GAUGE $(A^+ = 0) \Leftrightarrow A^0 + A^3 = 0$

P HAS LARGE POSITIVE Z- COMP.

$n \cdot A = 0$ WITH $n^\mu = \frac{1}{\sqrt{2}}(1, 0, 0, -1)$

$$\boxed{\sum_{\lambda=\pm 1} \varepsilon_\alpha(k,\lambda)\, \varepsilon^*_\beta(k,\lambda) = -g_{\alpha\beta} + \frac{k_\alpha n_\beta + k_\beta n_\alpha}{k \cdot n}}$$

$n \cdot \varepsilon = 0$.

$n^2 = 0$ (LIGHTLIKE VECTOR)

↳ $$w^{\mu\nu}_{q \to qg\,(a)} = \frac{1}{2} \cdot \frac{1}{(2\pi)^3}\, g^2\, \frac{4}{3}$$

$$\cdot \int \frac{d^3 \bar{k}}{2|\bar{k}|}\, \delta_+\left((P_q + q - k)^2\right) \cdot \frac{1}{(P_q - k)^4}$$

$$\cdot \Big\{ 2\, \mathrm{Tr}\Big\{ \gamma^\mu (\not{P}_q + \not{q} - \not{k}) \gamma^\nu \underbrace{(\not{P}_q - \not{k})\, \not{P}_q\, (\not{P}_q - \not{k})}_{2(k \cdot P_q)\not{k}} \Big\}$$

$$+ \frac{1}{k \cdot n}\, \mathrm{Tr}\Big\{ \gamma^\mu (\not{P}_q + \not{q} - \not{k}) \gamma^\nu (\not{P}_q - \not{k})\, \not{n}\, \underbrace{\not{P}_q\, \not{k}\, (\not{P}_q - \not{k})}_{2\, k \cdot P_q\, \not{P}_q} \Big\}$$

$$+ \frac{1}{k \cdot n}\, \mathrm{Tr}\Big\{ \gamma^\mu (\not{P}_q + \not{q} - \not{k}) \gamma^\nu \underbrace{(\not{P}_q - \not{k})\, \not{k}\, \not{P}_q}_{2k \cdot P_q\, \not{P}_q}\, \not{n}\, (\not{P}_q - \not{k}) \Big\} \Big\}$$

$$w^{\mu\nu}_{q \to qg\,(a)} = \frac{1}{2} \cdot \frac{1}{(2\pi)^3}\, g^2\, \frac{4}{3}$$

$$\cdot \int \frac{d^3 \bar{k}}{2|\bar{k}|}\, \delta_+\left((P_q + q - k)^2\right) \cdot \frac{1}{(P_q - k)^4} \cdot (-2)\, (P_q - k)^2$$

$$\cdot \mathrm{Tr}\Big\{ \gamma^\mu (\not{P}_q + \not{q} - \not{k}) \gamma^\nu \Big[\not{k} + \frac{1}{k \cdot n} \Big(n \cdot (P_q - k)\, \not{P}_q + n \cdot P_q\, (\not{P}_q - \not{k}) + k \cdot P_q\, \not{n} \Big) \Big] \Big\}$$

↳ FOR LEADING LOG TERM → DUE TO $\dfrac{1}{(P_q - k)^2}$

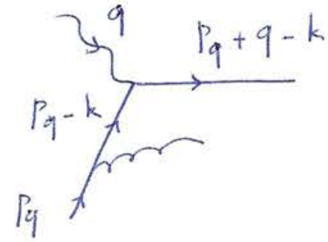
ONE CAN TAKE $\Theta = 0$ EVERYWHERE EXCEPT IN DENOMINATOR

CHOOSE FRAME

$$q = Q\,(0,0,0,-1)$$

$$P_q = \frac{Q}{2x}\,(1,0,0,1) \;\rightarrow\; P_q^2 = 0, \quad P_q \cdot q = \frac{Q^2}{2x}$$

$$k \approx k_{\Theta=0} = |\vec{k}|\,(1,0,0,1)$$



$$(P_q - k)^2 = 0$$

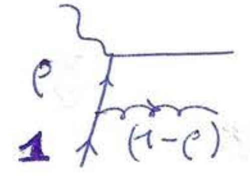
$$\delta_+\left((P_q + q - k)^2\right)$$

$$= \delta_+\left(\underbrace{(P_q - k)^2}_{0} - Q^2 + 2q\cdot(P_q - k)\right)$$

$$= \frac{1}{2P_q\cdot q}\,\delta\left(x - \frac{q\cdot(P_q - k)}{q\cdot P_q}\right)$$

$$\downarrow \qquad \boxed{\rho \equiv \frac{q\cdot(P_q - k)}{q\cdot P_q} = 1 - \frac{|\vec{k}|\,2x}{Q}}$$

$$= \frac{1}{2P_q\cdot q}\,\delta(x - \rho)$$



$$\therefore \quad k^\mu_{\Theta=0} = (1-\rho)\,P_q^\mu$$

$$\hookrightarrow \quad w^{\mu\nu}_{q\to qg\,(a)} = \frac{1}{2}\cdot\frac{g^2}{(2\pi)^3}\cdot\frac{4}{3}$$

$$\cdot\, 2\pi \int_{-1}^{1} d\cos\theta \int_0^\infty d|\bar{k}|\,\frac{|\bar{k}|}{4 P_q\cdot q}\cdot\frac{(-2)\,\delta(x-\rho)}{2|\bar{k}|\frac{Q}{2x}\cos\theta - \left(\mu^2 + 2|\bar{k}|\frac{Q}{2x}\right)}$$

$$\boxed{\cdot\left\{ (1-\rho)\,\mathrm{Tr}\left\{\gamma^\mu(\rho \not{P}_q + \not{q})\gamma^\nu \not{P}_q\right\} + \frac{1}{(1-\rho)}\,\mathrm{Tr}\left\{\gamma^\mu(\rho\not{P}_q+\not{q})\gamma^\nu(2\rho)\not{P}_q\right\}\right\}}$$

$$\Downarrow \quad \frac{1+\rho^2}{1-\rho}\cdot 4\left\{(\rho P_q+q)^\mu P_q^\nu + (\rho P_q+q)^\nu P_q^\mu - (\rho P_q+q)\cdot P_q\, g^{\mu\nu}\right\}$$

$$\Downarrow$$

$$w^{\mu\nu}_{q\to qg\,(a)} = -\frac{g^2}{(2\pi)^2}\cdot\frac{4}{3}\int_0 d|\bar{k}|\,\frac{x}{Q}\,\frac{\delta(x-\rho)}{4P_q\cdot q}\,\ln\left|\frac{2|\bar{k}|\frac{Q}{2x} - \left(\mu^2 + 2|\bar{k}|\frac{Q}{2x}\right)}{2|\bar{k}|\frac{Q}{2x} + \left(\mu^2+2|\bar{k}|\frac{Q}{2x}\right)}\right|$$

$$\cdot\,\frac{1+\rho^2}{1-\rho}\cdot 4\,(P_q\cdot q)\left\{\left(-g^{\mu\nu} - \frac{q^\mu q^\nu}{2\rho(P_q\cdot q)}\right) + \frac{2\rho}{P_q\cdot q}\left(p^\mu + \frac{q^\mu}{2\rho}\right)\left(p^\nu + \frac{q^\nu}{2\rho}\right)\right\}$$

$$\Downarrow \quad d|\bar{k}| = -\frac{Q}{2x}\,d\rho \qquad \rho = 0 \Rightarrow |\bar{k}| = \infty$$

$$\rho = 1 \Rightarrow |\bar{k}| = 0$$

$$w^{\mu\nu}_{q\to qg\,(a)} = -\frac{g^2}{(2\pi)^2}\,\frac{4}{3}\int_0^1 d\rho\;\frac{1}{2}\,\frac{1+\rho^2}{1-\rho}\,\delta(x-\rho)\,\ln\left|\frac{\mu^2}{\frac{Q^2}{2x^2}(1-\rho)^2+\mu^2}\right|$$

$$\cdot\left\{\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)\right.$$

$$\left.+\frac{2x}{p_q\cdot q}\left(p_q^\mu-\frac{p_q\cdot q}{q^2}\,q^\mu\right)\left(p_q^\nu-\frac{p_q\cdot q}{q^2}\,q^\nu\right)\right\}$$

$$\downarrow\qquad \ln\left[\left(\frac{Q^2}{2x^2}(1-x)^2+\mu^2\right)/\mu^2\right]\simeq \ln\frac{Q^2}{\mu^2}+\cdots$$

KEEP ONLY LEADING LOG TERM IF $Q^2 \gg \mu^2$

$$\simeq \frac{1}{4}\cdot\frac{4}{3}\cdot\frac{\alpha_s}{\pi}\int_0^1 d\rho\;\frac{1+\rho^2}{1-\rho}\,\delta(x-\rho)\,\ln\left(\frac{Q^2}{\mu^2}\right)$$

$$\cdot\,2\left\{\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)+\frac{2x}{p_q\cdot q}\left(p_q^\mu-\frac{p_q\cdot q}{q^2}\,q^\mu\right)\left(p_q^\nu-\frac{p_q\cdot q}{q^2}\,q^\nu\right)\right\}$$

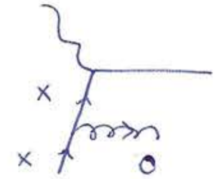
$$\Downarrow$$

$$\boxed{f_2(x,Q^2)\underset{\text{LEADING LOG}}{\simeq}\frac{1}{2}\,\frac{4}{3}\,\frac{\alpha_s}{\pi}\int_0^1 d\rho\;\frac{1+\rho^2}{1-\rho}\,\rho\,\delta(x-\rho)\,\ln\left(\frac{Q^2}{\mu^2}\right)}$$

$$= 2x\,f_1(x,Q^2)$$

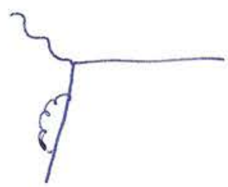
↳ SOFT - GLUON INFRARED SINGULARITY

$\rho \to 1$



$\frac{1}{1-\rho}$ SINGULAR

CANCELLED BY



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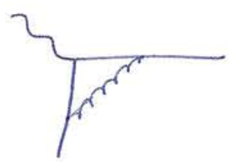


HAS SIMILAR STRUCTURE

BUT $\frac{1+\rho^2}{1-\rho} \longrightarrow \lambda\, \delta(1-\rho)$ (λ IS CONSTANT)

$$\boxed{f_2(x, Q^2) \underset{LL}{=} \frac{4}{3} \cdot \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu^2} \cdot \int_0^1 d\rho\, \rho\, \delta(x-\rho) \cdot \left[\frac{1+\rho^2}{(1-\rho)_+} + \lambda\, \delta(1-\rho) \right]}$$

WE INTRODUCED PRINCIPLE VALUE (i.e. SUBTRACT δ - FUNCTION)

$$\int_0^1 d\rho \frac{f(\rho)}{(1-\rho)_+} \equiv \int_0^1 d\rho \frac{f(\rho) - f(1)}{(1-\rho)}$$

↳ CORRESPONDS WITH FINITE PART

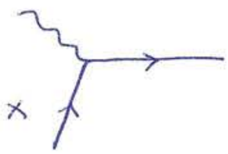
NOTE

$$\frac{1}{(1-\rho)_+} = \lim_{\varepsilon \to 0+} \left\{ \frac{1}{1-\rho}\, \theta(1-\rho-\varepsilon) - \delta(1-\rho) \int_0^{1-\varepsilon} d\rho' \frac{1}{1-\rho'} \right\}$$

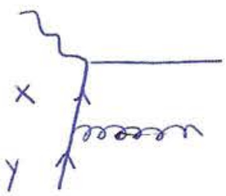
BY CHANGING $\frac{1}{1-\rho} \to \frac{1}{(1-\rho)_+}$ A PIECE OF δ- FUNCTION IS BROUGHT IN SINGULAR TERM TO CANCEL DIVERGENCE

* LEADING LOG CORRECTION TO VALENCE QUARK DISTRIBUTION (NON-SINGLET)

(NS)



$q_v(x)$



$q_v(x, Q^2)$ ↑ LOGARITHMIC $\ln \frac{Q^2}{\mu^2}$

QUARK STRUCK BY PHOTON HAS FRACTION $Z = \frac{x}{y}$ OF INITIAL QUARK MOMENTUM

$$\hookrightarrow \quad f_2^{NS}\left(\frac{x}{y}, Q^2\right) \underset{LL}{=} \frac{4}{3} \cdot \frac{\alpha_s}{2\pi} \cdot \ln \frac{Q^2}{\mu^2} \cdot \frac{x}{y} \left[\frac{1+z^2}{(1-z)_+} + \lambda\, \delta(1-z)\right]_{z=\frac{x}{y}}$$

$$f_2^{NS}\left(\frac{x}{y}, Q^2\right) = \frac{\alpha_s}{2\pi} \cdot \ln \frac{Q^2}{\mu^2} \cdot \frac{x}{y} \cdot P^{NS}_{q\to qg}\left(\frac{x}{y}\right)$$

NON-SINGLET SPLITTING FUNCTION

$$\boxed{P^{NS}_{q\to qg}(z) = \frac{4}{3} \cdot \left[\frac{1+z^2}{(1-z)_+} + \lambda\, \delta(1-z)\right]}$$

↳ $\delta q_v(x, Q^2)$ CORRECTION TO VALENCE QUARK DISTR. DUE TO GLUON EMISSION

$\delta q_v(x, Q^2)$ FROM $\frac{1}{x} \; F_2(x, Q^2)$

$$\boxed{\delta q_v(x, Q^2) = \frac{\alpha_s}{2\pi} \cdot \ln \frac{Q^2}{\mu^2} \int_x^1 \frac{dy}{y} \; P^{NS}_{q \to qg}\left(\frac{x}{y}\right) \cdot q_v(y, Q^2)}$$

$q_v(y, Q^2)$: QUARK DENSITY ∾ # QUARKS WITH MOMENTUM FRACTION y IN NUCLEON

$P_{q \to qg}\left(\frac{x}{y}\right)$: PROBABILITY THAT QUARK WITH MOM. FRAC y SPLITS INTO q & g, QUARK ENDS UP WITH MOM. FRAC. x $(x \leq y)$

LEADING LOG EVOLUTION (DGLAP EQUATION)

DOKSHITZER
GRIBOV
LIPATOV
ALTARELLI
PARISI

$$q_v(x, Q^2) = q_v(x) + \delta q_v(x, Q^2)$$

$$\Downarrow$$

$$\boxed{\frac{\partial}{\partial \ln Q^2} q_v(x, Q^2) = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y} P^{NS}_{q \to qg}\left(\frac{x}{y}\right) q_v(y, Q^2)}$$

DGLAP EQUATION

WITH $\boxed{P^{NS}_{q \to qg}(z) = \frac{1+z^2}{(1-z)_+} + \lambda \delta(1-z)}$

DETERMINE λ

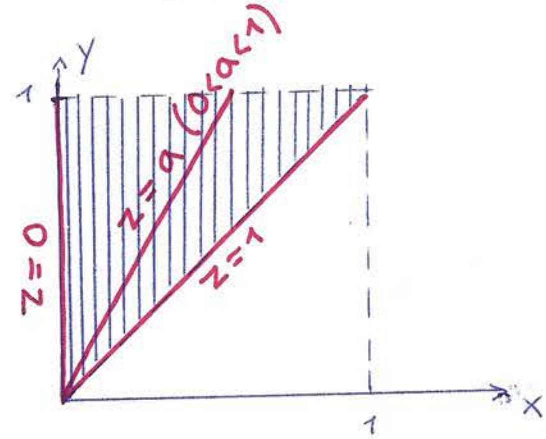
TOTAL NUMBER OF VALENCE QUARKS OF A GIVEN FLAVOR IS INDEPENDENT OF RESOLUTION SCALE Q

$$\frac{\partial}{\partial \ln Q^2} \int_0^1 dx \; q_v(x, Q^2) = 0$$

$$\Downarrow$$

$$0 = \frac{\alpha_s}{2\pi} \int_0^1 dx \int_x^1 \frac{dy}{y} \, P^{NS}\left(\frac{x}{y}\right) q_v(y, Q^2)$$

$$\downarrow \quad z = \frac{x}{y}$$



$$0 = \frac{\alpha_s}{2\pi} \int_0^1 dy \int_0^1 dz \; P^{NS}(z) \; q_v(y, Q^2)$$

$$= \frac{\alpha_s}{2\pi} \left(\int_0^1 dy \; q_v(y, Q^2) \right) \cdot \left(\int_0^1 dz \; P^{NS}(z) \right)$$

$$\Downarrow$$

$$\int_0^1 dz \; P^{NS}(z) = 0$$

$$\int_0^1 dz \left\{ \frac{1+z^2}{(1-z)_+} + \lambda \, \delta(1-z) \right\} = 0$$

$$\int_0^1 dz \left(\frac{1+z^2-2}{1-z} \right) + \lambda = 0 \quad \Longrightarrow \quad \boxed{\lambda = \frac{3}{2}}$$

↳ STRATEGY TO SOLVE DGLAP EQUATION

(INTEGRO - DIFFERENTIAL EQ.)

- START WITH ANSATZ OF $q_v(x, Q^2)$
  AT SOME LOW INPUT SCALE Q_0^2 (e.g. $Q_0^2 = 1\ \mathrm{GeV}^2$ TYPICAL)

$$q_v(x, Q_0^2) = A\, x^a (1-x)^b \left\{ 1 + c\sqrt{x} + d\, x \right\}$$

A DETERMINED FROM NORMALIZATION $\int_0^1 dx\, u_v(x, Q^2) = 2$

$$\int_0^1 dx\, d_v(x, Q^2) = 1$$

- USE DGLAP EQUATION
  TO CALCULATE $q_v(x, Q^2)$ AT LARGER Q^2
  Q^2 EVOLUTION : LOGARITHMIC

- FOR $SU(3)_F$ SINGLET DISTRIBUTION
  $\frac{1}{3}(u + d + s)$ CAN MIX WITH $g(x, Q^2)$ GLUON DISTR.

2 COUPLED EQUATIONS