Handout 9 (read by Dec. 18)

Preliminary note: In the following, let G be a compact Lie group. We consider continuous finite-dimensional representations $\Gamma = \{D(g)\}$, i.e., each $D_{ij}(g)$ depends continuously on g, and g depends continuously on a. The proofs of the following theorems require the substitutions

$$\sum_g \to \int_G d\mu(g) \quad \text{and} \quad |G| \to V$$

in section 2.3.

Theorem 3.2.4

1. 2.2.9 - 2.2.11: Every finite-dimensional representation of a compact Lie group is equivalent to a unitary representation. Therefore, it is possible to decompose the representation fully into a direct sum of irreducible representations.

2. 2.3.4: Let $\Gamma^{(\alpha)} = \{D^{(\alpha)}(g)\}$ be non-equivalent irreducible unitary finite-dimensional representations with dimension n_α. Then one has

$$\int_G d\mu(g) D^{(\alpha)}_{ir}(g) D^{(\beta)*}_{js}(g) = \frac{V}{n_\alpha}\delta^{\alpha\beta}\delta_{ij}\delta_{rs}.$$

3. 2.3.7: Let $\chi^{(\alpha)}$ be the character of $\Gamma^{(\alpha)}$:

$$\int_G d\mu(g)\chi^{(\alpha)}(g)\chi^{(\beta)*}(g) = V\delta^{\alpha\beta}.$$

4. 2.3.13: Let $\Gamma = \oplus_\alpha f_\alpha \Gamma^{(\alpha)}$, where f_α denotes the multiplicity of $\Gamma^{(\alpha)}$ in Γ. Then one has

$$f_\alpha = \frac{1}{V}\int_G d\mu(g)\chi^{(\alpha)*}(g)\chi(g),$$

   where χ is the character of Γ.

5. 2.3.17: Frobenius criterion for irreducibility:

   A finite-dimensional representation Γ with character χ is irreducible iff $\int_G d\mu(g)\chi^*(g)\chi(g) = V$.

Definition 3.4.2 Let $I = [a, b] \subseteq \mathbb{R}$ be an interval. A continuous function $g : I \to G$ is a *path in* G. The image $\Gamma = g(I)$ is called a *curve in* G. The function g is also called parameterization of the curve Γ.

If the group parameters a are continuous functions of a real variable then $g \circ a : I \to G$ is a path in G and

$$\Gamma = \{g(a(t))|t \in I \subseteq \mathbb{R} \wedge a(t) \, \text{continuous}\}$$

is a curve in G.

Definition 3.4.3

1. Two elements $g_1, g_2 \in G$ are called connected if they can be joined by a path in G.
2. G is called connected if every $g \in G$ can be joined with e by a path in G.

List of classical Lie groups

- general linear group: $\mathrm{GL}(n, \mathbb{K})$
- special linear group: $\mathrm{SL}(n, \mathbb{K}) := \{A \in \mathrm{GL}(n, \mathbb{K})|\det(A) = 1\}$
- $\mathrm{SL}_1(n, \mathbb{C}) := \{A \in \mathrm{GL}(n, \mathbb{C})|\det(A) \in \mathbb{R}\}$
- $\mathrm{SL}_2(n, \mathbb{C}) := \{A \in \mathrm{GL}(n, \mathbb{C})|\,|\det(A)| = 1\}$
- connected component of the identity in $\mathrm{GL}(n, \mathbb{R})$: $\mathrm{GL}^+(n, \mathbb{R}) := \{A \in \mathrm{GL}(n, \mathbb{R})|\det(A) > 0\}$
- orthogonal group: $\mathrm{O}(n) := \{A \in \mathrm{GL}(n, \mathbb{R})|A^T A = \mathbb{1}\}$
- complex orthogonal group: $\mathrm{O}(n, \mathbb{C}) := \{A \in \mathrm{GL}(n, \mathbb{C})|A^T A = \mathbb{1}\}$
- special orthogonal group: $\mathrm{SO}(n) := \{A \in \mathrm{O}(n)|\det(A) = 1\}$
- special complex orthogonal group: $\mathrm{SO}(n, \mathbb{C}) := \{A \in \mathrm{O}(n, \mathbb{C})|\det(A) = 1\}$
- unitary group: $\mathrm{U}(n) := \{A \in \mathrm{GL}(n, \mathbb{C})|A^\dagger A = \mathbb{1}\}$
- special unitary group: $\mathrm{SU}(n) := \{A \in \mathrm{U}(n)|\det(A) = 1\}$
- symplectic group: $\mathrm{Sp}(2n, \mathbb{K}) := \{A \in \mathrm{GL}(2n, \mathbb{K})|A^T J A = J\}$, where

$$J := \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$$

- $\mathrm{Sp}(2n) := \mathrm{Sp}(2n, \mathbb{C}) \cap \mathrm{U}(2n)$
- $\mathrm{O}(p, q) := \{A \in \mathrm{GL}(n, \mathbb{R})|A^T G(p, q) A = G(p, q)\}$, where

$$G(p, q) = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix}, \quad p + q = n$$

- $\mathrm{SO}(p,q) := \mathrm{O}(p,q) \cap \mathrm{SL}(n,\mathbb{R})$
- $\mathrm{U}(p,q) := \{A \in \mathrm{GL}(n,\mathbb{C}) | A^{\dagger}G(p,q)A = G(p,q)\}$
- $\mathrm{SU}(p,q) := \mathrm{U}(p,q) \cap \mathrm{SL}(n,\mathbb{C})$