Handout 8 (read by Dec. 11)

Preliminary note 3.2.1 Let $G = \{g_1, \ldots, g_n\}$ be a finite group. Moreover, let $f : G \to \mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, be a function. For any $h \in G$ and $M \subseteq G$ we have

$$\sum_{g\in M} f(g) = \sum_{g\in hM} f(h^{-1}g) = \sum_{g\in Mh} f(gh^{-1}).$$

Similarly, for any $h \in G$ and $G = M$ we have

$$\underline{\sum_{g\in G} f(g)} \;=\; \sum_{i=1}^{n} f(g_i) = \sum_{i=1}^{n} f(h^{-1}g_i) = \underline{\sum_{g\in G} f(h^{-1}g)} = \sum_{i=1}^{n} f(g_i h^{-1}) = \underline{\sum_{g\in G} f(gh^{-1})}. \quad (3.1)$$

Equation (3.1) was an essential ingredient in proving the fundamental orthogonality relation for matrices of irreducible representations (see theorem 2.3.4).

- Question: Is it possible to find a generalization of the sum to the case of Lie groups such that the property underlined in Eq. (3.1) still holds?
- Answer: The so-called Haar integral makes use of a suitably chosen measure such that for compact Lie groups an integration over the group parameters satisfies a relation analogous to Eq. (3.1).

We now investigate the properties of such an integral measure. To that end, we associate with the neighborhoods of the elements g and hg a *left-invariant* measure with the defining property

$$d\mu_L(g) = d\mu_L(hg), \quad (3.2)$$

such that

$$\int_M d\mu_L(g) f(g) = \int_{hM} d\mu_L(hg) f(h^{-1}g) \overset{(3.2)}{=} \int_{hM} d\mu_L(g) f(h^{-1}g).$$

For $M = G$, the equation

$$\int_G d\mu_L(g) f(g) = \int_G d\mu_L(g) f(h^{-1}g)$$

corresponds to

$$\sum_{g\in G} f(g) = \sum_{g\in G} f(h^{-1}g).$$

For constructing a right-invariant measure, we proceed analogously,

$$d\mu_R(g) = d\mu_R(gh),$$
$$\int_G d\mu_R(g) f(g) = \int_G d\mu_R(g) f(gh^{-1}).$$

- Construction

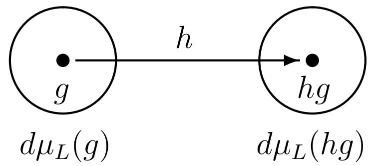

Figure 3.1: Left translation of a neighborhood of g by h.

For $g = g(a)$ let M be a neighborhood of g with $d\mu_L(g) = \rho_L(a)d^r a$. Furthermore, let $h = g(b)$ and $g(c) = g(b)g(a)$ with $c = \phi(b; a)$. We demand

$$\rho_L(a)d^r a = d\mu_L(g) \stackrel{(3.2)}{=} d\mu_L(hg) = \rho_L(c)d^r c.$$

- Determination of the density function ρ_L

Let M be a neighborhood of the identity $e = g(0)$ with volume $da_1 \dots da_r$. A left translation of M by $g(b)$ yields the neighborhood $M' = g(b)M$. From $g(b) = g(\phi(b; 0))$ we obtain

$$db_1 \dots db_r = \underbrace{\det \begin{pmatrix} \frac{\partial \phi_1(b;a)}{\partial a_1} & \cdots & \frac{\partial \phi_1(b;a)}{\partial a_r} \\ \vdots & & \vdots \\ \frac{\partial \phi_r(b;a)}{\partial a_1} & \cdots & \frac{\partial \phi_r(b;a)}{\partial a_r} \end{pmatrix}_{a=0}}_{=: J_L(b)} da_1 \dots da_r.$$

$\rho_L(0) \neq 0$ may be chosen arbitrarily. Setting

$$\rho_L(b) = \frac{\rho_L(0)}{J_L(b)},$$

we obtain

$$\rho_L(b)d^r b = \frac{\rho_L(0)}{J_L(b)} \underbrace{d^r b}_{J_L(b)d^r a} = \rho_L(0)d^r a.$$

Remarks 3.2.2

1. Consistency of the definition.

   To see the consistency of our definition, we consider the transition from a volume element near an arbitrary a to an arbitrary c in terms of $b : c = \phi(b; a)$:

   $$M_{g(c)} = g(b)M_{g(a)} = g(b)g(a)g^{-1}(a)M_{g(a)} = g(c)g^{-1}(a)M_{g(a)}.$$

   (a) The inverse transformation to a takes us from $a \to 0$.

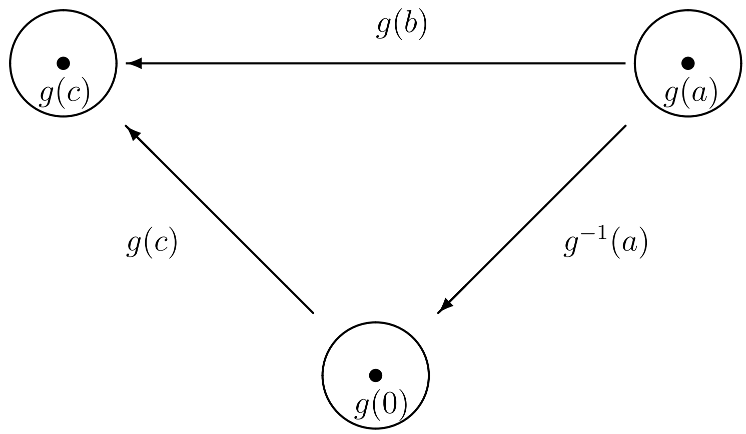

Figure 3.2: Left translation of a neighborhood $U_{g(a)}$ of $g(a)$ to a neighborhood $U_{g(c)}$ of $g(c)$ (1) in one step by $g(b)$ and (2) in two steps to a neighborhood $U_{g(0)}$ of the identity by $g^{-1}(a)$ and subsequently to $U_{g(c)}$ by $g(c)$.

(b) We then consider $0 \to c$.

(c) The definition is valid for the individual steps und thus also for executing the steps one after the other (see figure 3.2).

2. The result is unique up to a multiplicative constant.

3. The construction of a right-invariant measure proceeds analogously.

   Let M be a neighborhood of the identity with volume $da_1 \dots da_r$. A right translation of M by $g(b)$ yields the neighborhood $M' = Mg(b)$, where $g(b) = g(\phi(0;b))$.

$$db_1 \dots db_r = \underbrace{\det \begin{pmatrix} \frac{\partial \phi_1(a;b)}{\partial a_1} & \cdots & \frac{\partial \phi_1(a;b)}{\partial a_r} \\ \vdots & & \vdots \\ \frac{\partial \phi_r(a;b)}{\partial a_1} & \cdots & \frac{\partial \phi_r(a;b)}{\partial a_r} \end{pmatrix}_{a=0}}_{=: J_R(b)} da_1 \dots da_r.$$

   Put $\rho_R(b) = \frac{\rho_R(0)}{J_R(b)}$.

4. To summarize, we write

$$d\mu_L(g) = \rho_L(a) d^r a = \rho_L(0) \frac{1}{J_L(a)} d^r a$$

   and put $\rho_L(0) = 1$ to obtain

$$d\mu_L(g) = \frac{1}{J_L(a)} d^r a,$$

and proceed analogously for $d\mu_R(g)$,

$$d\mu_R(g) = \frac{1}{J_R(a)} d^r a.$$

5. $\int_G d\mu_{L/R}(g) f(g)$ are referred to as the left-invariant/right-invariant Haar integral.

6. In general, the left- and right-invariant Haar integrals differ.

7. Without proof:

   (a) Lie group G compact $\Leftrightarrow$ $V = \int_G d\mu_L(g)$ exists.
   Reminder: A subset N of a metric space M (this requires the definition of distance) is compact iff every sequence in N contains a convergent subsequence that converges towards a point in N.

   (b) V: so-called volume of the group

   (c) For compact Lie groups the left- and right-invariant measures coincide, i.e.

   $$d\mu_L(g) = d\mu_R(g) = d\mu(g).$$

8. In the above considerations we assumed that $a = 0$ corresponds to the identity of the group. If the parameters are chosen such that $a = a_0$ represents the identity, one needs to replace 0 by a_0 and "$a = 0$" by "$a = a_0$."