Handout 10 (read by Jan. 8)

Consider a Hilbert space with three Hermitian operators J_i satisfying the commutation relations (Einstein's summation convention implied)

$$[J_i, J_j] = i\epsilon_{ijk} J_k. \tag{4.3}$$

In terms of the definitions

$$\begin{aligned} \vec{J}^2 &:= J_i J_i, \\ J_\pm &:= J_1 \pm i J_2, \end{aligned} \tag{4.4}$$

we obtain a number of useful relations which we will apply in the construction of irreducible representations:

$$\begin{aligned} [\vec{J}^2, J_j] &= [J_i J_i, J_j] \\ &= J_i [J_i, J_j] + [J_i, J_j] J_i \\ &= J_i i\epsilon_{ijk} J_k + i\epsilon_{ijk} J_k J_i \\ &= i\epsilon_{ijk} J_i J_k - i\epsilon_{ijk} J_i J_k \\ &= 0. \end{aligned} \tag{4.5}$$

According to definition 3.3.7, $\vec{J}^2$ is a Casimir operator. In particular, Eq. (4.5) implies

$$[\vec{J}^2, J_\pm] = 0. \tag{4.6}$$

Moreover, we find

$$[J_3, J_\pm] = [J_3, J_1 \pm i J_2] = i J_2 \pm J_1 = \pm J_\pm, \tag{4.7}$$

$$[J_+, J_-] = [J_1 + i J_2, J_1 - i J_2] = 2 J_3. \tag{4.8}$$

Finally, we will need various ways to express the Casimir operator in terms of $J_\pm$ and J_3:

$$\begin{aligned} \vec{J}^2 &= \frac{1}{2}(J_+ J_- + J_- J_+) + J_3^2 \\ &= J_+ J_- - \frac{1}{2}[J_+, J_-] + J_3^2 \\ &= J_+ J_- - J_3 + J_3^2 && (4.9) \\ &= J_- J_+ - \frac{1}{2}[J_-, J_+] + J_3^2 \\ &= J_- J_+ + J_3 + J_3^2. && (4.10) \end{aligned}$$

- 1. step: Since commuting operators have a common set of eigenstates, we simultaneously diagonalize J_3 and $\vec{J}^2$. We denote the corresponding eigenstates initially by $|\lambda, \mu\rangle$:

$$J_3 |\lambda, \mu\rangle = \mu |\lambda, \mu\rangle, \tag{4.11}$$

$$\vec{J}^2 |\lambda, \mu\rangle = \lambda |\lambda, \mu\rangle, \tag{4.12}$$

  with real μ and λ, because $\vec{J}^2$ and J_3 are Hermitian. Moreover, eigenstates corresponding to different eigenvalues are orthogonal such that, after a suitable normalization, we can assume

$$\langle \lambda, \mu | \lambda', \mu' \rangle = \delta_{\lambda\lambda'} \delta_{\mu\mu'}.$$

Before proceeding to the next step, we need the following theorem:

Theorem 4.2.1 *Let A and B be linear operators satisfying $A|a\rangle = a|a\rangle$ and $[A, B] = \alpha B$. Then applies:*

$B|a\rangle$ is eigenstate of A with eigenvalue $a + \alpha$, provided $B|a\rangle \neq 0$.[1]

Proof:

$$A(B|a\rangle) = (AB)|a\rangle = (BA + [A, B])|a\rangle = BA|a\rangle + \alpha B|a\rangle = (a + \alpha)(B|a\rangle).$$

- 2. step: We apply theorem 4.2.1 using $A = \vec{J}^2$ or $A = J_3$ and $B = J_\pm$ and make use of the commutation relations of Eqs. (4.6) and (4.7). We then obtain $J_\pm|\lambda, \mu\rangle$ as further eigenstates of $\vec{J}^2$ and J_3 with

$$\vec{J}^2(J_\pm|\lambda, \mu\rangle) = \lambda(J_\pm|\lambda, \mu\rangle),$$
$$J_3(J_\pm|\lambda, \mu\rangle) = (\mu \pm 1)(J_\pm|\lambda, \mu\rangle),$$

provided $J_\pm|\lambda, \mu\rangle \neq 0$. $J_\pm$ is *raising* or rather *lowering operator* for J_3. The eigenvalues of J_3 are ordered in spacings of 1.

Theorem 4.2.2 *Let A be a Hermitian operator and $|\psi\rangle$ be an arbitrary normalized state. Then applies:*

$$\langle\psi|A^2|\psi\rangle \geq 0.$$

Proof: Let $|\psi'\rangle = A|\psi\rangle$. As a result of the properties of the scalar product we obtain

$$0 \leq \langle\psi'|\psi'\rangle = \langle\psi|A^\dagger A|\psi\rangle = \langle\psi|A^2|\psi\rangle$$

for $A = A^\dagger$.

- 3. step: We establish a relation between λ and a maximal eigenvalue j of J_3.

  Consider an eigenstate $|\lambda, \mu\rangle$ of $\vec{J}^2$ and J_3. Because $\vec{J}^2$ is the sum of squares of Hermitian operators, we obtain, using theorem 4.2.2,

$$\lambda = \langle\lambda, \mu|\vec{J}^2|\lambda, \mu\rangle = \langle\lambda, \mu|J_1^2|\lambda, \mu\rangle + \langle\lambda, \mu|J_2^2|\lambda, \mu\rangle + \mu^2\langle\lambda, \mu|\lambda, \mu\rangle \geq \mu^2.$$

  Applying J_+ to $|\lambda, \mu\rangle$ and normalizing the result, we find analogously

$$\lambda \geq (\mu + 1)^2$$

  and, after n successive applications,

$$\lambda \geq (\mu + n)^2.$$

  For sufficiently large n this leads to a contradiction unless, for a maximal value $\mu_{\max} =: j$, one has

$$J_+|\lambda, j\rangle = 0.$$

[1]By definition the zero vector does not qualify as an eigenvector.

Using Eq. (4.10), we find for such a state

$$\lambda|\lambda, j\rangle = \vec{J}^{\,2}|\lambda, j\rangle = \big(J_- J_+ + J_3(J_3+1)\big)|\lambda, j\rangle = j(j+1)|\lambda, j\rangle,$$

i.e., $\lambda = j(j+1)$.

In the following we write $|j, m\rangle$ for the eigenstates, in particular, for denoting the state we make use of the quantum number j instead of the eigenvalue $j(j+1)$.

- 4. step: We now prove the existence of a minimal eigenvalue $-j$ of J_3.

  We start from $|j, j\rangle$. We apply J_- to this state, normalize the result and proceed as above:

  $$j(j+1) \geq (j-1)^2.$$

  Applying J_- n times yields

  $$j(j+1) \geq (j-n)^2 = (n-j)^2,$$

  which, again, for sufficiently large n results in a contradiction, unless a $\mu_{\rm min}$ exists such that

  $$J_-|j, \mu_{\rm min}\rangle = 0.$$

  Using Eq. (4.9) we obtain

  $$j(j+1)|j, \mu_{\rm min}\rangle = \big(J_+ J_- + J_3(J_3-1)\big)|j, \mu_{\rm min}\rangle = \mu_{\rm min}(\mu_{\rm min}-1)|j, \mu_{\rm min}\rangle,$$

  with the solutions $\mu_{\rm min} = -j$ and $\mu_{\rm min} = j+1$. Because of $\mu_{\rm max} = j$ we can discard the second solution.

- 5. step: We now turn to the question which values for j are possible.

  For a given j, the eigenvalues of J_3 extend from $-j$ to j in steps of 1.[2] The number of eigenvalues is $2j+1$ and corresponds to the dimension of the representation. It, therefore, has to be a natural number:

  $$2j+1 \in \mathbb{N} = \{1, 2, 3, 4, \ldots\} \quad \Rightarrow \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots.$$

  For a given j, the set consisting of the states $|j, \mu\rangle$ with $-j \leq \mu \leq j$ is referred to as a *multiplet*.

- 6. step: Finally, we establish a convention fixing the relative phases between the states of a multiplet.

  To that end we consider the squared norm of the state $J_\pm|j, m\rangle$:

  $$\| J_\pm |j, m\rangle \|^2 = \langle j, m| J_\pm^\dagger J_\pm |j, m\rangle = \langle j, m| J_\mp J_\pm |j, m\rangle. \qquad (*)$$

  Using Eqs. (4.9) and (4.10), we write

  $$J_\mp J_\pm = \vec{J}^{\,2} - J_3^2 \mp J_3$$

[2] The set of all eigenvalues of a linear operator A is the *spectrum of A*.

and evaluate $(*)$:

$$\begin{aligned}\|J_\pm|j,m\rangle\|^2 &= \langle j,m|(\vec{J}^2 - J_3^2 \mp J_3)|j,m\rangle \\ &= j(j+1) - m(m\pm 1) \\ &= (j\mp m)(j\pm m+1).\end{aligned}$$

We make use of the so-called *Condon-Shortley phase convention* for states with the same j and different m:

$$J_\pm|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle \tag{4.13}$$

$$= \sqrt{j(j+1) - m(m\pm 1)}|j,m\pm 1\rangle. \tag{4.14}$$