Name:

1	2	3	4	5	6	7	8	9	$\sum$

S. Scherer Deadline: Jan. 17, 1 pm, 05-124

Symmetries in Physics (WS 2018/2019)
Exercise 5

1. We consider the so-called vector representation of the group D_3 on the vector space $\mathbb{R}^3$. This representation is defined in terms of a rotation by 120° about the z-axis and a rotation by 180° about the x-axis, i.e.,

$$D_3^V(c) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D_3^V(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

   (a) **[2]** Determine the matrices $D_3^V(c^2)$, $D_3^V(bc)$, and $D_3^V(bc^2)$. Note that D_3^V is a representation.

   (b) **[2]** $D_3^V(bc)$ and $D_3^V(bc^2)$ represent rotations by 180° about the rotation axes $\hat{n}(bc)$ and $\hat{n}(bc^2)$, respectively. Determine the rotation axes. Hint: Interpret the rotation axes as eivenvectors with eigenvalue 1.

   (c) **[1]** Determine for the three conjugacy classes K_i the character of D_3^V.

   (d) **[2]** Determine the coefficients a_μ^V of the decomposition

$$D_3^V = a_1^V D^{(1)} \oplus a_2^V D^{(2)} \oplus a_3^V D^{(3)}.$$

2. **[3]** Consider $G = D_3$. Determine in analogy to the lecture the coefficients $a_\sigma^{\mu\nu}$, $\mu, \nu, \sigma = 1, 2, 3$, of the Clebsch-Gordan decomposition of the inner tensor product representation

$$D^{(\mu)} \otimes D^{(\nu)} = a_1^{\mu\nu} D^{(1)} \oplus a_2^{\mu\nu} D^{(2)} \oplus a_3^{\mu\nu} D^{(3)}.$$

   Hint: Make use of the orthogonality relation for characters of irreducible representations. The character of the inner tensor product representation is equal to the product of the characters.

3. **[5]** How many real parameters does one need to describe the groups (explain your result)

   - $\mathrm{SL}(n, \mathbb{C}) = \{A | A \in \mathrm{GL}(n, \mathbb{C}), \det(A) = 1\}$,
   - $\mathrm{SL}(n, \mathbb{R}) = \{A | A \in \mathrm{GL}(n, \mathbb{R}), \det(A) = 1\}$,
   - $\mathrm{SU}(n) = \{A | A \in \mathrm{U}(n), \det(A) = 1\}$,
   - $\mathrm{O}(n, \mathbb{C}) = \{A | A \in \mathrm{GL}(n, \mathbb{C}), A^T A = A A^T = \mathbb{1}_{n\times n}\}$,
   - $\mathrm{O}(n, \mathbb{R}) = \{A | A \in \mathrm{GL}(n, \mathbb{R}), A^T A = A A^T = \mathbb{1}_{n\times n}\}$?

4. **[2]** We consider the group O(2). Let A be an element of the branch SO(2) and B be an element of the branch S_1SO(2) (see example 1.3.6, Handout 2), where

$$A = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \quad B = \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ -\sin(\beta) & -\cos(\beta) \end{pmatrix}, \quad 0 \le \alpha, \beta < 2\pi.$$

Calculate the distance

$$d(A,B) := ||A-B|| = \left[\sum_{i,j=1}^{2}(A_{ij}-B_{ij})^2\right]^{\frac{1}{2}}.$$

5. Consider $\mathrm{so}(n) := \{B \in \mathrm{gl}(n,\mathbb{R}) | B^T = -B\}$, i.e. the set of all skew symmetric real $n \times n$ matrices ($n \geq 2$).
   (a) **[1]** Show that $\mathrm{so}(n)$ is, in combination with matrix addition and scalar multiplication, a real vector space.
   (b) **[1]** Verify that
   $$[A,B] := AB - BA \quad \forall A, B \in \mathrm{so}(n)$$
   defines a closed bilinear multiplication, where AB refers to standard matrix multiplication (examine closure and linearity in both factors).
   (c) **[1]** Verify the Jacobi identity.
   (d) **[1]** What is the dimension of $\mathrm{so}(3)$? Specify explicitly corresponding basis vectors.

6. **[1]** Consider the angular momentum algebra $[J_i, J_j] = i\epsilon_{ijk}J_k$. Show that the product is not associative,
$$[J_i,[J_j,J_k]] \neq [[J_i,J_j],J_k].$$

7. Let $\mathcal{L}$ be a Lie algebra with basis $\{L_\alpha\}$ and commutation relations
$$[L_\alpha, L_\beta] = C_{\alpha\beta\gamma}L_\gamma.$$
   (a) **[1]** Verify $C_{\alpha\beta\gamma} = -C_{\beta\alpha\gamma}$.
   (b) **[2]** Verify $C_{\beta\gamma\mu}C_{\alpha\mu\delta} + C_{\gamma\alpha\mu}C_{\beta\mu\delta} + C_{\alpha\beta\mu}C_{\gamma\mu\delta} = 0$.

8. **[2]** Let G be a classical Lie group with Lie algebra $\mathcal{L}G$. Using a Taylor series expansion, verify for $C, D \in \mathcal{L}G$ and $t \in \mathbb{R}$
$$\exp(tC)\exp(tD)\exp(-tC)\exp(-tD) = I + t^2[C,D] + \mathcal{O}(t^3), \quad t \to 0.$$

9. **[2]** Consider the angular momentum algbra $[J_i, J_j] = i\epsilon_{ijk}J_k$. Using
$$\begin{aligned} J_\pm|j,m\rangle &= \sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle, \quad J_\pm = J_1 \pm iJ_2, \\ J_3|j,m\rangle &= m|j,m\rangle, \\ \langle j,m|j,m'\rangle &= \delta_{mm'}, \end{aligned}$$
construct the three-dimensional representation $\Psi^{(1)}(J_i)$ ($i = 1,2,3$) of the angular momentum algebra.