Name:

1	2	3	4	5	6	$\sum$

S. Scherer

Deadline: Dec. 20, 1 pm, 05-124

Symmetries in Physics (WS 2018/2019)
Exercise 4

1. Consider the Hamilton operator of the two-electron system of Example 2.2.22:

$$H = \underbrace{\frac{\vec{p}^{\,2}(1)}{2m} + V_1(r(1))}_{H_0(1)} + \underbrace{\frac{\vec{p}^{\,2}(2)}{2m} + V_1(r(2))}_{H_0(2)} + V_2(r_{12}) =: H_0 + V_2$$

with $(n = 1, 2)$

$$r(n) \;=\; |\vec{r}(n)|, \quad r_{12} = |\vec{r}(1) - \vec{r}(2)|, \quad V_1(r(n)) = -\frac{Z\alpha}{r(n)}, \quad V_2(r_{12}) = \frac{\alpha}{r_{12}}$$

and the commutation relations $(m, n \in \{1, 2\};\ i, j \in \{1, 2, 3\})$

$$[x_i(m), x_j(n)] \;=\; 0, \quad [p_i(m), p_j(n)] = 0, \quad [x_i(m), p_j(n)] = i\delta_{ij}\delta_{mn}.$$

   (a) **[3]** Evaluate $[\ell_i(n), V_2(r_{12})]$ for $n = 1$ and $n = 2$.

   (b) **[1]** Verify $[L_i, V_2(r_{12})] = 0$, where $\vec{L} = \vec{\ell}(1) + \vec{\ell}(2)$.

2. Consider the Hilbert space $\mathcal{H}_{\frac{1}{2}}$ of a spin-1/2 particle with basis

$$\left\{ |\uparrow\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\downarrow\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

   (a) **[1]** Let σ_i, $i = 1, 2, 3$, be the Pauli matrices and $\sigma_\pm := \sigma_1 \pm i\sigma_2$. Evaluate

$$\sigma_\pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \sigma_\pm \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \sigma_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \sigma_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

   Define $\mathcal{H} := \mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_{\frac{1}{2}}$ and

$$|1\rangle = |\uparrow\rangle \otimes |\uparrow\rangle, \quad |2\rangle = |\uparrow\rangle \otimes |\downarrow\rangle, \quad |3\rangle = |\downarrow\rangle \otimes |\uparrow\rangle, \quad |4\rangle = |\downarrow\rangle \otimes |\downarrow\rangle.$$

   We introduce the following new basis:

$$|1, 1\rangle := |1\rangle, \quad |1, 0\rangle := \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle), \quad |1, -1\rangle := |4\rangle,$$
$$|0, 0\rangle := \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle).$$

   We define the operator for the total spin as $\vec{S} = \frac{\vec{\sigma}}{2} \otimes \mathbb{1} + \mathbb{1} \otimes \frac{\vec{\sigma}}{2} = \frac{\vec{\sigma}(1)}{2} + \frac{\vec{\sigma}(2)}{2} = \vec{S}(1) + \vec{S}(2)$.

   (b) **[2]** Evaluate S_3 applied to $|1, 1\rangle$, $|1, 0\rangle$, $|1, -1\rangle$, and $|0, 0\rangle$.

(c) [**4**] Determine $\vec{S}^{\,2}$ applied to $|1,1\rangle$ and $|0,0\rangle$.
Hints:

$$\vec{S}^{\,2} = \left(\frac{\sigma_i}{2}\otimes \mathbb{1} + \mathbb{1}\otimes\frac{\sigma_i}{2}\right)\left(\frac{\sigma_i}{2}\otimes \mathbb{1} + \mathbb{1}\otimes\frac{\sigma_i}{2}\right) = \frac{1}{4}\left(\vec{\sigma}^{\,2}\otimes\mathbb{1} + 2\sigma_i\otimes\sigma_i + \mathbb{1}\otimes\vec{\sigma}^{\,2}\right).$$

Physics notation:

$$\vec{S}^{\,2} = \frac{1}{4}\left(\vec{\sigma}^{\,2}(1) + 2\sigma_i(1)\sigma_i(2) + \vec{\sigma}^{\,2}(2)\right).$$

Express $\sigma_i(1)\sigma_i(2)$ in terms of the operators $\sigma_\pm$ and σ_3 and make use of the results of (a).

3. **Removal of degeneracy**. [**4**] Consider the following Hamilton operator for two spin-1/2 particles (we are no longer using the direct product notation)

$$H = \underbrace{a(\vec{S}^{\,2}(1) + \vec{S}^{\,2}(2))}_{H_0} + \underbrace{2b\vec{S}(1)\cdot\vec{S}(2)}_{H_1} + \underbrace{c(S_3(1) + S_3(2))}_{H_2}$$

with $0 < c \ll b \ll a$, $\vec{S}(i) = \frac{\vec{\sigma}(i)}{2}$, $i = 1,2$. H_0 is invariant under SU(2) $\times$ SU(2), H_1 is invariant under $\{(g,g)|g \in \mathrm{SU}(2)\} \cong \mathrm{SU}(2)$, and H_2 is invariant under U(1). Determine the eigenstates and eigenvalues of H_0, H_0+H_1, and $H_0+H_1+H_2$. Sketch the spectrum. Make use of $\vec{S} = \vec{S}(1) + \vec{S}(2)$ and the results of problem 2.

4. [**1**] Let $D^{(1)}$ and $D^{(2)}$ be two equivalent, finite-dimensional representations of a group G on the vector spaces V_1 and V_2. Verify for the characters $\chi^{(1)}(g) = \chi^{(2)}(g)\ \forall\ g \in G$.

5. [**4**] Construct in analogy to Example 2.3.15 the regular representations of the groups $C_4 = \{e, c, c^2, c^3\}$ with $c^4 = e$ and of the Klein four-group $V = \{e, a, b, c\}$ with $a^2 = b^2 = c^2 = e$ (see Exercise 1, problem 2).

Hint: The matrices of the regular representation are defined as

$$g g_i = \sum_{j=1}^{|G|} D_{ji}(g) g_j, \quad i = 1, \ldots, |G|.$$

Enumerate $g_1 = e$, $g_2 = c$, $g_3 = c^2$, $g_4 = c^3$ and $g_1 = e$, $g_2 = a$, $g_3 = b$, $g_4 = c$.

---

6. Consider the abstract group D_3 with six elements $\{e, c, c^2, b, bc, bc^2\}$ (see Example 1.2.10).

(a) [**2**] Construct a two-dimensional representation on $\mathbb{R}^2$ by interpreting the group elements geometrically as active rotations about the origin or as reflections.
Example:

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(c) = \begin{pmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

Choose for b a mirror reflection over the x-axis.

(b) [**1**] Determine the character of the 6 group elements.

(c) [**4**] Determine the 4 six-dimensional vectors $(D_{ij}(e), D_{ij}(c), ..., D_{ij}(bc^2))^T$. Verify explicitly the $4+3+2+1 = 10$ orthogonality relations

$$\sum_g D_{ir}(g) D^*_{js}(g) = \frac{6}{2}\delta_{ij}\delta_{rs}.$$