Handout 4 (read by Nov. 13)

Example 1.3.16 Elements of SO(3) with a given fixed angle of rotation form a conjugacy class.

Explanation: Each rotation D may be characterized by a rotation axis $\hat{n}$ and an angle of rotation $0 \leq \varphi < 2\pi$, where

$$D\hat{n} = \hat{n}, \quad \mathrm{Tr}(D) = 1 + 2\cos(\varphi).$$

Let $\mathrm{Tr}(D_1) = \mathrm{Tr}(D_2)$. For a suitable basis, $\hat{n}_1 = \hat{e}_1$ and $D_1\hat{n}_1 = \hat{n}_1$, i.e.,

$$D_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi_1) & -\sin(\varphi_1) \\ 0 & \sin(\varphi_1) & \cos(\varphi_1) \end{pmatrix} \text{ with } 0 \leq \varphi_1 < 2\pi.$$

Let $T \in \mathrm{SO}(3)$ be such that the rotation axis $\hat{n}_1$ is mapped under T on the rotation axis $\hat{n}_2$, i.e., $\hat{n}_2 = T\hat{n}_1$ and $\hat{n}_1 = T^{-1}\hat{n}_2$. From

$$T^{-1}D_2T\hat{n}_1 = T^{-1}D_2\hat{n}_2 = T^{-1}\hat{n}_2 = \hat{n}_1$$

we conclude that $T^{-1}D_2T$ is also a rotation about the axis $\hat{n}_1$. Let

$$T^{-1}D_2T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi_2) & -\sin(\varphi_2) \\ 0 & \sin(\varphi_2) & \cos(\varphi_2) \end{pmatrix}.$$

We now consider the following trace,

$$\mathrm{Tr}(T^{-1}D_2T) = \mathrm{Tr}(TT^{-1}D_2) = \mathrm{Tr}(D_2) = 1 + 2\cos(\varphi_2) = \mathrm{Tr}(D_1) = 1 + 2\cos(\varphi_1).$$

From this we conclude

$$\cos(\varphi_1) = \cos(\varphi_2) \Rightarrow \varphi_2 = \pm\varphi_1.$$

We distinguish two different cases:

1. $\varphi_2 = \varphi_1$: In this case we have $D_1 = T^{-1}D_2T$.

2. $\varphi_2 = -\varphi_1$: We first determine

$$T^{-1}D_2T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi_1) & \sin(\varphi_1) \\ 0 & -\sin(\varphi_1) & \cos(\varphi_1) \end{pmatrix}.$$

   Multiplication from the right by

$$S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in \mathrm{SO}(3)$$

and from the left by $S^{-1} = S$ yields

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi_1) & \sin(\varphi_1) \\ 0 & -\sin(\varphi_1) & \cos(\varphi_1) \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi_1) & -\sin(\varphi_1) \\ 0 & \sin(\varphi_1) & \cos(\varphi_1). \end{pmatrix}.$$

We thus find

$$D_1 = S^{-1}T^{-1}D_2TS = (TS)^{-1}D_2(TS).$$