Name:

1	2	3	4	5	6	$\sum$

S. Scherer

Deadline: Dec. 6, 1 pm, 05-124

Symmetries in Physics (WS 2018/2019)
Exercise 3

1. Consider
$$\mathrm{SU}(2) \ni U_j(\theta) = \exp\left(-i\theta\frac{\sigma_j}{2}\right), \quad j = 1,2,3,$$
with the Pauli matrices
$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
   (a) **[3]** Using the series expansion of the exponential function, verify
$$U_j(\theta) = \cos\left(\frac{\theta}{2}\right)\mathbb{1} - i\sin\left(\frac{\theta}{2}\right)\sigma_j.$$
   Hint: $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathbb{1}$.

   (b) **[2]** Verify
$$U_3(\gamma) = \begin{pmatrix} \exp\left(-i\frac{\gamma}{2}\right) & 0 \\ 0 & \exp\left(i\frac{\gamma}{2}\right) \end{pmatrix}, \quad U_2(\beta) = \begin{pmatrix} \cos\left(\frac{\beta}{2}\right) & -\sin\left(\frac{\beta}{2}\right) \\ \sin\left(\frac{\beta}{2}\right) & \cos\left(\frac{\beta}{2}\right) \end{pmatrix}.$$
   Remark: Later on, the entries of $U_2(\beta)$ will be denoted by $d^{(\frac{1}{2})}_{mm'}(\beta)$.

   (c) **[1]** Evaluate $U_3(2\pi)$ and $U_3(2\pi + \theta)$? When do you reach the identity.

2. Consider the state $|n, l, m\rangle = |2, 1, 1\rangle$ of the hydrgen atom,
$$\Psi_{211}(\vec{x}\,) = R_{21}(r)Y_{11}(\theta, \phi).$$
   (a) **[2]** Evaluate the expectation value $\langle 2, 1, 1|\ell_x|2, 1, 1\rangle$.
$$\text{Hints:} \qquad \int_0^\infty \mathrm{d}r\, r^2 R_{21}^2(r) = 1, \quad Y_{11}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin(\theta)e^{i\phi},$$
$$\ell_x = i\left[\sin(\phi)\frac{\partial}{\partial\theta} + \cot(\theta)\cos(\phi)\frac{\partial}{\partial\phi}\right].$$
   (b) **[2]** Consider a rotation about the y axis with rotation angle β in terms of
$$\mathcal{R}(0, \beta, 0) = \exp(-i\beta\ell_y).$$
   Using the Baker-Campbell-Hausdorff formula
$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \ldots$$
   for $A = i\beta\ell_y$ and $B = \ell_x$ in combination with canonical commutation relations for the angular momentum operators, derive
$$\exp(i\beta\ell_y)\ell_x \exp(-i\beta\ell_y) = \cos(\beta)\ell_x + \sin(\beta)\ell_z.$$

(c) **[2]** Now consider the state which results from a rotation about the y axis with rotation angle $\pi/2$:

$$|2,1,1\rangle' = \exp\left(-i\frac{\pi}{2}\ell_y\right)|2,1,1\rangle.$$

Evaluate ${}'\langle 2,1,1|\ell_x|2,1,1\rangle'$.

(d) **[2]** Using the canonical commutation relations

$$[x_i, x_j] = 0, \quad [p_i, p_j] = 0, \quad [x_i, p_j] = i\delta_{ij},$$

in combination with the definition $\ell_i = \epsilon_{ijk}x_j p_k$, derive the commutation relations $[x_i, \ell_j]$ and $[p_i, \ell_j]$.

(e) **[2]** Now evaluate

$$\exp(i\beta\ell_y)x\exp(-i\beta\ell_y) \quad \text{and} \quad \exp(i\beta\ell_y)p_x\exp(-i\beta\ell_y).$$

3. **[2]** Let G be a group and define $M := \{D|D\,\text{representation of}\ G\}$. Verify that the definition $D_1 : V_1 \to V_1 \sim D_2 : V_2 \to V_2$, if a bijective $S : V_1 \to V_2$ exists with $SD_1(g)S^{-1} = D_2(g)\,\forall\, g \in G$, indeed provides an equivalence relation.

4. **[2]** Consider the following three-dimensional representation $D : \mathbb{R}^3 \to \mathbb{R}^3$ of the permutation group S_3:

$$D(P_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D(P_2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D(P_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$D(P_4) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad D(P_5) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad D(P_6) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Show that the representation is reducible.

Hint: Find a common eigenvector of all $D(g)$ and thus a non-trivial invariant vector subspace.

5. **[3]** Let $D = \{D(g)\}$ be a representation of a group G in terms of matrices with $g \mapsto D(g)$. Verify that

$$(1) \quad g \mapsto D^*(g), \quad (2) \quad g \mapsto D^T(g^{-1}), \quad (3) \quad g \mapsto D^\dagger(g^{-1})$$

also define representations.

Hint: First argue why the matrices (1), (2), and (3) are invertible. Then verify the homomorphism property.

6. **[2]** Consider a two-dimensional representation of SU(2) over $\mathbb{C}^2$, using elements of the form

$$U(\theta, \hat{n}) = \exp\left(-i\frac{\theta}{2}\,\hat{n}\cdot\vec{\sigma}\right),$$

where σ_i $(i = 1, 2, 3)$ are Pauli matrices.

Verify that $\{U^*(\theta, \hat{n})\}$ is an equivalent representation.

Hint: Determine σ_i^*. Using $S = -i\sigma_2$, verify

$$U^*(\theta, \hat{n}) = S\,U(\theta, \hat{n})\,S^{-1}.$$