Name:	1	2	3	4	5	6	7	8	9	$\sum$

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Deadline: Nov. 22, 1 pm, 05-124

Symmetries in Physics (WS 2018/2019)
Exercise 2

1. Consider a Lorentz transformation $\Lambda \in L_+^\uparrow$ satisfying $\Lambda t = t$, where

$$t = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

   (a) **[2]** Verify that the set $\{\Lambda \in L_+^\uparrow | \Lambda t = t\}$ is a subgroup of $L_+^\uparrow$.
   Remark: The group is referred to as the little group of t.

   (b) **[5]** Show that Λ is isomorphic to a proper rotation.
   Hint: Express Λ as

$$\Lambda = \left(\begin{array}{c|ccc} \Lambda_{00} & \Lambda_{01} & \Lambda_{02} & \Lambda_{03} \\ \hline \Lambda_{10} & & & \\ \Lambda_{20} & & D_{3\times 3} & \\ \Lambda_{30} & & & \end{array}\right)$$

   and make use of $\Lambda t = t$. From this you will obtain conditions for Λ_{00} and Λ_{i0}, $i = 1, 2, 3$. Then insert the new form of Λ into $G = \Lambda^T G \Lambda$. From this you will obtain conditions for Λ_{0j}, $j = 1, 2, 3$ and $D_{3\times 3}$. Finally, make use of $\det(\Lambda) = 1$.

2. Let Λ_1 and Λ_2 be proper, orthochronous Lorentz transformations, i.e., $\Lambda_i \in L_+^\uparrow$. Verify for $\Lambda_3 = \Lambda_2 \Lambda_1$

   (a) **[1]** $\Lambda_3^T G \Lambda_3 = G$,

   (b) **[1]** $\det(\Lambda_3) = +1$.

3. We consider the group $G = \mathrm{SU}(2)$. Which of the following maps A defines an action of G on M?

   (a) **[1]**

$$M := \{m = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} | z_i \in \mathbb{C}\},\ A_i(g, m) = g_{ij} z_j \text{ or } A(g, m) = gm.$$

   (b) **[1]**

$$M := \{m = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} | x_i \in \mathbb{R}\},\ A_i(g, m) = g_{ij} x_j \text{ or } A(g, m) = gm.$$

(c) **[1]**

$$M := \{m \in \mathrm{SU}(2)\},\ A_{ij}(g,m) = g_{ik}m_{kj} \text{ or } A(g,m) = gm.$$

(d) **[1]**

$$M := \{m \in \mathrm{SU}(2)\},\ A_{ij}(g,m) = g_{ik}m_{kl}g^*_{jl} \text{ or } A(g,m) = gmg^\dagger.$$

(e) **[1]**

$$M := \{m \in \mathrm{SU}(2)\},\ A_{ij}(g,m) = g_{ik}m_{kl}g_{lj} \text{ or } A(g,m) = gmg.$$

(f) **[1]**

$$M := \{m | m \text{ Hermitian } 2 \times 2 \text{ matrix}\},$$
$$A_{ij}(g,m) = g_{ik}m_{kl}g^*_{jl} \text{ or } A(g,m) = gmg^\dagger.$$

4. The centre Z of a group G consists of all elements $z \in G$ which commute with all elements of the group:

$$Z := \{z \in G | zg = gz \ \forall \ g \in G\}.$$

   (a) **[1]** Verify that Z is an Abelian subgroup of G.

   (b) **[1]** Verify that Z is a normal subgroup of G.

5. **[3]** Verify the group axioms $(G1) - (G3)$ for the factor group G/H.

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6. Let $G = \{C = (A,B) | A \in \mathrm{SU}(n), B \in \mathrm{SU}(n)\}$ with $C_1C_2 = (A_1A_2, B_1B_2)$. Which of the following sets is a subgroup of G? Is any of the subgroups a normal subgroup?

   (a) **[1]** $\{X = (A,A) | A \in \mathrm{SU}(n)\}$,

   (b) **[1]** $\{X = (A, \mathbb{1}_{n\times n}) | A \in \mathrm{SU}(n)\}$,

   (c) **[1]** $\{X = (A, A^\dagger) | A \in \mathrm{SU}(n)\}$.

7. **[3]** Consider the group $D_3 = \langle c, b \rangle$ with the defining relations $c^3 = b^2 = (bc)^2 = e$ and its normal subgroup $C_3 = \langle c \rangle$ with $c^3 = e$. Identify the elements of the factor group D_3/C_3. Explicitly calculate the products of the elements of the factor group.

8. **[2]** Consider the subgroup $C_2 = \langle b \rangle$ with $b^2 = e$. Note that C_2 is not a normal subgroup. Determine the set of all left cosets gC_2 and the set of all right cosets C_2g. What is the difference in comparison with the set of left and right cosets of a normal subgroup?

9. **[2]** Verify that the group $\mathrm{O}(3)$ is the internal direct product of $\mathrm{SO}(3)$ and $\{e, p\} = \{\mathbb{1}_{3\times 3}, -\mathbb{1}_{3\times 3}\}$.