Chiral Dynamics I + II

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Introduction and Motivation

Literature Search

- http://www.slac.stanford.edu/spires/

- http://www.arxiv.org/

General Literature on Chiral Perturbation Theory and Effective Field Theory

<u>Classical Papers:</u>

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System of Units

- SI (Système International) units
  - reduced Planck constant
    $\hbar = 1.054\,571\,68(18) \times 10^{-34}$ J s,

$1 \text{ J} = 1 \text{ kg m}^2 \text{ s}^{-2}$

– electron mass

$m_e = 9.109\,3826(16) \times 10^{-31}$ kg

– electron charge magnitude

$e = 1.602\,176\,53(14) \times 10^{-19}$ C,

1 C= 1 A s

– permittivity of free space

$$\begin{aligned} \epsilon_0 &= \frac{1}{\mu_0 c^2} \\ &= 8.854\,187\,817 \ldots \times 10^{-12} \text{ C V}^{-1} \text{ m}^{-1} \end{aligned}$$

1 V = 1 J / C

– speed of light in vacuum

$c = 299\,792\,458$ m s^{-1}

– basic length scale of sub-nuclear physics

$1 \text{ fm} = 10^{-15} \text{ m}$

– unit of cross sections

$1 \text{ barn} = 10^2 \, \text{fm}^2$

- Natural units

  – Set $\hbar = c = 1 = \epsilon_0 = \mu_0$

  – fine-structure constant

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = 1/137.035\,999\,11(46)$$
$$\rightarrow e^2/4\pi$$
$$\approx \frac{1}{137}$$

  – conversion constant

$\hbar c = 197.326\,968(17) \text{ MeV fm} \rightarrow 1$

allows one to express energies in terms of inverse lengths and viceversa

- Maxwell equations (in the vaccuum)

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \rho, \\
\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} &= \vec{J}, \\
\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0, \\
\vec{\nabla} \cdot \vec{B} &= 0.
\end{aligned}
$$

- Continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0.$$

- Lorentz force

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}).$$

1.1 Motivation and Keywords

- Chiral perturbation theory (**ChPT**) is the effective field theory (**EFT**) of the Standard Model/strong interactions at low energies.

- EFTs are **low-energy approximations** to (more) fundamental theories.

- Instead of solving the underlying theory, low-energy physics is described with a set of variables (**effective degrees of freedom**) that is suited for the particular energy region of interest.

- ChPT: **Pions and nucleons** instead of the more fundamental quarks and gluons of QCD.

  Later: Δ resonance, vector and axial-vector mesons.

- Calculate physical quantities in terms of an **expansion in** p/Λ, where p stands for momenta or masses that are smaller than a certain momentum scale Λ.

Compare with QED: Perturbation theory in small coupling constant.

- There exists a regime where both fundamental and effective theories yield the same results.
- EFTs are based on the **most general Lagrangian**, which includes all terms that are compatible with the symmetries of the underlying theory.

  $\Rightarrow$ **Infinite number of terms**.

  Each term is accompanied by a low-energy coupling constant (**LEC**).

  Compare with "fundamental" QED: 2 parameters, e and m_e.

- Method that allows one to decide which terms contribute in a calculation up to a certain accuracy:

  Weinberg's power counting.

- In actual calculations only a finite number of terms in the expansion in p/Λ has to be considered.

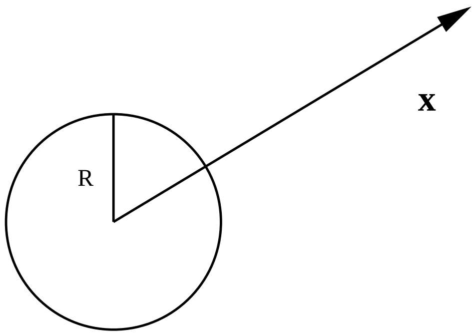
  $\Rightarrow$ **Predictive power**.

- EFTs are non-renormalizable in the traditional sense. However, consider **all terms that are allowed by the symmetries**. $\Rightarrow$ Ultraviolet divergences that occur in calculations up to any given order of p/Λ can be renormalized by redefining fields and parameters of the Lagrangian of the EFT.

  The so-called non-renormalizable theories are actually just as renormalizable as renormalizable theories.

1.2 Example from Electrostatics Illustrating the Idea of a (Distance) Scale

Consider charge distribution $\rho(\vec{x}\,')$ which is localized inside a sphere of radius R:



Potential from solution to Poisson equation,

$$\Delta\phi = -\rho,$$

reads

$$\phi(\vec{x}) = \frac{1}{4\pi}\int \frac{\rho(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}\, \mathrm{d}^3 x'.$$

Make use of

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi).$$

$\Rightarrow$

- Solution for $|\vec{x}| \lesssim R$ complicated.
- Solution for $|\vec{x}| \gg R$ simple, because

$$\phi(\vec{x}) = \sum_{l,m} \underbrace{\left[\int Y_{lm}^*(\theta', \phi') r'^l \rho(\vec{x}\,')\, \mathrm{d}^3 x'\right]}_{\text{multipole moment } q_{lm}} \frac{1}{2l+1} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}.$$

In Cartesian coordinates

$$4\pi\phi(\vec{x}) = \frac{q}{r} + \frac{\vec{p} \cdot \vec{x}}{r^3} + \frac{1}{2} \sum_{i,j} Q_{ij} \frac{x_i x_j}{r^5} + \cdots,$$

1. total charge

$$q = \int \rho(\vec{x}\,') \,\mathrm{d}^3x'$$

2. electric dipole moment

$$\vec{p} = \int \vec{x}\,'\rho(\vec{x}\,') \,\mathrm{d}^3x'$$

3. traceless quadrupole moment tensor

$$Q_{ij} = \int (3x'_i x'_j - r'^2\delta_{ij})\rho(\vec{x}\,') \,\mathrm{d}^3x'$$

4. etc.

- Infinite number of terms.
- However, far away, details of charge distribution not important, knowledge of the leading-order terms sufficient.

- Systematic improvement possible.
- For smaller r, higher multipoles become more important.
- q_{lm} parameterize short-distance physics.
- $\frac{1}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$ determine the long-distance effects of short-distance physics.
- (Simplified) analogies[1]

Multipole expansion	EFT
q_{lm}	LECs
$\frac{1}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$	Structures of most general $\mathcal{L}_{\text{EFT}}$

- Here: Simple separation of scales (R).
- ChPT: Scales depend on underlying dynamics and masses of the participating particles.

[1] Observables will be calculated in perturbation theory using $\mathcal{L}_{\text{EFT}}$.

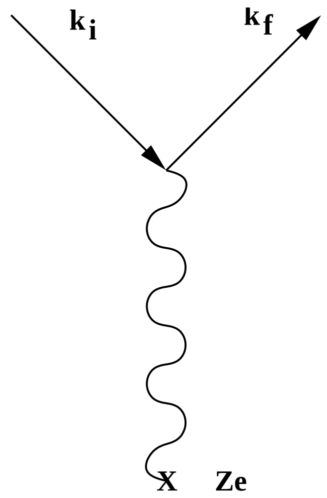
Outlook: Important scales in ChPT

1. $4\pi F_\pi$, where $F_\pi = 92.4$ MeV is the pion-decay constant.
2. $m_\rho = (775.5 \pm 0.4)$ MeV is the rho-meson mass.

- Examples
  - Strong interaction: Root mean square electric radius of the proton $r_E^p = (0.870 \pm 0.008)$ fm.
  - Electromagnetic interaction: Bohr radius $0.5291772108(18) \times 10^{-10}$ m.

1.3 Electron Scattering off a Static Charge Distribution

Consider electron scattering in the Coulomb potential of an infinitely heavy point charge:



Kinematics

$$
\begin{aligned}
k &= |\vec{k}_i| = |\vec{k}_f|, \\
\vec{q} &= \vec{k}_f - \vec{k}_i, \\
\vec{q}^{\,2} &= \vec{k}_f^2 + \vec{k}_i^2 - 2\vec{k}_f \cdot \vec{k}_i \\
&= 2k^2[1 - \cos(\theta)] = 4k^2 \sin^2\left(\frac{\theta}{2}\right),
\end{aligned}
$$

$$v = \frac{k}{E}.$$

Without proof: Mott cross section

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{(Z\alpha)^2 E^2}{4k^4 \sin^4\left(\frac{\theta}{2}\right)} \left[1 - v^2 \sin^2\left(\frac{\theta}{2}\right)\right].$$

Nonrelativistic limit

$$\begin{aligned} \frac{k}{E} &\to 0, \\ E &\to m_e. \end{aligned}$$

$\Rightarrow$ Rutherford formula

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} = \left(\frac{m_e Z\alpha}{2k^2 \sin^2\left(\frac{\theta}{2}\right)}\right)^2.$$

Extension from point charge to static, spinless charge distribution $Ze\rho(\vec{x})$ with normalization

$$\int \rho(\vec{x})\,\mathrm{d}^3x = 1.$$

$\Rightarrow$ Cross section

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} |F(\vec{q})|^2$$

where

$$F(\vec{q}) = \int e^{i\vec{q}\cdot\vec{x}}\rho(\vec{x})\,\mathrm{d}^3x.$$

Expand exponential function

$$e^{i\vec{q}\cdot\vec{x}} = 1 + i\vec{q}\cdot\vec{x} - \frac{1}{2}(\vec{q}\cdot\vec{x})^2 + \cdots,$$

Result

$$F(\vec{q}) \;=\; \int \rho(\vec{x})\,\mathrm{d}^3x + i\vec{q}\cdot\int \vec{x}\rho(\vec{x})\,\mathrm{d}^3x$$

$$-\frac{1}{2} q_i q_j \int \underbrace{x_i x_j}_{\frac{1}{3}(3x_i x_j - r^2\delta_{ij} + r^2\delta_{ij})} \rho(\vec{x})\, \mathrm{d}^3 x + \cdots$$

$$= 1 + i\vec{q} \cdot \frac{\vec{p}}{Ze} - \frac{1}{6} q_i q_j \frac{Q_{ij}}{Ze} - \frac{1}{6} \vec{q}^{\,2} \langle r^2 \rangle + \cdots$$

$\langle r^2 \rangle$ mean squared radius + multipole moments

- Conclusions
  - More and more details can be resolved with increasing $|\vec{q}|$.
  - In reverse, a fixed upper value of $|\vec{q}|$ sets a limit on the physics phenomena that can be studied at small distances.

1.4 Weinberg's Effective Field Theory Program

Foundations of EFT as a Quantum Field Theory
S. Weinberg, Physica A **96**, 327 (1979)

... if one writes down the *most general possible Lagrangian*, including *all* terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian *to any given order of perturbation theory*, the result will simply be the most general possible S–matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles.

Explanation of terms

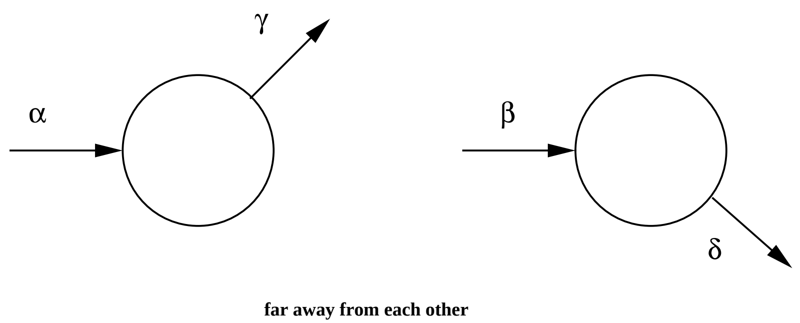
- Symmetries: Poincaré invariance, discrete symmetries C, P, T, but also internal symmetries such as isospin symmetry, chiral symmetry including the possibility of a spontaneous symmetry breakdown.
- Analyticity $\leftrightarrow$ Causality.
- Unitarity: the sum over the probabilities of the final states must

yield exactly 1,

$$\sum_f |\langle f|S|i\rangle|^2 = 1.$$

- Cluster decomposition (Weinberg, Vol. 1, chapter 4): loosely speaking, distant experiments must yield uncorrelated results:

$$S_{\gamma+\delta\leftarrow\alpha+\beta} \to S_{\delta\leftarrow\beta} S_{\gamma\leftarrow\alpha}.$$



1.5 Classification of Effective Field Theories

See G. Ecker, arXiv:hep-ph/0507056, for more details.
A first classification of EFTs is based on the structure of the transitions from the "fundamental" (energies $> \Lambda$) to the "effective" level (energies $< \Lambda$).

1. Complete decoupling

   The fundamental theory contains heavy and light degrees of freedom. For energies $\ll \Lambda$ the heavy particles are never produced.

   In modern jargon (obtained from the path integral formalism): the heavy degrees of freedom have been integrated out (simplified discussion below).

   Typical xxample: weak interactions in the Standard Model.

   E.g., neutron beta decay $n \to p\, e^-\, \bar{\nu}_e$ with $q = p_n - p_p$:

$$|(p_n - p_p)^2| \ll M_W^2$$

$\Rightarrow$

$$\frac{1}{q^2 - M_W^2} \to -\frac{1}{M_W^2}.$$

$\Rightarrow$ effective $V - A$ theory of the weak interactions.

Example: "Integrating out" a heavy degree of freedom in a toy model ($m \ll M$)

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \\
\mathcal{L}_0 &= \frac{1}{2}(\partial_\mu \Phi \partial^\mu \Phi - M^2 \Phi^2) + \frac{1}{2}(\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2), \\
\mathcal{L}_{\text{int}} &= -\frac{\lambda}{2} \Phi \varphi^2.
\end{aligned}$$

Equations of motion:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi} - \frac{\partial \mathcal{L}}{\partial \Phi} = \Box\Phi + M^2\Phi + \frac{\lambda}{2}\varphi^2 \;=\; 0, \qquad (*)$$

$$\Box\varphi + m^2\varphi + \lambda\varphi\Phi \;=\; 0. \qquad (**)$$

Formally solve $(*)$:

$$\Phi = -\frac{\lambda}{2M^2}\frac{1}{1+\frac{\Box}{M^2}}\varphi^2.$$

Insert solution into $(**)$. $\Rightarrow$

$$\Box\varphi + m^2\varphi - \frac{\lambda^2}{2M^2}\varphi\frac{1}{1+\frac{\Box}{M^2}}\varphi^2 = 0.$$

Expand to leading order in $1/M^2$:

$$\Box\varphi + m^2\varphi - \frac{\lambda^2}{2M^2}\varphi^3 = 0. \qquad (***)$$

Q: Which effective Lagrangian generates $(***)$?

$$\text{A:} \quad \mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2) + \frac{\lambda^2}{8M^2}\varphi^4.$$

Compare with original Lagrangian:

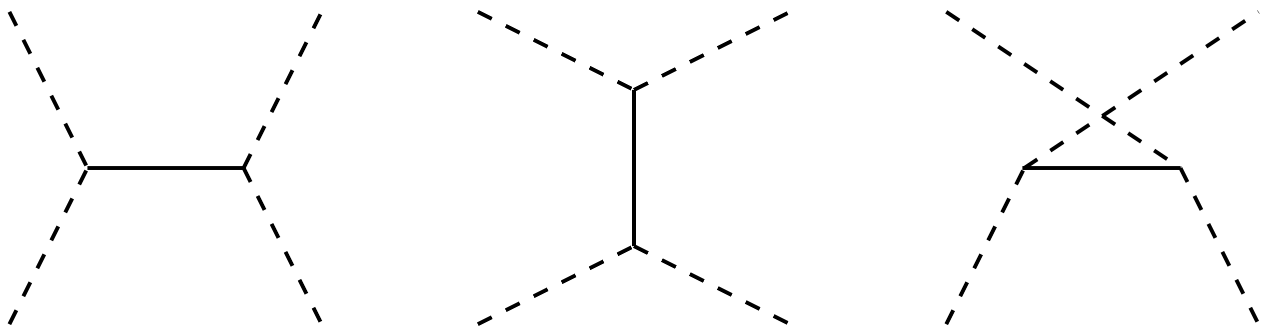
- Heavy degree of freedom is gone.
- A different interaction term has appeared.

Q: Do the two Lagrangians produce the same low-energy scattering amplitude for $\varphi(p_1) + \varphi(p_2) \to \varphi(p_3) + \varphi(p_4)$?

A: Yes!

- Calculation with original Lagrangian.

Dotted line: light particle; solid line: heavy particle.



Mandelstam variables

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2, \\ t &= (p_1 - p_3)^2 = (p_4 - p_2)^2, \\ u &= (p_1 - p_4)^2 = (p_3 - p_2)^2, \\ s + t + u &= 4m^2. \end{aligned}$$

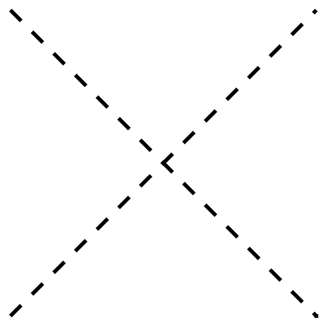
Condition: $\{s, |t|, |u|\} \ll M^2 = \Lambda^2. \qquad (*)$

Result:

$$\mathcal{M}_{\text{fund}} = (-i\lambda)^2 \left(\frac{i}{s - M^2 + i0^+} + \frac{i}{t - M^2 + i0^+} + \frac{i}{u - M^2 + i0^+} \right)$$

$$\overset{(*)}{=} \frac{3i\lambda^2}{M^2}\left[1+\mathcal{O}\left(\frac{\{s,t,u\}}{M^2}\right)\right].$$

- Effective theory: description in terms of contact interaction



$$\mathcal{M}_{\text{eff}} = \frac{i\lambda^2 4!}{8M^2} = \frac{3i\lambda^2}{M^2}.$$

Both calculations yield the same result!

EFT calculation simpler.

2. Partial decoupling

   The heavy fields do not disappear completely from the EFT but only their high-momentum modes.

Application: physics of heavy quarks

3. Spontaneous symmetry breaking

   The fundamental theory exhibits spontaneous symmetry breaking giving rise to massless Goldstone bosons. EFT written in terms of Goldstone boson degrees of freedom. A spontaneously broken symmetry relates processes with different numbers of Goldstone bosons. (Compare with Wigner-Eckart theorem, relating processes of the same type.) The corresponding effective Lagrangian is not renormalizable in the traditional sense.

   Example: chiral symmetry of QCD for N_f massless quark flavors.

Another classification of EFTs is related to the status of their coupling constants.

1. Coupling constants can be determined by matching the EFT with the underlying theory at short distances.

Example: physics of heavy quarks.

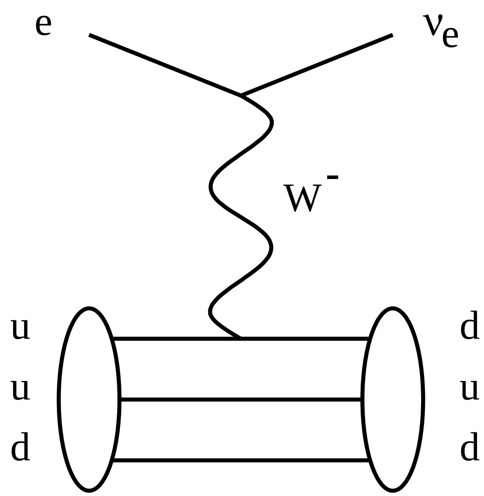
2. Coupling constants are constrained by symmetries only.

   - The underlying theory is not known. The coupling constants parameterize so-called new physics.
     Example: physics beyond the Standard Model.
   - The matching cannot be performed in perturbation theory even though the underlying theory is known.
     Example: QCD $\leftrightarrow$ ChPT.
     However, nonperturbative methods such as Lattice QCD start to predict LECs.

Example for two EFTs in operation

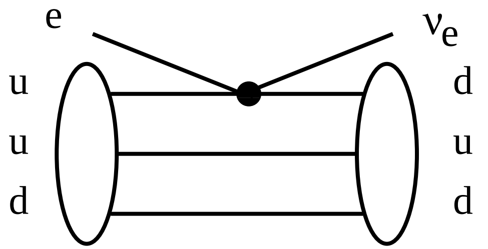
$$e^-(k_i) + p(p_i) \to \nu_e(k_f) + n(p_f)$$

At the "fundamental" level



$$
\begin{aligned}
q &= p_f - p_i, \\
\mathcal{M} &\sim \frac{1}{q^2 - M_W^2} f(q^2), \\
M_W &= (80.425 \pm 0.038)\,\mathrm{GeV}.
\end{aligned}
$$

Transition to $V-A$ description of the weak interaction for $|q^2| \ll M_W^2$:

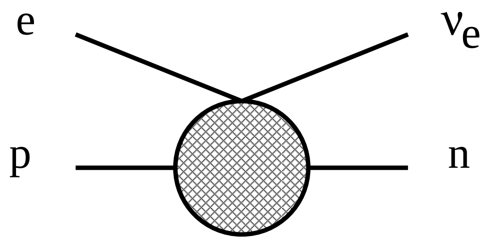


$$\sim \underbrace{\sqrt{2}}_{\text{Clebsch-Gordan coefficient}} \underbrace{\frac{g}{2\sqrt{2}}}_{\text{hadronic vertex}} \frac{1}{M_W^2} \underbrace{\frac{g}{2\sqrt{2}}}_{\text{leptonic vertex}}$$

$$= \frac{g^2}{4\sqrt{2}M_W^2} = G_F = 1.16639(1) \times 10^{-5}\text{GeV}^{-2}.$$

G_F: Fermi constant, $e = g\sin(\theta_W)$, θ_W: weak angle; $\sin^2(\theta_W) = 0.23149(15)$ (the numerical value depends on the definition, here effective angle).

Transition to effective field theory of the strong interaction (ChPT):



1.6 Aim of these lectures

Most general description of the strong interactions at low energies: $\pi\pi$, πN, etc., taking the spontaneous breakdown of chiral symmetry in QCD into account.

- Challenge: we need the
  1. the most general Lagrangian;
  2. a consistent power counting scheme to perform perturbative calculations.

Chapter 2

Lagrangian Formalism and Noether Theorem

2.1 Lagrangian Formalism of Fields

Q: Why Lagrangian formalism?
A: Weinberg, Vol. I, p 292

> ... it makes it easy to satisfy Lorentz invariance and other symmetries: a classical theory with a Lorentz-invariant Lagrangian will when canonically quantized lead to a Lorentz-invariant quantum theory.

Continuous systems: Introduce **fields** as dynamical variables.
Both time and space coordinates, $x^\mu = (t, \vec{x})$, are regarded as parameters.
Consider the Lagrangian (density) $\mathcal{L}$ of a scalar field, $\Phi(x) = \Phi(t, \vec{x})$,

$$\mathcal{L} = \mathcal{L}(\Phi(x), \partial_\mu \Phi(x)), \tag{2.1}$$

$$\partial_\mu \Phi = \frac{\partial \Phi}{\partial x^\mu} = \left(\frac{\partial \Phi}{\partial t}, \frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z} \right).$$

(Here: explicit dependence of $\mathcal{L}$ on x excluded.)
Lagrange function $L(t)$

$$L(t) = \int_{\mathbb{R}^3} \mathrm{d}^3 x \, \mathcal{L}(\Phi(x), \partial_\mu \Phi(x)). \tag{2.2}$$

Classical equation of motion (EOM) for $\Phi(x)$ from Hamilton's **princi-**

ple of critical action applied to

$$S[\Phi] = \underbrace{\int_{t_1}^{t_2} \mathrm{d}t \int_{\mathbb{R}^3}}_{\int_R \mathrm{d}^4x} \mathrm{d}^3x \, \mathcal{L}(\Phi(x), \partial_\mu \Phi(x)). \tag{2.3}$$

Define

$$\Phi_\epsilon(x) = \Phi(x) + \epsilon h(x), \tag{2.4}$$

where $h(x) = 0$ for $x \in \partial R$ (∂R boundary of R).
Let

$$F(\epsilon) = \int_R \mathrm{d}^4x \, \mathcal{L}(\Phi(x) + \epsilon h(x), \partial_\mu \Phi(x) + \epsilon \partial_\mu h(x)), \tag{2.5}$$

so that $F(0) = S[\Phi]$.
Expand F

$$F(\epsilon) \; = \; \int_R \mathrm{d}^4x \, \mathcal{L}(\Phi, \partial_\mu \Phi) + \epsilon \int_R \mathrm{d}^4x \left(h \frac{\partial \mathcal{L}}{\partial \Phi} + \partial_\mu h \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi} \right)$$

$$+O(\epsilon^2).$$

Principle of critical action

$$\delta S[\Phi] \equiv F'(0) \overset{!}{=} 0, \tag{2.6}$$

i.e.

$$0 = \int_R \mathrm{d}^4 x \left(h \frac{\partial \mathcal{L}}{\partial \Phi} + \partial_\mu h \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi} \right).$$

Apply product rule to second term

$$\partial_\mu h \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi} = \mathcal{D}_\mu \left(h \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi} \right) - h \mathcal{D}_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi},$$

where

$$\mathcal{D}_\mu \equiv \frac{\partial}{\partial x^\mu} + \partial_\mu \Phi \frac{\partial}{\partial \Phi} + \partial_\mu \partial_\nu \Phi \frac{\partial}{\partial \partial_\nu \Phi}.$$

Intermediate step

$$\begin{aligned}
\int_R \mathrm{d}^4x\, \mathcal{D}_\mu \left(h \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi} \right) &= \int_{\mathbb{R}^3} \mathrm{d}^3x \int_{t_1}^{t_2} \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t} \left(h \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} \right) \\
&\quad + \int_{t_1}^{t_2} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}z \int_{-\infty}^{\infty} \mathrm{d}x \frac{\mathrm{d}}{\mathrm{d}x} \left(h \frac{\partial \mathcal{L}}{\partial \frac{\partial \Phi}{\partial x}} \right) + \cdots \\
&= \int_{\mathbb{R}^3} \mathrm{d}^3x \underbrace{\left[h \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} \right]_{t_1}^{t_2}}_{0,\ \text{since } h(t_1, \vec{x}) = h(t_2, \vec{x}) = 0} \\
&\quad + \int_{t_1}^{t_2} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}z \underbrace{\left[h \frac{\partial \mathcal{L}}{\partial \frac{\partial \Phi}{\partial x}} \right]_{-\infty}^{\infty}}_{0,\ \text{since } h(t, \vec{x}) = 0 \text{ for } x \to \pm\infty} + \cdots \\
&= 0.
\end{aligned}$$

$\Rightarrow$

$$0 = \int_R \mathrm{d}^4x\, h(x) \left(\frac{\partial \mathcal{L}}{\partial \Phi} - \mathcal{D}_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi} \right). \tag{2.7}$$

Apply fundamental lemma of variational calculus:

> If the integral $\int_{x_1}^{x_2} \mathrm{d}x h(x) g(x)$ vanishes for arbitrary continuous functions $h(x)$ with continuous derivatives and $h(x_1) = h(x_2) = 0$, then $g(x) = 0$ in $[x_1, x_2]$.

$\Rightarrow$ **Euler-Lagrange equation (of motion) (EOM)**

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \mathcal{D}_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi} = 0, \tag{2.8}$$

or written out explicitly

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \Phi}{\partial t} \right)} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \Phi}{\partial x} \right)} - \cdots = 0.$$

- Remark: Almost every physics text writes ∂_μ instead of $\mathcal{D}_\mu$ with the understanding that differentiation with respect to both *explicit* and *implicit* x dependence is meant. From now on we follow this convention.

Generalization to n fields $\Phi_i(x)$ straightforward: perform n independent variations.
Define

$$\Phi_{i,\epsilon_i}(x) = \Phi_i(x) + \epsilon_i h_i(x), \quad i = 1, \cdots, n \tag{2.9}$$

and

$$F(\epsilon_1, \cdots, \epsilon_n) = S[\Phi_{i,\epsilon_i}]. \tag{2.10}$$

Demand critical value

$$\left.\frac{\partial F}{\partial \epsilon_i}\right|_{\vec{\epsilon}=0} = 0, \quad i = 1, \cdots, n, \tag{2.11}$$

$\Rightarrow n$ EOM

$$\frac{\partial \mathcal{L}}{\partial \Phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} = 0, \quad i = 1, \cdots, n. \tag{2.12}$$

2.2 Examples

1. Free scalar field Φ:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi \partial^\mu \Phi - m^2 \Phi^2) = \frac{1}{2}(g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - m^2 \Phi^2). \tag{2.13}$$

The metric tensor allows one to lower and raise indices:

$$\begin{aligned} g \; &= \; (g_{\mu\nu}) = (g^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\ g_{\mu\nu} \; &= \; g_{\nu\mu}, \end{aligned}$$

$$
\begin{aligned}
a_\mu &= g_{\mu\nu} a^\nu = g_{\nu\mu} a^\nu = g_\mu{}^\nu a_\nu = g^\nu{}_\mu a_\nu, \\
\frac{\partial \partial_\mu \Phi}{\partial \partial_\rho \Phi} &= g_\mu{}^\rho = g^\rho{}_\mu.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \Phi} &= -m^2 \Phi, \\
\frac{\partial \mathcal{L}}{\partial \partial_\rho \Phi} &= \frac{1}{2} g^{\mu\nu} (g_\mu{}^\rho \partial_\nu \Phi + \partial_\mu \Phi g_\nu{}^\rho) \\
&= \frac{1}{2} (g_\mu{}^\rho \partial^\mu \Phi + \partial^\nu \Phi g_\nu{}^\rho) \\
&= \frac{1}{2} (\partial^\rho \Phi + \partial^\rho \Phi) = \partial^\rho \Phi = f^\rho(x, \Phi, \partial_\mu \Phi),
\end{aligned}
$$

$$
\begin{aligned}
\partial_\rho \frac{\partial \mathcal{L}}{\partial \partial_\rho \Phi} &= \underbrace{\frac{\partial f^\rho}{\partial x^\rho}}_{0} + \partial_\rho \Phi \underbrace{\frac{\partial f^\rho}{\partial \Phi}}_{0} + \partial_\rho \partial_\sigma \Phi \quad \underbrace{\frac{\partial f^\rho}{\partial \partial_\sigma \Phi}}_{\frac{\partial \partial^\rho \Phi}{\partial \partial_\sigma \Phi} = g^{\rho\sigma}} \\
&= \Box \Phi.
\end{aligned}
$$

"Quick" way:

$$
\partial_\rho \frac{\partial \mathcal{L}}{\partial \partial_\rho \Phi} = \partial_\rho \partial^\rho \Phi = \Box \Phi.
$$

Klein-Gordon equation:

$$
(\Box + m^2)\Phi = 0. \tag{2.14}
$$

- Side note: Both approaches yield the same answer.

Consider general structure

$$
\Phi^m \partial_{\mu_1} \Phi \cdots \partial_{\mu_n} \Phi.
$$

"Quick" method including application of chain/product rule:

$$
\begin{aligned}
&\partial_\mu(\Phi^m \partial_{\mu_1}\Phi \cdots \partial_{\mu_n}\Phi) \\
&= \; m\Phi^{m-1}\partial_\mu\Phi\partial_{\mu_1}\Phi \cdots \partial_{\mu_n}\Phi \\
&\quad +\Phi^m \underbrace{(\partial_\mu\partial_{\mu_1}\Phi \cdots \partial_{\mu_n}\Phi + \cdots + \partial_{\mu_1}\Phi \cdots \partial_\mu\partial_{\mu_n}\Phi)}_{n \text{ terms}}.
\end{aligned}
$$

Second method:

$$
\begin{aligned}
&(\partial_\mu\phi\frac{\partial}{\partial\Phi} + \partial_\mu\partial_\nu\Phi\frac{\partial}{\partial\partial_\nu\Phi})(\Phi^m \partial_{\mu_1}\Phi \cdots \partial_{\mu_n}\Phi) \\
&= \; m\Phi^{m-1}\partial_\mu\Phi\partial_{\mu_1}\Phi \cdots \partial_{\mu_n}\Phi \\
&\quad + \underbrace{\Phi^m\partial_\mu\partial_\nu\Phi({g_{\mu_1}}^\nu \cdots \partial_{\mu_n}\Phi + \cdots + \partial_{\mu_1}\Phi \cdots {g_{\mu_n}}^\nu)}_{\Phi^m(\partial_\mu\partial_{\mu_1}\Phi \cdots \partial_{\mu_n}\Phi + \cdots + \partial_{\mu_1}\Phi \cdots \partial_\mu\partial_{\mu_n}\Phi)}.
\end{aligned}
$$

Solutions of the Klein-Gordon equation can be written as

$$\Phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega(\vec{k}\,)} \left[a(\vec{k}\,)e^{-ik\cdot x} + a^*(\vec{k}\,)e^{ik\cdot x}\right],$$

where

$$k \cdot x = k_0 x_0 - \vec{k} \cdot \vec{x}, \quad k_0 = \omega(\vec{k}\,) = \sqrt{m^2 + \vec{k}^{\,2}}.$$

2. Free Dirac field Ψ with mass m:

$$\mathcal{L} = \bar{\Psi} i \gamma^\mu \partial_\mu \Psi - m \bar{\Psi}\Psi. \tag{2.15}$$

Recall:

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix},$$

where Ψ_i continuously differentiable complex functions.

We will use so-called standard or Dirac representation of gamma matrices

$$\gamma^0 = \gamma_0 = \begin{pmatrix} 1_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & -1_{2\times2} \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0_{2\times2} & \vec{\sigma} \\ -\vec{\sigma} & 0_{2\times2} \end{pmatrix},$$

σ_i: Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (2.16)$$

Notation

$$\bar{\Psi} = \Psi^\dagger \gamma_0 = (\Psi_1^* \; \Psi_2^* \; -\Psi_3^* \; -\Psi_4^*).$$

EOM:

$$\frac{\partial \mathcal{L}}{\partial \Psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Psi} = -m\bar{\Psi} - \partial_\mu \bar{\Psi} i \gamma^\mu = 0.$$

Take adjoint, make use of $\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$, and multiply by γ^0. $\Rightarrow$

Dirac equation:

$$\gamma^0(-m \underbrace{\gamma^{0\dagger}}_{\underbrace{\gamma^0\gamma^0}_{\mathbb{1}}\gamma^0} \Psi + i\gamma^0\gamma^\mu \underbrace{\gamma^0 \underbrace{\gamma^{0\dagger}}_{\gamma^0}}_{\mathbb{1}} \partial_\mu\Psi) = (i\partial\!\!\!/ - m)\Psi = 0. \quad (2.17)$$

Feynman slash: $a\!\!\!/ = a_\mu\gamma^\mu$.

Solutions of the Dirac equation are of the type

$$u^{(r)}(\vec{p})e^{-ip\cdot x}, \quad r = 1, 2, \quad \text{(so-called positive-energy solutions)}$$
$$v^{(r)}(\vec{p})e^{ip\cdot x}, \quad r = 1, 2, \quad \text{(so-called negative-energy solutions)}$$

where $\vec{p} \in \mathbb{R}^3$ and $p_0 = E(\vec{p}) = \sqrt{m^2 + \vec{p}^2}$.

Properties of the Dirac spinors:

$$\begin{aligned} (p\!\!\!/ - m)u^{(r)}(\vec{p}) &= 0, \\ (p\!\!\!/ + m)v^{(r)}(\vec{p}) &= 0, \\ \bar{u}^{(r)}(\vec{p})(p\!\!\!/ - m) &= 0, \end{aligned}$$

$$
\begin{aligned}
\bar{v}^{(r)}(\vec{p})(\not{p}+m) &= 0, \\
\bar{u}^{(r)}(\vec{p})u^{(s)}(\vec{p}) &= -\bar{v}^{(r)}(\vec{p})v^{(s)}(\vec{p}) = 2m\delta_{rs}, \\
u^{(r)\dagger}(\vec{p})u^{(s)}(\vec{p}) &= v^{(r)\dagger}(\vec{p})v^{(s)}(\vec{p}) = 2E(\vec{p})\delta_{rs}, \\
u^{(r)\dagger}(\vec{p})v^{(s)}(-\vec{p}) &= 0, \\
\sum_{r=1}^{2} u_{\alpha}^{(r)}(\vec{p})\bar{u}_{\beta}^{(r)}(\vec{p}) &= (\not{p}+m)_{\alpha\beta}, \\
\sum_{r=1}^{2} v_{\alpha}^{(r)}(\vec{p})\bar{v}_{\beta}^{(r)}(\vec{p}) &= (\not{p}-m)_{\alpha\beta}, \\
\sum_{r=1}^{2}[u_{\alpha}^{(r)}(\vec{p})\bar{u}_{\beta}^{(r)}(\vec{p}) - v_{\alpha}^{(r)}(\vec{p})\bar{v}_{\beta}^{(r)}(\vec{p})] &= 2m\delta_{\alpha\beta}.
\end{aligned}
$$

Explicit representation

$$u^{(r)}(\vec{p}) \;=\; \sqrt{E(\vec{p})+m}\begin{pmatrix} \chi_r \\ \frac{\vec{\sigma}\cdot\vec{p}}{E(\vec{p})+m}\chi_r \end{pmatrix},$$
$$v^{(r)}(\vec{p}) \;=\; \sqrt{E(\vec{p})+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E(\vec{p})+m}\chi_r \\ \chi_r \end{pmatrix},$$

where

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Solutions to the Dirac equation can be written as

$$\Psi(x) = \sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^3 2E(\vec{p})} \left[b_r(\vec{p}) u^{(r)}(\vec{p}) e^{-ip\cdot x} + d_r^*(\vec{p}) v^{(r)}(\vec{p}) e^{ip\cdot x} \right].$$

3. Pseudoscalar pion-nucleon interaction:

Define isospin doublet

$$\Psi = \begin{pmatrix} p \\ n \end{pmatrix}$$

(p and n are both four-component Dirac fields) and isospin triplet of real pseudoscalar fields

$$\vec{\Phi} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix}.$$

Terminology pseudoscalar refers to behavior under parity:

$$\Phi_i(t, \vec{x}) \mapsto -\Phi_i(t, -\vec{x})$$

Fields corresponding to physical (charged) states

$$\pi^+ = \frac{1}{\sqrt{2}}(\Phi_1 - i\Phi_2), \quad \pi^- = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2), \quad \pi^0 = \Phi_3. \quad (2.18)$$

Using the Pauli matrices τ_i (for isospin)

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (2.19)$$

$\Rightarrow$

$$\vec{\tau} \cdot \vec{\Phi} = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}.$$

Note

$$(\vec{\tau} \cdot \vec{\Phi})^\dagger = \vec{\tau} \cdot \vec{\Phi},$$

because τ_i Hermitian and Φ_i real.

Define $a \overleftrightarrow{\partial}_\mu b \equiv a\partial_\mu b - (\partial_\mu a)b$ and consider Lagrangian

$$\mathcal{L} = \bar{\Psi} \left(\frac{i}{2} \overleftrightarrow{\not{\partial}} - m_N \right) \Psi + \frac{1}{2} \left(\partial_\mu \vec{\Phi} \cdot \partial^\mu \vec{\Phi} - M_\pi^2 \vec{\Phi}^2 \right) - ig\bar{\Psi}\gamma_5 \vec{\tau} \cdot \vec{\Phi}\Psi, \qquad (2.20)$$

where

$$\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0_{2\times 2} & 1_{2\times 2} \\ 1_{2\times 2} & 0_{2\times 2} \end{pmatrix}, \quad \{\gamma_5, \gamma^\mu\} = 0.$$

Parameters:

$$\begin{aligned} & m_N = m_p = m_n, \quad m_p = 938.3\,\mathrm{MeV}, \quad m_n = 939.6\,\mathrm{MeV}, \\ & M_\pi = M_{\pi^\pm} = M_{\pi^0}, \quad M_{\pi^\pm} = 139.6\,\mathrm{MeV}, \quad M_{\pi^0} = 135.0\,\mathrm{MeV}, \\ & g = g_{\pi N}, \quad g_{\pi N} = 13.2. \end{aligned}$$

Introduce components of the nucleon field

$$\Psi_{f,\alpha}$$

$f = 1, 2$: isospin index (p, n),
$\alpha = 1, 2, 3, 4$: bispinor or Dirac index.

Eq. (2.20) is compact version of

$$
\begin{aligned}
\mathcal{L} \; = \; & \frac{1}{2} \sum_{f',f=1}^{2} \sum_{\alpha',\alpha=1}^{4} i(\bar{\Psi}_{f',\alpha'} {\gamma^{\mu}}_{\alpha'\alpha} \mathbb{1}_{f'f} \partial_{\mu} \Psi_{f,\alpha} - \partial_{\mu} \bar{\Psi}_{f',\alpha'} {\gamma^{\mu}}_{\alpha'\alpha} \mathbb{1}_{f'f} \Psi_{f,\alpha}) \\
& - m_N \sum_{f',f=1}^{2} \sum_{\alpha',\alpha=1}^{4} \bar{\Psi}_{f',\alpha'} \mathbb{1}_{\alpha'\alpha} \mathbb{1}_{f'f} \Psi_{f,\alpha} \\
& + \frac{1}{2} \sum_{i=1}^{3} \left(\partial_{\mu} \Phi_i \partial^{\mu} \Phi_i - M_{\pi}^2 \Phi_i^2 \right) \\
& - ig \sum_{f',f=1}^{2} \sum_{\alpha',\alpha=1}^{4} \sum_{i=1}^{3} \bar{\Psi}_{f',\alpha'} {\gamma_5}_{\alpha'\alpha} {\tau_i}_{f'f} \Psi_{f,\alpha} \Phi_i .
\end{aligned}
$$

Remarks:

- Unit matrices (in isospin and Dirac space) are usually omitted.
- Matrices operating in different spaces commute.

Example: $\tau_i\gamma_5 = \gamma_5\tau_i$.

EOM

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\bar{\Psi}} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\bar{\Psi}} &= \frac{i}{2}\not{\partial}\Psi - m_N\Psi - ig\gamma_5\vec{\tau}\cdot\vec{\Phi}\Psi - \partial_\mu\left(-\frac{i}{2}\gamma^\mu\Psi\right) = 0, \\
\Rightarrow \quad (i\not{\partial} - m_N)\Psi &= ig\gamma_5\vec{\tau}\cdot\vec{\Phi}\Psi, \qquad (2.21) \\
\frac{\partial\mathcal{L}}{\partial\vec{\Phi}} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\vec{\Phi}} &= -M_\pi^2\vec{\Phi} - ig\bar{\Psi}\gamma_5\vec{\tau}\Psi - \Box\vec{\Phi} = 0, \\
\Rightarrow \quad (\Box + M_\pi^2)\vec{\Phi} &= -ig\bar{\Psi}\gamma_5\vec{\tau}\Psi. \qquad (2.22)
\end{aligned}$$

Remarks:

- Set of coupled partial differential equations.
- Lagrangians of interacting systems are of the type

$$\mathcal{L} = \sum_i \mathcal{L}_{i,\text{free}} + \mathcal{L}_{\text{int}}.$$

Correspondingly, EOM are of the type:

free		including interaction
$(\Box + M_\pi^2)\Phi_i = 0$	$\rightarrow$	$(\Box + M_\pi^2)\Phi_i =$ "source"
$(i\partial\!\!\!/ - m_N)\Psi = 0$	$\rightarrow$	$(i\partial\!\!\!/ - m_N)\Psi =$ "source′"

Compare with electrostatics

$$\Delta\phi = -\rho.$$

4. * Interaction of electromagnetic field with external four-current (density) J^μ:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu, \tag{2.23}$$

where

$$(F^{\mu\nu}) = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} . \qquad (2.24)$$

Symbolically

$$(T^{\mu\nu}) \;=\; \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right), \quad (T^\mu{}_\nu) = (T^{\mu\rho} g_{\rho\nu}) = \left(\begin{array}{c|c} a & -b \\ \hline c & -d \end{array}\right),$$

$$(T_\mu{}^\nu) \;=\; \left(\begin{array}{c|c} a & b \\ \hline -c & -d \end{array}\right), \quad (T_{\mu\nu}) = \left(\begin{array}{c|c} a & -b \\ \hline -c & d \end{array}\right),$$

and thus

$$(F_{\mu\nu}) = \left(\begin{array}{r|rrr} 0 & E_x & E_y & E_z \\ \hline -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{array}\right).$$

In words: $F_{\mu\nu}$ results from $F^{\mu\nu}$ by the substitutions $\vec{E} \to -\vec{E}$ and $\vec{B} \to \vec{B}$.

Recall:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\Phi, \quad \vec{B} = \vec{\nabla} \times \vec{A}. \tag{2.25}$$

Using

$$\begin{aligned} \frac{\partial F_{\mu\nu}}{\partial \partial_\sigma A_\rho} &= \frac{\partial}{\partial \partial_\sigma A_\rho} (\partial_\mu A_\nu - \partial_\nu A_\mu) = g_\mu{}^\sigma g_\nu{}^\rho - g_\nu{}^\sigma g_\mu{}^\rho, \\ \frac{\partial}{\partial \partial_\sigma A_\rho}(F_{\mu\nu} F^{\mu\nu}) &= 2(g_\mu{}^\sigma g_\nu{}^\rho - g_\nu{}^\sigma g_\mu{}^\rho) F^{\mu\nu} = 4F^{\sigma\rho} \end{aligned}$$

we obtain

$$\frac{\partial \mathcal{L}}{\partial A_\rho} - \partial_\sigma \frac{\partial \mathcal{L}}{\partial \partial_\sigma A_\rho} = -J^\rho + \partial_\sigma F^{\sigma\rho} = 0.$$

Covariant version of the inhomogeneous Maxwell equations

$$\partial_\sigma F^{\sigma\rho} = J^\rho. \tag{2.26}$$

Corresponds to

$$\vec{\nabla} \cdot \vec{E} = \rho, \quad \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J}. \tag{2.27}$$

- Q: Where are the homogeneous equations?

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0. \tag{2.28}$$

endequation

- A: Because of Eq. (2.25) automatically satisfied.

Introducing dual tensor

$$\tilde{F}^{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \tag{2.29}$$

where $\epsilon_{\mu\nu\rho\sigma}$ denotes the totally antisymmetric Levi-Civita tensor,

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \{\mu,\nu,\rho,\sigma\} \text{ is an even permutation of } \{0,1,2,3\} \\ -1 & \text{if } \{\mu,\nu,\rho,\sigma\} \text{ is an odd permutation of } \{0,1,2,3\} \\ 0 & \text{otherwise} \end{cases} ,$$

$\epsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}$, one finds

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}, \tag{2.30}$$

i.e., one obtains $\tilde{F}^{\mu\nu}$ from $F^{\mu\nu}$ via the substitutions $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$ in Eq. (2.24). (e.g., $\tilde{F}^{01} = -\frac{1}{2}((-1)F_{23} + F_{32}) = F_{23} = -B_x$).

Homogeneous equations are automatically satisfied:

$$\partial_\mu \tilde{F}^{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu(\partial_\rho A_\sigma - \partial_\sigma A_\rho) = -\epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\rho A_\sigma = 0,$$

since $\epsilon^{\mu\nu\rho\sigma}$ is antisymmetric under $\mu \leftrightarrow \rho$ and $\partial_\mu \partial_\rho A_\sigma$ is symmetric under $\mu \leftrightarrow \rho$.

5. Free spin-1 field of mass m:

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu V_\nu - \partial_\nu V_\mu, && (2.31) \\ \mathcal{L} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 V_\mu V^\mu. && (2.32) \end{aligned}$$

Note sign of mass term!

$$\frac{\partial \mathcal{L}}{\partial V_\rho} - \partial_\sigma \frac{\partial \mathcal{L}}{\partial \partial_\sigma V_\rho} = m^2 V^\rho + \partial_\sigma F^{\sigma\rho} = 0. \tag{2.33}$$

Take ∂_ρ of Eq. (2.33). Antisymmetry of $F^{\sigma\rho}$ under $\sigma \leftrightarrow \rho$. $\Rightarrow$

$$m^2 \partial_\rho V^\rho = 0,$$

so that, for $m^2 \neq 0$, we obtain an additional condition

$$\partial_\rho V^\rho = 0. \tag{2.34}$$

Insert into EOM, Eq. (2.33). $\Rightarrow$

$$(\Box + m^2)V^\rho \;=\; 0, \qquad (2.35)$$

$$\partial_\rho V^\rho \;=\; 0. \qquad (2.36)$$

Second equation yields condition among the four components of V^ρ.
$\Rightarrow$ Three independent degrees of freedom (spin-1 object).

Solutions of Proca equation can be decomposed into plane waves as

$$V^\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\omega(\vec{k}\,)} \sum_{r=1}^{3} \epsilon_r^\mu(\vec{k}\,) a_r(\vec{k}\,) e^{-ik\cdot x} \quad + H.c.,$$

where $k_0 = \omega(\vec{k}\,) = \sqrt{m^2 + \vec{k}^{\,2}}$.

Polarization vectors $\epsilon_r(\vec{k}\,)$ satisfy, for any $\vec{k}$,

$$\epsilon_r(\vec{k}) \cdot \epsilon_s(\vec{k}\,) = -\delta_{rs}.$$

Moreover, $\partial_\mu V^\mu(x) = 0$ implies

$$k_\mu \epsilon_r^\mu(\vec{k}\,) = 0.$$

Example:

$$
\begin{aligned}
k &= (\omega(\vec{k}\,), 0, 0, |\vec{k}|), \\
\epsilon_1(\vec{k}\,) &= (0, 1, 0, 0), \\
\epsilon_2(\vec{k}\,) &= (0, 0, 1, 0), \\
\epsilon_3(\vec{k}\,) &= (|\vec{k}|, 0, 0, \omega(\vec{k}\,))/m.
\end{aligned}
$$

"Completeness" relation (for any $\vec{k}$)

$$\sum_{r=1}^{3} \epsilon_r^\mu(\vec{k}\,)\epsilon_r^\nu(\vec{k}\,) = -g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}.$$

2.3 Effective Lagrangians Involving Higher-Derivative Terms

Lagrangians discussed so far contained fields and their first partial derivatives.

EFT: Most general Lagrangian containing arbitrarily high partial derivatives.

Symbolically

$$\mathcal{L}(\Phi, \partial_\mu \Phi, \partial_\mu \partial_\nu \Phi, \cdots). \tag{2.37}$$

Define

$$S[\Phi] = \int_R \mathrm{d}^4 x \, \mathcal{L}(\Phi, \partial_\mu \Phi, \partial_\mu \partial_\nu \Phi, \cdots) \tag{2.38}$$

and consider test functions $\Phi_\epsilon(x) = \Phi(x) + \epsilon h(x)$, where $h(x) = 0$, $\partial_\mu h(x) = 0$, $\cdots$, for $x \in \partial R$. Define

$$\begin{aligned} F(\epsilon) &= \int_R \mathrm{d}^4 x \, \mathcal{L}(\Phi + \epsilon h, \partial_\mu \Phi + \epsilon \partial_\mu h, \partial_\mu \partial_\nu \Phi + \epsilon \partial_\mu \partial_\nu h, \cdots) \\ &= F(0) + \epsilon \int_R \mathrm{d}^4 x \left[\frac{\partial \mathcal{L}}{\partial \Phi} h + \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi} \partial_\mu h + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \Phi} \partial_\mu \partial_\nu h + \cdots \right] \\ &\quad + O(\epsilon^2). \end{aligned} \tag{2.39}$$

Apply Hamilton's principle

$$0 \stackrel{!}{=} F'(0) = \int_R \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \Phi} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi}\partial_\mu h + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \Phi}\partial_\mu \partial_\nu h + \cdots\right]. \quad (2.40)$$

Consider, e.g., third term on right-hand side of Eq. (2.40):

$$\begin{aligned} \int_R \mathrm{d}^4x \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \Phi}\partial_\mu \partial_\nu h &= \underbrace{\int_R \mathrm{d}^4x\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \Phi}\partial_\nu h\right)}_{0\text{, since } \partial_\nu h(\partial R) = 0} - \int_R \mathrm{d}^4x \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \Phi}\right)\partial_\nu h \\ &= -\underbrace{\int_R \mathrm{d}^4x\, \partial_\nu \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \Phi} h\right)}_{0\text{, since } h(\partial R) = 0} + \int_R \mathrm{d}^4x\, h \partial_\nu \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \Phi}. \end{aligned}$$

Systematics as follows: Using partial integration move derivatives off h, make use of boundary condition.

$\Rightarrow$ Final result is of the type

$$\int_R \mathrm{d}^4x\, h[\quad] = 0.$$

Apply fundamental lemma of variational calculus.
$\Rightarrow$ (generalized) EOM

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi} + \partial_\nu \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \Phi} - \cdots = 0. \tag{2.41}$$

Alternating signs originate from odd or even number of partial integrations, respectively.

- Toy model:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \Phi_1 \partial^\mu \Phi_1 - m_1^2 \Phi_1^2\right) + \frac{1}{2}\left(\partial_\mu \Phi_2 \partial^\mu \Phi_2 - m_2^2 \Phi_2^2\right) - g(\Box \Phi_1)^2 \Phi_2^2. \tag{2.42}$$

- Remark on dimension of coupling constant g:
  $\dim[\mathcal{L}] = E^4$, $\dim[\Phi_i] = E$, $\dim[\partial_\mu] = E$. $\Rightarrow \dim[g] = 1/E^4$.
  (The quantized theory is not renormalizable in the usual sense.)
- EOM

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\Phi_1} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_1} + \partial_\nu\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\Phi_1} &= -m_1^2\Phi_1 - \Box\Phi_1 - g\Box(2\Box\Phi_1\Phi_2^2) = 0, \\
\Rightarrow \quad (\Box + m_1^2)\Phi_1 &= -2g\Box(\Box\Phi_1\Phi_2^2), \qquad (2.43) \\
\frac{\partial\mathcal{L}}{\partial\Phi_2} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_2} &= -m_2^2\Phi_2 - \Box\Phi_2 - 2g(\Box\Phi_1)^2\Phi_2 = 0, \\
\Rightarrow \quad (\Box + m_2^2)\Phi_2 &= -2g(\Box\Phi_1)^2\Phi_2. \qquad (2.44)
\end{aligned}$$

2.4 Noether Theorem

Continuous symmetries $\leftrightarrow$ conserved quantities

Consider Lagrangian $\mathcal{L}$ depending on n independent fields Φ_i and their first partial derivatives.
Extension to higher-order derivatives is also possible.
Typically $n \geq 2$ for bosons, and $n \geq 1$ for fermions, e.g. U(1).

$$\mathcal{L} = \mathcal{L}(\Phi_i, \partial_\mu \Phi_i). \tag{2.45}$$

$\Rightarrow n$ EOM

$$\frac{\partial \mathcal{L}}{\partial \Phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} = 0, \quad i = 1, \cdots, n. \tag{2.46}$$

Consider infinitesimal transformations which depend on r real local parameters $\epsilon_a(x)$ (method of Gell-Mann and Lévy, Nuovo Cim. **16**, 705

(1960))

$$\Phi_i(x) \mapsto \Phi_i'(x) = \Phi_i(x) + \delta\Phi_i(x) = \Phi_i(x) - i\epsilon_a(x) F_i^a[\Phi_j(x)]. \quad (2.47)$$

Variation of the Lagrangian

$$\delta\mathcal{L} = \mathcal{L}(\Phi_i', \partial_\mu \Phi_i') - \mathcal{L}(\Phi_i, \partial_\mu \Phi_i) = \frac{\partial \mathcal{L}}{\partial \Phi_i}\delta\Phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i}\partial_\mu \delta\Phi_i$$

$$\partial_\mu \delta\Phi_i = -i[\partial_\mu \epsilon_a(x)] F_i^a - i\epsilon_a(x)\partial_\mu F_i^a,$$

$$\begin{aligned} \cdots &= \epsilon_a(x)\left(-i\frac{\partial \mathcal{L}}{\partial \Phi_i} F_i^a - i\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i}\partial_\mu F_i^a\right) + \partial_\mu \epsilon_a(x)\left(-i\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} F_i^a\right) \\ &\equiv \epsilon_a(x)\partial_\mu J^{\mu,a} + \partial_\mu \epsilon_a(x) J^{\mu,a}. \end{aligned} \quad (2.48)$$

Define for each infinitesimal transformation a four-current density as

$$J^{\mu,a} = -i\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} F_i^a. \quad (2.49)$$

Consistency for solutions of the EOM

$$\partial_\mu J^{\mu,a} = -i\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i}\right) F_i^a - i\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i}\partial_\mu F_i^a = -i\frac{\partial \mathcal{L}}{\partial \Phi_i}F_i^a - i\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i}\partial_\mu F_i^a.$$

Currents and divergences of currents from variation

$$J^{\mu,a} = \frac{\partial \delta \mathcal{L}}{\partial \partial_\mu \epsilon_a}, \tag{2.50}$$

$$\partial_\mu J^{\mu,a} = \frac{\partial \delta \mathcal{L}}{\partial \epsilon_a}. \tag{2.51}$$

Assume Lagrangian to be invariant under a **global** transformation:

$$\delta \mathcal{L} = 0 \quad \wedge \quad \partial_\mu \epsilon_a(x) J^{\mu,a} = 0.$$

$\Rightarrow$ Current $J^{\mu,a}$ is conserved

$$\partial_\mu J^{\mu,a} = 0.$$

Charge

$$Q^a(t) = \int \mathrm{d}^3 x J^{0,a}(t, \vec{x}) \tag{2.52}$$

is time independent, i.e., a constant of the motion:

$$\frac{\mathrm{d}Q^a(t)}{\mathrm{d}t} = \int \mathrm{d}^3x \, \frac{\partial J^{0,a}(t,\vec{x})}{\partial t}$$

Make use of

$$\int \mathrm{d}^3x \vec{\nabla} \cdot \vec{J}^a = \int \mathrm{d}\vec{F} \cdot \vec{J}^a = \lim_{R\to\infty} R^2 \int \mathrm{d}\Omega \hat{e}_r \cdot \vec{J}^a = 0.$$

Current density $\vec{J}^a(t,\vec{x})$ must fall off faster than $1/r^2$ as $r = |\vec{x}| \to \infty$. Usually the case, except in the presence of massless "charged" particles [see J. Bernstein, Rev. Mod. Phys. **46**, 1 (1974)].

$$\begin{aligned} &= \int \mathrm{d}^3x \left(\frac{\partial J^{0,a}(t,\vec{x})}{\partial t} + \vec{\nabla} \cdot \vec{J}^a(t,\vec{x}) \right) = \int \mathrm{d}^3x \, \partial_\mu J^{\mu,a}(t,\vec{x}) \\ &= \int \mathrm{d}^3x \frac{\partial \delta\mathcal{L}}{\partial \epsilon_a} = 0 \quad \text{for} \quad \delta\mathcal{L} = 0. \end{aligned} \tag{2.53}$$

- Remark: So far **classical** fields. Charge Q^a is not yet quantized and can have any continuous value.

Different versions of classical conservation laws (Weinberg, Vol. 1, chapter 7.3)

$\Phi \mapsto \Phi + \delta\Phi = \Phi + \epsilon\delta\tilde{\Phi}$

Invariant quantity	conservation law	current density or charge
$\delta\mathcal{L} = 0$	$\partial_\mu J^\mu = 0$	$J^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi}\delta\tilde{\Phi}$
$\delta\mathcal{L} = \epsilon\partial_\mu\mathcal{J}^\mu$	$\partial_\mu J^\mu = 0$	$J^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi}\delta\tilde{\Phi} - \mathcal{J}^\mu$
$\delta L = 0$	$\frac{dQ(t)}{dt} = 0$	$Q = \int \mathrm{d}^3x\frac{\partial\mathcal{L}}{\partial\partial_0\Phi}\delta\tilde{\Phi}$
$\delta L = \epsilon\frac{d\mathcal{Q}(t)}{dt}$	$\frac{dQ(t)}{dt} = 0$	$Q = \int \mathrm{d}^3x\frac{\partial\mathcal{L}}{\partial\partial_0\Phi}\delta\tilde{\Phi} - \mathcal{Q}$
$\delta S = 0$	$\partial_\mu J^\mu = 0$	explicit form of J^μ not known

• Example from Assignment 1:

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu \Phi_1 \partial^\mu \Phi_1 + \partial_\mu \Phi_2 \partial^\mu \Phi_2 - m^2(\Phi_1^2 + \Phi_2^2)\right] - \frac{\lambda}{4}\left(\Phi_1^2 + \Phi_2^2\right)^2 . \tag{2.54}$$

Perform infinitesimal, active rotation of the fields by angle $\epsilon(x)$ (1 local parameter) (After replacing the fields by field operators, the corresponding Hilbert space states must be transformed oppositely.)

$$\Phi_1' = \Phi_1 + \delta\Phi_1 = \Phi_1 - \epsilon(x)\Phi_2, \quad \Phi_2' = \Phi_2 + \delta\Phi_2 = \Phi_2 + \epsilon(x)\Phi_1. \tag{2.55}$$

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \Phi_i}\delta\Phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i}\partial_\mu \delta\Phi_i \\ &= \underbrace{-m^2\Phi_1[-\epsilon(x)]\Phi_2 - m^2\Phi_2\epsilon(x)\Phi_1}_{0} \\ &\quad - \underbrace{\lambda(\Phi_1^2 + \Phi_2^2)\{\Phi_1[-\epsilon(x)]\Phi_2 + \Phi_2\epsilon(x)\Phi_1\}}_{0} \end{aligned}$$

$$
\begin{aligned}
&+\partial^\mu \Phi_1 \partial_\mu[-\epsilon(x)\Phi_2] + \partial^\mu \Phi_2 \partial_\mu[\epsilon(x)\Phi_1] \\
&= \partial_\mu \epsilon(x)(-\partial^\mu \Phi_1 \Phi_2 + \Phi_1 \partial^\mu \Phi_2). \qquad (2.56)
\end{aligned}
$$

$$
\Rightarrow \quad J^\mu = \frac{\partial \delta \mathcal{L}}{\partial \partial_\mu \epsilon} = \Phi_1 \partial^\mu \Phi_2 - \partial^\mu \Phi_1 \Phi_2, \quad \partial_\mu J^\mu = \frac{\partial \delta \mathcal{L}}{\partial \epsilon} = 0. \qquad (2.57)
$$

- Conclusion: Lagrangian of Eq. (2.54) is invariant under global rotation of fields Φ_1 and Φ_2. Underlying symmetry group is $\mathrm{O}(2) = \mathrm{SO}(2) \cup S\mathrm{SO}(2)$, where

  $$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

  Elements of SO(2) may be described by a continuous parameter $0 \leq \varphi < 2\pi$:

  $$R(\varphi) = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}.$$

  Conserved current of Eq. (2.57) is associated with this invariance.

Remark: The groups SO(2) and U(1) are isomorphic.

2.5 Canonical Quantization of a Scalar Field

Literature

- Itzykson and Zuber, chapter 3.1
- Ryder, chapter 4.1

So far: Free scalar field as classical system.

Quantum-mechanical interpretation of Klein-Gordon equation as relativistic single-particle equation faces two difficulties:

1. Klein-Gordon equation allows for solutions of "negative energy," if one interprets $i\partial^\mu$ as operator corresponding to four-momentum.
2. It is not possible to define a positive definite probability density ρ.

Both problems disappear in quantum field theory.

Hamiltonian formulation

Introduce generalized momentum Π conjugate to field Φ

$$p = \frac{\partial L}{\partial \dot{q}} \quad \rightarrow \quad \Pi = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} \tag{2.58}$$

and Hamiltonian density

$$H = p\dot{q} - L \quad \rightarrow \quad \mathcal{H} = \Pi\dot{\Phi} - \mathcal{L}. \tag{2.59}$$

Hamilton function

$$H = \int \mathrm{d}^3x\, \mathcal{H}. \tag{2.60}$$

Free scalar field

$$\begin{aligned} \Pi &= \dot{\Phi}, \\ \mathcal{H} &= \dot{\Phi}^2 - \frac{1}{2}\left(\dot{\Phi}\dot{\Phi} - \vec{\nabla}\Phi \cdot \vec{\nabla}\Phi - m^2\Phi^2\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\Pi^2 + \vec{\nabla}\Phi \cdot \vec{\nabla}\Phi + m^2\Phi^2\right), \\
H &= \frac{1}{2}\int \mathrm{d}^3x \left(\Pi^2 + \vec{\nabla}\Phi \cdot \vec{\nabla}\Phi + m^2\Phi^2\right). \qquad (2.61)
\end{aligned}
$$

Note: $H(t) \geq 0$ (integrand always positive), i.e., problem of "negative energies" does not show up in classical field theory.

Canonical quantization

Interpret $\Phi(t, \vec{x})$ as Hermitian operator in analogy to transition from classical mechanics to quantum mechanics.
Corresponding Hilbert space still needs to be identified (see below).
Operator Φ plays role similar to that of position operator q of nonrelativistic quantum mechanics.
Interpret $\vec{x}$ as some sort of parameter, resulting in (uncountably) infinite number of degrees of freedom, i.e., at each $\vec{x}$ one has a dynamical degree

of freedom $\Phi(\vec{x})$ (which is function of t).

Visualization of quantization procedure

1. Divide three-dimensional space into cells of volume δV.
2. Denote each cell by a triplet $\vec{r}$ of integers (see Fig. 2.1).

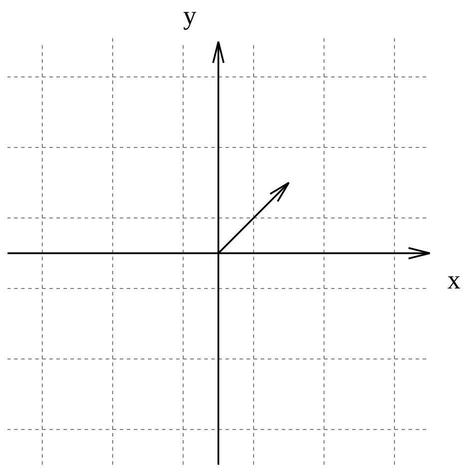


Figure 2.1: Two-dimensional illustration of the cells.

3. $\Phi_{\vec{r}}$: average value of Φ in cell $\vec{r}$ (function of t).

4. $\mathcal{L}_{\vec{r}}$: average of Lagrangian density in cell $\vec{r}$.

5. Lagrange function reads

$$L = \sum_{\vec{r}} L_{\vec{r}} = \sum_{\vec{r}} \delta V \mathcal{L}_{\vec{r}} \overset{\delta V \to 0}{\to} \int \mathrm{d}^3 x \, \mathcal{L}. \tag{2.62}$$

6. Define momentum $p_{\vec{r}}$ conjugate to $\Phi_{\vec{r}}$ as

$$p_{\vec{r}} = \frac{\partial L}{\partial \dot{\Phi}_{\vec{r}}} = \frac{\delta V \partial \mathcal{L}_{\vec{r}}}{\partial \dot{\Phi}_{\vec{r}}} \equiv \delta V \Pi_{\vec{r}}, \tag{2.63}$$

with continuum limit Eq. (2.58).

7. Quantization rule: Consider $\Phi_{\vec{r}}$ and $p_{\vec{r}}$ as operators in the Heisenberg picture

Recall

$$O_H(t) = e^{iHt} O_S e^{-iHt}$$

$$\begin{aligned}
[q_i, p_j] = i\delta_{ij} = [q_{Hi}(t), p_{Hj}(t)] \quad &\rightarrow \quad [\Phi_{\vec{r}}(t), p_{\vec{s}}(t)] = i\delta_{\vec{r}\vec{s}}, \\
[q_{Hi}(t), q_{Hj}(t)] = 0 \quad &\rightarrow \quad [\Phi_{\vec{r}}(t), \Phi_{\vec{s}}(t)] = 0, \\
[p_{Hi}(t), p_{Hj(t)}] = 0 \quad &\rightarrow \quad [p_{\vec{r}}(t), p_{\vec{s}}(t)] = 0. \quad (2.64)
\end{aligned}$$

Take limit $\delta V \rightarrow 0$,

$$\begin{aligned}
\lim_{\delta V \to 0} \Phi_{\vec{r}}(t) &= \Phi(t, \vec{x}), \\
\lim_{\delta V \to 0} \frac{p_{\vec{s}}(t)}{\delta V} &= \Pi(t, \vec{y}), \\
\lim_{\delta V \to 0} \frac{\delta_{\vec{r}\vec{s}}}{\delta V} &= \delta^3(\vec{x} - \vec{y}),
\end{aligned}$$

$\Rightarrow$ so-called canonical equal-time commutation relations (ETCR) of the operators Φ and Π:

$$\begin{aligned} [\Phi(t,\vec{x}),\Pi(t,\vec{y})] &= i\delta^3(\vec{x}-\vec{y}), \\ [\Phi(t,\vec{x}),\Phi(t,\vec{y})] = [\Pi(t,\vec{x}),\Pi(t,\vec{y})] &= 0. \end{aligned} \tag{2.65}$$

Note: Real field has "turned into" Hermitian operator.
Operator Φ must satisfy ETCR + EOM obtained from Hamilton's variational principle,

$$(\Box + m^2)\Phi(x) = 0. \tag{2.66}$$

Solution: Consider Fourier decomposition

$$\Phi(t,\vec{x}) = \int \underbrace{\frac{\mathrm{d}^3k}{(2\pi)^3 2\omega(\vec{k}\,)}}_{\widetilde{\mathrm{d}^3k}} [a(\vec{k}\,)e^{-ik\cdot x} + a^\dagger(\vec{k}\,)e^{ik\cdot x}] = \Phi^\dagger(t,\vec{x}), \tag{2.67}$$

where

1. $k_0 = \omega(\vec{k}\,) = \sqrt{m^2 + \vec{k}^{\,2}}$ (Assignment 2, 6. (a)).

2. $a(\vec{k}\,)$ and $a^\dagger(\vec{k}\,)$ operators.

Assignment 3, 1. (a): ETCR imply commutation relations

$$[a(\vec{k}\,), a^\dagger(\vec{k}')] = (2\pi)^3 2\omega(\vec{k}\,)\delta^3(\vec{k}-\vec{k}'), \quad [a(\vec{k}\,), a(\vec{k}')] = [a^\dagger(\vec{k}\,), a^\dagger(\vec{k}')] = 0. \tag{2.68}$$

Interpretation of operators $a(\vec{k}\,)$ and $a^\dagger(\vec{k}\,)$:

Assignment 3, 1. (b):

$$H = \frac{1}{2}\int \widetilde{\mathrm{d}^3k}\,\omega(\vec{k}\,)\left(a^\dagger(\vec{k}\,)a(\vec{k}\,) + a(\vec{k}\,)a^\dagger(\vec{k}\,)\right). \tag{2.69}$$

Let

$$H|E\rangle = E|E\rangle.$$

Consider

$$Ha(\vec{k}\,)|E\rangle = \left(a(\vec{k}\,)H + [H, a(\vec{k}\,)]\right)|E\rangle.$$

$$
\begin{aligned}
[H, a(\vec{k}\,)] &= \frac{1}{2}\int \widetilde{\mathrm{d}^3k'}\,\omega(\vec{k}')[a^\dagger(\vec{k}')a(\vec{k}') + a(\vec{k}')a^\dagger(\vec{k}'), a(\vec{k}\,)] \\
&= \frac{1}{2}\int \widetilde{\mathrm{d}^3k'}\,\omega(\vec{k}')\{a^\dagger(\vec{k}')\underbrace{[a(\vec{k}'), a(\vec{k}\,)]}_{0} + \underbrace{[a^\dagger(\vec{k}'), a(\vec{k}\,)]}_{-(2\pi)^3 2\omega(\vec{k})\delta^3(\vec{k}-\vec{k}')}\, a(\vec{k}') \\
&\quad + a(\vec{k}')\underbrace{[a^\dagger(\vec{k}'), a(\vec{k}\,)]}_{\text{s.a.}} + \underbrace{[a(\vec{k}'), a(\vec{k}\,)]}_{0}\, a^\dagger(\vec{k}')\} \\
&= -\omega(\vec{k})a(\vec{k}\,)
\end{aligned}
$$

$$
= \left[E - \omega(\vec{k})\right] a(\vec{k}\,)|E\rangle. \tag{2.70}
$$

Analogously

$$
Ha^\dagger(\vec{k}\,)|E\rangle = \left[E + \omega(\vec{k}\,)\right] a^\dagger(\vec{k}\,)|E\rangle. \tag{2.71}
$$

Repeat for momentum operator

$$\vec{P} = \int \widetilde{\mathrm{d}^3k}\, \vec{k} a^\dagger(\vec{k}\,)a(\vec{k}\,). \tag{2.72}$$

Conclusion: Operators $a^\dagger(\vec{k}\,)$ and $a(\vec{k}\,)$ create and destroy a quantum of energy $\omega(\vec{k}\,)$ and momentum $\vec{k}$.

Remark on normal ordering:

Let $|0\rangle$ denote ground state (vacuum) with

$$a(\vec{k}\,)|0\rangle = 0, \quad \langle 0|a^\dagger(\vec{k}\,) = 0 \quad \forall \quad \vec{k}. \tag{2.73}$$

Consider vacuum expectation value (VEV) of Hamilton operator of Eq. (2.69):

$$\langle 0|H|0\rangle \;=\; \frac{1}{2}\int \widetilde{\mathrm{d}^3k}\, \omega(\vec{k}\,)\langle 0| \left(a^\dagger(\vec{k}\,)a(\vec{k}\,) + a(\vec{k}\,)a^\dagger(\vec{k}\,)\right) |0\rangle$$

$$
\begin{aligned}
&= \frac{1}{2}\int \widetilde{\mathrm{d}^3 k}\, \omega(\vec{k}\,)\langle 0|(2a^\dagger(\vec{k}\,) \underbrace{a(\vec{k}\,)}_{a(\vec{k}\,)|0\rangle = 0} + [a(\vec{k}\,), a^\dagger(\vec{k}\,)]|0\rangle \\
&= \infty, \qquad (2.74)
\end{aligned}
$$

because $[a(\vec{k}\,), a^\dagger(\vec{k}\,)] \sim \delta^3(0)$ and, in addition, one needs to integrate over $\vec{k}$.
Interpretation of infinite constant: infinite sum over oscillator ground-state energies.

Redefinition of Hamilton operator such that ground state corresponds to energy eigenvalue $E_0 = 0$:

$$
\begin{aligned}
H &= \frac{1}{2}\int \widetilde{\mathrm{d}^3 k}\, \omega(\vec{k}\,) \Big[a^\dagger(\vec{k}\,)a(\vec{k}\,) + a(\vec{k}\,)a^\dagger(\vec{k}\,) \\
&\quad - \langle 0| \left(a^\dagger(\vec{k}\,)a(\vec{k}\,) + a(\vec{k}\,)a^\dagger(\vec{k}\,)\right)|0\rangle\Big]
\end{aligned}
$$

$$= \frac{1}{2}\int \widetilde{\mathrm{d}^3 k}\, \omega(\vec{k}\,) : a^\dagger(\vec{k}\,)a(\vec{k}\,) + a(\vec{k}\,)a^\dagger(\vec{k}\,) :$$
$$= \int \widetilde{\mathrm{d}^3 k}\, \omega(\vec{k}\,)a^\dagger(\vec{k}\,)a(\vec{k}\,). \tag{2.75}$$

$:\ \ :$ denotes symbol for **normal ordering**: annihilation operators are always to the right of creation operators. For bosons, creation and annihilation operators commute, when they are written inside a normal product.

Example

$$: a(\vec{k}\,)a^\dagger(\vec{q}\,)a^\dagger(\vec{p}\,)a(\vec{r}\,) : \overset{\text{e.g.}}{=} : a^\dagger(\vec{q}\,)a(\vec{r}\,)a^\dagger(\vec{p}\,)a(\vec{k}\,) := a^\dagger(\vec{q}\,)a^\dagger(\vec{p}\,)a(\vec{r}\,)a(\vec{k}\,).$$

Construction of Hilbert space

Define number operator as

$$N = \int \widetilde{\mathrm{d}^3 k}\, a^\dagger(\vec{k}\,)a(\vec{k}\,). \tag{2.76}$$

Properties (Assignment 3, 1. (c))

$$\begin{aligned} N &= N^\dagger, \\ [N, H] &= 0, \\ [N, \vec{P}] &= 0. \end{aligned}$$

Consider one-particle state

$$|\vec{k}\,\rangle = a^\dagger(\vec{k}\,)|0\rangle \tag{2.77}$$

normalized as

$$\begin{aligned} \langle \vec{k}\,'|\vec{k}\,\rangle &= \langle 0|a(\vec{k}\,')a^\dagger(\vec{k}\,)|0\rangle \\ &= \langle 0|a^\dagger(\vec{k}\,)a(\vec{k}\,') + [a(\vec{k}\,'), a^\dagger(\vec{k}\,)]|0\rangle \\ &= (2\pi)^3 2\omega(\vec{k}\,)\delta^3(\vec{k} - \vec{k}\,'). \end{aligned} \tag{2.78}$$

Basis of complete Hilbert space (so-called Fock space) is constructed as

$$a^\dagger(\vec{k}_1)\cdots a^\dagger(\vec{k}_n)|0\rangle, \tag{2.79}$$

where the $\vec{k}_i$ are not necessarily different. Recall:

$$\begin{aligned} H a^\dagger(\vec{k}_1)\cdots a^\dagger(\vec{k}_n)|0\rangle &= \Big(a^\dagger(\vec{k}_1)H + [H, a^\dagger(\vec{k}_1)]\Big)\cdots a^\dagger(\vec{k}_n)|0\rangle \\ &= a^\dagger(\vec{k}_1)\Big(H + \omega(\vec{k}_1)\Big)\cdots a^\dagger(\vec{k}_n)|0\rangle \\ &= \Big(\omega(\vec{k}_1) + \cdots + \omega(\vec{k}_n)\Big)\, a^\dagger(\vec{k}_1)\cdots a^\dagger(\vec{k}_n)|0\rangle, \\ \vec{P}\cdots &= \Big(\vec{k}_1 + \cdots + \vec{k}_n\Big)\cdots, \\ N\cdots &= n\cdots. \end{aligned} \tag{2.80}$$

Basis states described in terms of occupation numbers $n(\vec{k}\,)$ corresponding to eigenstate of momentum $\vec{k}$:

$$|n(\vec{k}_1),\cdots,n(\vec{k}_n)\rangle = \prod_{\vec{k}_i} \frac{1}{\sqrt{n(\vec{k}_i)!}}\Big(a^\dagger(\vec{k}_i)\Big)^{n(\vec{k}_i)}|0\rangle. \tag{2.81}$$

Eq. (2.68) $\Rightarrow$ all creation operators commute.

Arbitrary normalized state as superposition

$$
\begin{aligned}
|\Phi\rangle \;=\; & \Bigg(c_0 + \int \widetilde{\mathrm{d}^3k_1}\, c_1(\vec{k}_1)a^\dagger(\vec{k}_1) + \frac{1}{\sqrt{2!}}\int \widetilde{\mathrm{d}^3k_1}\widetilde{\mathrm{d}^3k_2}\, c_2(\vec{k}_1,\vec{k}_2)a^\dagger(\vec{k}_1)a^\dagger(\vec{k}_2) \\
& + \frac{1}{\sqrt{3!}}\int \widetilde{\mathrm{d}^3k_1}\widetilde{\mathrm{d}^3k_2}\widetilde{\mathrm{d}^3k_3} c_3(\vec{k}_1,\vec{k}_2,\vec{k}_3)a^\dagger(\vec{k}_1)a^\dagger(\vec{k}_2)a^\dagger(\vec{k}_3) + \cdots\Bigg)|0\rangle .
\end{aligned}
\tag{2.82}
$$

(Recall: n objects may be ordered in $n!$ different ways.) Normalization condition $\Rightarrow$

$$
1 = \langle\Phi|\Phi\rangle = |c_0|^2 + \int \widetilde{\mathrm{d}^3k_1}|c_1(\vec{k}_1)|^2 + \int \widetilde{\mathrm{d}^3k_1}\widetilde{\mathrm{d}^3k_2}|c_2(\vec{k}_1,\vec{k}_2)|^2 + \cdots .
\tag{2.83}
$$

c_n: momentum distribution of the component containing n quanta ("wave function in momentum space").

$[a^\dagger(\vec{k}\,), a^\dagger(\vec{k}')] = 0 \Rightarrow$ momentum space wave functions are symmetric under interchange of any two arguments.

Probability of finding "particle 1" with momentum $\vec{k}$ and "particle 2" with momentum $\vec{k}'$ is the same as finding "particle 1" with momentum $\vec{k}'$ and "particle 2" with momentum $\vec{k}$. The so-called Bose-Einstein statistics originates from the commutation relations of Eq. (2.68).

2.6 Quantization of the Dirac Field

Literature

- Itzykson and Zuber, chapter 3.3
- Ryder, chapter 4.3

Empirical fact: Spin-$\frac{1}{2}$ particles obey Fermi-Dirac statistics and the Pauli exclusion principle.

Decompose Ψ and $\bar{\Psi}$ into plane-wave solutions

$$\begin{aligned}
\Psi(x) &= \sum_{r=1}^{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E(\vec{p})} \left[b_r(\vec{p}) u^{(r)}(\vec{p}) e^{-ip\cdot x} + d_r^\dagger(\vec{p}) v^{(r)}(\vec{p}) e^{ip\cdot x} \right] \\
&=: \Psi^{(+)}(x) + \Psi^{(-)}(x), \qquad (2.84) \\
\bar{\Psi}(x) &= \sum_{r=1}^{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E(\vec{p})} \left[b_r^\dagger(\vec{p}) \bar{u}^{(r)}(\vec{p}) e^{ip\cdot x} + d_r(\vec{p}) \bar{v}^{(r)}(\vec{p}) e^{-ip\cdot x} \right] \\
&= \bar{\Psi}^{(-)}(x) + \bar{\Psi}^{(+)}(x) = \overline{\Psi^{(+)}}(x) + \overline{\Psi^{(-)}}(x), \qquad (2.85)
\end{aligned}$$

where $p_0 = E(\vec{p}) = \sqrt{m^2 + \vec{p}^{\,2}}$.

In order to satisfy Pauli exclusion principle, postulate anti commutation relations for creation operators $b_r^\dagger$ and $d_r^\dagger$ (annihilation operators b_r and d_r) for particles and antiparticles

$$\{b_r(\vec{p}), b_s^\dagger(\vec{p}\,')\} = (2\pi)^3 2E(\vec{p}) \delta^3(\vec{p} - \vec{p}\,') \delta_{rs}, \qquad (2.86)$$

$$\{d_r(\vec{p}), d_s^\dagger(\vec{p}\,')\} \;=\; (2\pi)^3 2E(\vec{p})\delta^3(\vec{p}-\vec{p}\,')\delta_{rs}. \tag{2.87}$$

All remaining anti commutation relations vanish:

$$\begin{aligned} \{b_r(\vec{p}), b_s(\vec{p}\,')\} &= 0, \\ \{b_r^\dagger(\vec{p}), b_s^\dagger(\vec{p}\,')\} &= 0, \end{aligned} \tag{2.88}$$

$$\begin{aligned} \{d_r(\vec{p}), d_s(\vec{p}\,')\} &= 0, \\ \{d_r^\dagger(\vec{p}), d_s^\dagger(\vec{p}\,')\} &= 0, \end{aligned} \tag{2.89}$$

$$\begin{aligned} \{b_r(\vec{p}), d_s(\vec{p}\,')\} &= 0, \\ \{b_r^\dagger(\vec{p}), d_s^\dagger(\vec{p}\,')\} &= 0, \end{aligned} \tag{2.90}$$

$$\begin{aligned} \{b_r(\vec{p}), d_s^\dagger(\vec{p}\,')\} &= 0, \\ \{d_r(\vec{p}), b_s^\dagger(\vec{p}\,')\} &= 0. \end{aligned} \tag{2.91}$$

Simplification of notation:

$$b_r(\vec{p}) \;\mapsto\; b_i, \quad \text{etc.}$$

$$\sum_{r=1}^{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E(\vec{p})} \mapsto \sum_i ,$$

$$\{b_r(\vec{p}), b_s^\dagger(\vec{p}\,')\} = (2\pi)^3 2E(\vec{p})\delta^3(\vec{p} - \vec{p}\,')\delta_{rs} \mapsto \{b_i, b_j^\dagger\} = \delta_{ij}, \quad \text{etc.}$$

Substitute back at end of calculation.

Example: Consider superposition of two particles

$$\begin{aligned} |\Phi\rangle &= \frac{1}{\sqrt{2}} \sum_{i,j} c_2(i,j) b_i^\dagger b_j^\dagger |0\rangle \\ &= \frac{1}{\sqrt{2}} \sum_{i,j} c_2(i,j)(-1) b_j^\dagger b_i^\dagger |0\rangle \\ &= -\frac{1}{\sqrt{2}} \sum_{i,j} c_2(j,i) b_i^\dagger b_j^\dagger |0\rangle, \end{aligned}$$

i. e. wave function is antisymmetric under exchange of two arguments:

$$c_2(i,j) = -c_2(j,i).$$

In particular: 0 for $i = j$ (Pauli principle).
Normalization

$$\langle \Phi | \Phi \rangle = 1$$

leads to

$$\begin{aligned} 1 &= \sum_{i,j} |c_2(i,j)|^2 \\ &\mapsto \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E(\vec{p})} \sum_{r=1}^{2} \int \frac{\mathrm{d}^3 p'}{(2\pi)^3 2E(\vec{p}\,')} \sum_{r'=1}^{2} c_2(\vec{p}, r; \vec{p}\,', r') = 1. \end{aligned}$$

Anti commutation relations of Dirac fields from

- anti commutation relations of Eqs. (2.86) - (2.91)
- properties of Dirac spinors of chapter 2.2, example 2.

$$\{\Psi_\alpha(t,\vec{x}\,), \Psi^\dagger_\beta(t,\vec{y}\,)\} = \delta^3(\vec{x}-\vec{y}\,)\delta_{\alpha\beta}, \tag{2.92}$$

$$\{\Psi_\alpha(t,\vec{x}\,), \Psi_\beta(t,\vec{y}\,)\} = 0, \tag{2.93}$$

$$\{\Psi^\dagger_\alpha(t,\vec{x}\,), \Psi^\dagger_\beta(t,\vec{y}\,)\} = 0, \tag{2.94}$$

α and β: Dirac indices (1,2,3,4).

Example

$$\{\Psi_\alpha(t,\vec{x}\,), \Psi^\dagger_\beta(t,\vec{y}\,)\}$$

Rewrite symbolically

$$\Psi(x) = \sum_i \left(b_i u_i(x) + d_i^\dagger v_i(x)\right),$$

$$\Psi^\dagger(y) = \sum_i \left(b_i^\dagger u_i^\dagger(y) + d_i v_i^\dagger(y)\right).$$

Thus

$$\begin{aligned}
\{\Psi_\alpha(x), \Psi^\dagger_\beta(y)\}_{x_0=y_0} &= \sum_{i,j} \{b_i u_{i\alpha}(x) + d^\dagger_i v_{i\alpha}(x), b^\dagger_j u^\dagger_{j\beta}(y) + d_j v^\dagger_{j\beta}(y)\}_{x_0=y_0} \\
&= \sum_{i,j} (u_{i\alpha}(x) u^\dagger_{j\beta}(y) \underbrace{\{b_i, b^\dagger_j\}}_{\delta_{ij}} + u_{i\alpha}(x) v^\dagger_{j\beta}(y) \underbrace{\{b_i, d_j\}}_{0} \\
&\quad + v_{i\alpha}(x) u^\dagger_{j\beta}(y) \underbrace{\{d^\dagger_i, b^\dagger_j\}}_{0} + v_{i\alpha}(x) v^\dagger_{j\beta}(y) \underbrace{\{d^\dagger_i, d_j\}}_{\delta_{ij}})_{x_0=y_0} \\
&= \sum_i (u_{i\alpha}(x) u^\dagger_{i\beta}(y) + v_{i\alpha}(x) v^\dagger_{i\beta}(y))_{x_0=y_0}.
\end{aligned}$$

Continuous notation

$$\cdots = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E(\vec{p})} \sum_{r=1}^{2} \left(u^{(r)}_\alpha(\vec{p}) e^{-ip\cdot x} u^{(r)\dagger}_\beta(\vec{p}) e^{ip\cdot y} + v^{(r)}_\alpha(\vec{p}) e^{ip\cdot x} v^{(r)\dagger}_\beta(\vec{p}) e^{-ip\cdot y} \right)_{x_0=y_0}$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E(\vec{p})} \sum_{r=1}^{2} \left(u_\alpha^{(r)}(\vec{p}) u_\beta^{(r)\dagger}(\vec{p}) e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + v_\alpha^{(r)}(\vec{p}) v_\beta^{(r)\dagger}(\vec{p}) e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \right).$$

Make use of

$$\sum_{r=1}^{2} u^{(r)}(\vec{p}) u^{(r)\dagger}(\vec{p}) = (E+m) \sum_{r=1}^{2} \begin{pmatrix} 1_{2\times 2}\chi_r \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_r \end{pmatrix} \begin{pmatrix} \chi_r^\dagger 1_{2\times 2} & \chi_r^\dagger \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}$$

and

$$\sum_{r=1}^{2} \chi_r \chi_r^\dagger = 1_{2\times 2}$$

$$\begin{aligned} \cdots &= (E+m) \begin{pmatrix} 1_{2\times 2} & \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} & \frac{\vec{p}^{\,2}}{(E+m)^2} \end{pmatrix} = \begin{pmatrix} (E+m) & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & (E-m) \end{pmatrix} \\ &= (\not{p} + m)\gamma_0 \quad (p_0 = E(\vec{p})) \end{aligned}$$

Completely analogous

$$\sum_{r=1}^{2} v^{(r)}(\vec{p})v^{(r)\dagger}(\vec{p}) = (\not{p} - m)\gamma_0 \quad (p_0 = E(\vec{p})).$$

Put together

$$\{\Psi_\alpha(x), \Psi_\beta^\dagger(y)\}_{x_0=y_0} = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E(\vec{p})} \Big[(\not{p} + m)\gamma_0 e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + (\not{p} - m)\gamma_0 e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\Big]_{\alpha\beta}.$$

Substitution in the second term $\vec{p} \to -\vec{p}$;
Note $E(-\vec{p}) = E(\vec{p})$ and $\gamma_0^2 = 1$:

$$\cdots = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\delta_{\alpha\beta}e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = \delta_{\alpha\beta}\delta^3(\vec{x} - \vec{y}).$$

Hamilton operator

Start from Lagrangian

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi$$

define canonically conjugate momentum

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\Psi}} = i\Psi^\dagger$$

Hamiltonian density

$$\mathcal{H} = \Pi\dot{\Psi} - \mathcal{L} = i\Psi^\dagger\dot{\Psi} - \bar{\Psi}(i\gamma_0\partial_0 + i\vec{\gamma}\cdot\vec{\nabla} - m)\Psi = \bar{\Psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\Psi = \Psi^\dagger i\dot{\Psi}$$

for solutions of the Dirac equation.

Remark: Interpreting the Dirac equation as a field equation is not sufficient to remove the problem of "negative energies." Quantization in terms of anti commutation relations required.

Consider Hamilton operator

$$
\begin{aligned}
H &= \int \mathrm{d}^3x \mathcal{H} \\
&= \int \mathrm{d}^3x \underbrace{\int \frac{\mathrm{d}^3p}{(2\pi)^3 2E(\vec{p})} \sum_{r=1}^{2}}_{\sum\!\!\!\!\int} \underbrace{\int \frac{\mathrm{d}^3p\,'}{(2\pi)^3 2E(\vec{p}\,')} \sum_{r'=1}^{2}}_{\sum\!\!\!\!\int\,'} \\
&\quad \times [b_r^\dagger(\vec{p}) u^{(r)\dagger}(\vec{p}) e^{ip\cdot x} + d_r(\vec{p}) v^{(r)\dagger}(\vec{p}) e^{-ip\cdot x}] \\
&\quad \times i[-iE(\vec{p}\,')][b_{r\,'}(\vec{p}\,') u^{(r\,')}(\vec{p}\,') e^{-ip\,'\cdot x} \underbrace{-}_{\text{sign!}} d_{r\,'}^\dagger(\vec{p}\,') v^{(r\,')}(\vec{p}\,') e^{ip\,'\cdot x}].
\end{aligned}
$$

Perform integration $\int \mathrm{d}^3x$

$$
\begin{aligned}
\cdots &= \sum\!\!\!\!\int \sum\!\!\!\!\int\,' E(\vec{p}\,') \\
&\quad \Big[b_r^\dagger(\vec{p}) b_{r\,'}(\vec{p}\,') u^{(r)\dagger}(\vec{p}) u^{(r\,')}(\vec{p}\,') e^{ix_0[E(\vec{p})-E(\vec{p}\,')]} (2\pi)^3 \delta^3(\vec{p}-\vec{p}\,')
\end{aligned}
$$

$$
\begin{aligned}
&-b_r^\dagger(\vec{p})d_{r'}^\dagger(\vec{p}')u^{(r)\dagger}(\vec{p})v^{(r')}(\vec{p}')e^{ix_0[E(\vec{p})+E(\vec{p}')]}(2\pi)^3\delta^3(\vec{p}+\vec{p}') \\
&+d_r(\vec{p})b_{r'}(\vec{p}')v^{(r)\dagger}(\vec{p})u^{(r')}(\vec{p}')e^{-ix_0[E(\vec{p})+E(\vec{p}')]}(2\pi)^3\delta^3(\vec{p}+\vec{p}') \\
&-d_r(\vec{p})d_{r'}^\dagger(\vec{p}')v^{(r)\dagger}(\vec{p})v^{(r')}(\vec{p}')e^{-ix_0[E(\vec{p})-E(\vec{p}')]}(2\pi)^3\delta^3(\vec{p}-\vec{p}')\Big].
\end{aligned}
$$

Perform integration $\int \mathrm{d}^3p$

$$
\begin{aligned}
\cdots = \sum_{r=1}^{2}\sum\!\!\!\!\!\int{}' E(\vec{p}')\\
\times\Bigg[&\frac{1}{2E(\vec{p}')}b_r^\dagger(\vec{p}')b_{r'}(\vec{p}')\underbrace{u^{(r)\dagger}(\vec{p}')u^{(r')}(\vec{p}')}_{2E(\vec{p}')\delta_{rr'}}\underbrace{e^{ix_0[E(\vec{p}')-E(\vec{p}')]}}_{1} \\
&-\frac{1}{2E(-\vec{p}')}b_r^\dagger(-\vec{p}')d_{r'}^\dagger(\vec{p}')\underbrace{u^{(r)\dagger}(-\vec{p}')v^{(r')}(\vec{p}')}_{0}e^{ix_0(E(-\vec{p}')+E(\vec{p}'))}
\end{aligned}
$$

$$+\frac{1}{2E(-\vec{p}\,')}d_r(-\vec{p}\,')b_{r\,'}(\vec{p}\,')\underbrace{v^{(r)\dagger}(-\vec{p}\,')u^{(r\,')}(\vec{p}\,')}_{0}e^{-ix_0[E(-\vec{p}\,')+E(\vec{p}\,')]}$$

$$\left.-\frac{1}{2E(\vec{p}\,')}d_r(\vec{p}\,')d^\dagger_{r\,'}(\vec{p}\,')\underbrace{v^{(r)\dagger}(\vec{p}\,')v^{(r\,')}(\vec{p}\,')}_{2E(\vec{p}\,')\delta_{rr\,'}}\underbrace{e^{-ix_0[E(\vec{p}\,')-E(\vec{p}\,')]}}_{1}\right] .$$

- Perform sum $\sum_{r\,'}$;
- rename $\vec{p}\,' \to \vec{p}$:

$$\cdots = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E(\vec{p})}\sum_{r=1}^{2} E(\vec{p})[b^\dagger_r(\vec{p})b_r(\vec{p}) - d_r(\vec{p})d^\dagger_r(\vec{p})].$$

So far: We only made use of the properties of the Dirac spinors. We made sure to write creation and annihilation operators in the order as they appear in the multiplication.

Introduce **normal ordering** including a sign change for each interchange of fermion operators.
Example

$$
\begin{aligned}
:\bar{\Psi}\Gamma\Psi: &= :(\bar{\Psi}_\alpha^{(-)} + \bar{\Psi}_\alpha^{(+)})\Gamma_{\alpha\beta}(\Psi_\beta^{(+)} + \Psi_\beta^{(-)}): \\
&= \bar{\Psi}_\alpha^{(-)}\Gamma_{\alpha\beta}\Psi_\beta^{(+)} + \bar{\Psi}_\alpha^{(-)}\Gamma_{\alpha\beta}\Psi_\beta^{(-)} + \bar{\Psi}_\alpha^{(+)}\Gamma_{\alpha\beta}\Psi_\beta^{(+)} - \Psi_\beta^{(-)}\Gamma_{\alpha\beta}\bar{\Psi}_\alpha^{(+)},
\end{aligned}
\tag{2.95}
$$

with Γ an arbitrary 4×4 matrix.
Recall:

$$
\begin{aligned}
(+) &\sim \text{annihilation operator} \times e^{-ip\cdot x}, \\
(-) &\sim \text{creation operator} \times e^{+ip\cdot x}.
\end{aligned}
$$

Result for normal-ordered Hamilton operator

$$
H = \int \mathrm{d}^3x : \Psi^\dagger(x) i\dot{\Psi}(x) :
$$

$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E(\vec{p})} \sum_{r=1}^{2} E(\vec{p})[b_r^\dagger(\vec{p})b_r(\vec{p}) + d_r^\dagger(\vec{p})d_r(\vec{p})].$$

Sign change in second term $\Rightarrow$ Hamilton-Operator positive definite. Using commutation instead of anti commutation relations would have led to Hamilton operator which is not bounded from below. In other words, existence of a stable ground state requires that Dirac equation be quantized according to Dirac-Fermi statistics (anti commutators). Special example of the so-called spin-statistics theorem, according to which fermions/bosons are quantized with anti commutation/commutation relations (see, e.g., W. Pauli, Phys. Rev. **58**, 716 (1940)).

2.7 More on Noether's Theorem in Quantum (Field) Theory

So far: Noether's theorem on the classical level.
In principle, charges $Q^a(t)$ can have any continuous real value.
Now: Transition to a Quantum (Field) Theory.

Analogy: Point mass m in a central potential $V(\vec{r}) = V(r)$.
Lagrange and Hamilton functions are rotationally invariant.
Consequence: Angular momentum $\vec{l} = \vec{r} \times \vec{p}$ is a constant of the motion.

Transition to quantum mechanics. $\Rightarrow$ Operators

$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij}, \quad [\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{p}_i, \hat{p}_j] = 0.$$

Components of angular momentum operator

$$\hat{l}_i \;=\; \epsilon_{ijk}\hat{x}_j\hat{p}_k = -i\hat{p}_j \underbrace{(-i\epsilon_{ijk})}_{(L_i^{\mathrm{ad}})_{jk}} \hat{x}_k,$$

$$L_i^{\mathrm{ad}}: \qquad 3\times 3 \text{ matrices satisfying } [L_i^{\mathrm{ad}}, L_j^{\mathrm{ad}}] = i\epsilon_{ijk}L_k^{\mathrm{ad}},$$

satisfy commutation relations

$$[\hat{l}_i, \hat{l}_j] = i\epsilon_{ijk}\hat{l}_k$$

i.e., they cannot simultaneously be diagonalized.
Recall: Angular momentum operators are generators of rotations:

$$|\Psi'\rangle = \exp(-i\alpha_i\hat{l}_i)|\Psi\rangle.$$

Rotational invariance of the quantum system

$$[\hat{H}, \hat{l}_i] = 0$$

i.e., $\hat{l}_i$ are still constants of the motion.
Simultaneously diagonalize $\hat{H}$, $\hat{\vec{l}}^{\,2}$, and $\hat{l}_3$.

Example: Hydrogen atom

$$E_n = -\frac{\alpha^2 m}{2n^2} \approx -\frac{13.6}{n^2}\,\mathrm{eV},$$

where $n = n' + l + 1$; $n' \geq 0$ denotes principal quantum number. Degeneracy of an energy level is given by n^2 (spin neglected).

- Value E_1 and spacing of levels determined by *dynamics* of the system, i.e., specific form of potential.

- Multiplets with eigenvalues $l(l+1)$ and $m = -l, \cdots, l$ $(l = 0, 1, 2, \cdots)$.
  Multiplicities of energy levels are consequence of underlying rotational *symmetry*. (In fact, accidental degeneracy for $n \geq 2$ is result of even higher symmetry of $1/r$ Hamiltonian, namely SO(4) symmetry.)

In short: Multiplicities depend on underlying symmetry; energy eigenvalues depend on V (dynamics).

What happens in QFT?

After canonical quantization: Fields Φ_i and conjugate momenta $\Pi_i = \partial\mathcal{L}/\partial(\partial_0\Phi_i)$ are considered as operators subject to ETCR

$$[\Phi_i(t,\vec{x}),\Pi_j(t,\vec{y})] = i\delta^3(\vec{x}-\vec{y})\delta_{ij} \;\leftrightarrow\; [\hat{x}_i,\hat{p}_j] = i\delta_{ij}$$
$$[\Phi_i(t,\vec{x}),\Phi_j(t,\vec{y})] = 0 \;\leftrightarrow\; [\hat{x}_i,\hat{x}_j] = 0$$
$$[\Pi_i(t,\vec{x}),\Pi_j(t,\vec{y})] = 0 \;\leftrightarrow\; [\hat{p}_i,\hat{p}_j] = 0$$

Consider as special case of Eq. (2.47) infinitesimal transformations which are **linear** in fields

$$\Phi_i(x) \mapsto \Phi_i'(x) = \Phi_i(x) - i\epsilon_a(x)t^a_{ij}\Phi_j(x) \quad \leftrightarrow \quad \hat{x}_i \mapsto \hat{x}_i - i\epsilon_k(-i\epsilon_{kij})\hat{x}_j \tag{2.96}$$

t^a_{ij} are constants generating mixing of fields

$$J^{\mu,a}(x) \;=\; -i\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\,t^a_{ij}\,\Phi_j, \tag{2.97}$$
$$Q^a(t) \;=\; -i\int \mathrm{d}^3x\,\Pi_i(x)\,t^a_{ij}\,\Phi_j(x) \;\leftrightarrow\; \hat{l}_k = -i\hat{p}_i(-i\epsilon_{kij})\hat{x}_j = \epsilon_{kij}\hat{x}_i\hat{p}_j,$$

(2.98)

where $J^{\mu,a}(x)$ and $Q^a(t)$ are now operators.

Transformation behavior of field operators

$$\begin{aligned}
[Q^a(t), \Phi_k(t,\vec{y})] &= -it^a_{ij} \int \mathrm{d}^3x\, [\Pi_i(t,\vec{x})\Phi_j(t,\vec{x}), \Phi_k(t,\vec{y})] \\
&= -it^a_{ij} \int \mathrm{d}^3x\, (\Pi_i(t,\vec{x})[\Phi_j(t,\vec{x}), \Phi_k(t,\vec{y})] \\
&\qquad + [\Pi_i(t,\vec{x}), \Phi_k(t,\vec{y})]\Phi_j(t,\vec{x})) \\
&= -it^a_{ij} \int \mathrm{d}^3x\, (-i\delta^3(\vec{x}-\vec{y})\delta_{ik}\Phi_j(t,\vec{x})) \\
&= -t^a_{kj}\Phi_j(t,\vec{y}) \qquad (2.99) \\
&\leftrightarrow [\hat{l}_k, \hat{x}_i] = i\epsilon_{kij}\hat{x}_j \qquad (2.100)
\end{aligned}$$

Q^a are generators of the transformations acting on the states of Hilbert space.

Interpretation of charge operators

1. Time-independent charge operators satisfy (Heisenberg picture)

$$\frac{\mathrm{d}Q^a}{\mathrm{d}t} = i[Q^a, H] = 0,$$

i.e., H and Q^a may simultaneously be diagonalized.
Group theory: Degeneracy of an energy level is associated with dimensionality of irreducible representations of underlying symmetry group.
$\Rightarrow$ Investigation of particle spectrum yields information on underlying symmetry.

Example: Isospin multiplets

Name	I	I_3	Y	Mass [MeV]	Life time [s] Width [MeV]	(Main-) Decay
pion	1	± 1 0	0 0	140 135	$2.6 \cdot 10^{-8}$s $8.4 \cdot 10^{-17}$s	$\pi^+ \to \mu^+ + \nu_\mu$ $\pi^0 \to \gamma\gamma$
η	0	0	0	547	1.2 keV	$\pi\pi\pi$
ρ	1	$1, 0, -1$	0	770	151 MeV	$\pi\pi$
nucleon	$\frac{1}{2}$	$p : \frac{1}{2}$ $n : -\frac{1}{2}$	1 1	938 940	stable ($> 2.1 \times 10^{29}$ years) 886 s	 $pe^-\bar{\nu}_e$
Δ	$\frac{3}{2}$	$\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$	1	1232	120 MeV	$N\pi$

$Q = I_3 + \frac{1}{2}Y$. $(200\,\text{MeV})^{-1} \approx \frac{1}{3} \cdot 10^{-23}\,$s.

2. Symmetries relate scattering amplitudes of various processes (of the same type).
   To be discussed later. Also: Green functions of different types: Ward identities.

   Example: Isospin symmetry $\Rightarrow$ πN scattering may be described in terms of two scattering amplitudes $T^{\frac{1}{2}}$ and $T^{\frac{3}{2}}$ (Wigner-Eckart theorem):

$$\langle I', I_3'|T|I, I_3\rangle = T^I \delta_{II'}\delta_{I_3 I_3'}.$$

Interpret constants t^a_{ij} as entries in ith row and jth column of an $n \times n$ matrix T^a,

$$T^a = \begin{pmatrix} t^a_{11} & \cdots & t^a_{1n} \\ \vdots & & \vdots \\ t^a_{n1} & \cdots & t^a_{nn} \end{pmatrix}.$$

Assumption: Matrices form n-dimensional representation of a Lie algebra,

$$[T^a, T^b] = iC_{abc}T^c \tag{2.101}$$

with structure constants C_{abc}. $\Rightarrow$ Charge operators $Q^a(t)$ form a Lie algebra

$$[Q^a(t), Q^b(t)] = iC_{abc}Q^c(t). \tag{2.102}$$

Infinitesimal, linear transformations of fields Φ_i may then be written in compact form,

$$\begin{pmatrix} \Phi_1(x) \\ \vdots \\ \Phi_n(x) \end{pmatrix} = \Phi(x) \mapsto \Phi'(x) = (1 - i\epsilon_a T^a)\Phi(x). \tag{2.103}$$

T_a may be brought into block-diagonal form. $\Rightarrow$ Only fields belonging to the same multiplet transform into each other under the symmetry group.

Example 1: U(1) (or O(2))

Scalar field theory with a global U(1) invariance:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu \Phi_1 \partial^\mu \Phi_1 + \partial_\mu \Phi_2 \partial^\mu \Phi_2) - \frac{m^2}{2}(\Phi_1^2 + \Phi_2^2) - \frac{\lambda}{4}(\Phi_1^2 + \Phi_2^2)^2 \\ &= \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi - \lambda(\Phi^\dagger \Phi)^2, \end{aligned} \tag{2.104}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2}}[\Phi_1(x) + i\Phi_2(x)], \quad \Phi^\dagger(x) = \frac{1}{\sqrt{2}}[\Phi_1(x) - i\Phi_2(x)],$$

with real scalar fields Φ_1 and Φ_2.
$\mathcal{L}$ is invariant under the global transformations

$$\begin{aligned} \Phi' &= (1 + i\epsilon)\Phi, \\ \Phi'^\dagger &= (1 - i\epsilon)\Phi^\dagger, \end{aligned}$$

or

$$\begin{pmatrix} \Phi' \\ \Phi'^{\dagger} \end{pmatrix} = (1 - i\epsilon \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{T}) \begin{pmatrix} \Phi' \\ \Phi'^{\dagger} \end{pmatrix}, \tag{2.105}$$

with ϵ infinitesimal real parameter.
Apply method of Gell-Mann and Lévy with *local* parameter $\epsilon(x)$,

$$\delta\mathcal{L} = \partial_\mu \epsilon(x)(i\partial^\mu \Phi^\dagger \Phi - i\Phi^\dagger \partial^\mu \Phi), \tag{2.106}$$

$\Rightarrow$ current density

$$J^\mu = \frac{\partial \delta\mathcal{L}}{\partial \partial_\mu \epsilon} = (i\partial^\mu \Phi^\dagger \Phi - i\Phi^\dagger \partial^\mu \Phi), \tag{2.107}$$

$$\partial_\mu J^\mu = \frac{\partial \delta\mathcal{L}}{\partial \epsilon} = 0. \tag{2.108}$$

Extend analysis to *quantum* field theory.
Define conjugate momenta,

$$\Pi = \frac{\partial \mathcal{L}}{\partial \partial_0 \Phi} = \dot{\Phi}^\dagger, \quad \Pi^\dagger = \frac{\partial \mathcal{L}}{\partial \partial_0 \Phi^\dagger} = \dot{\Phi}. \tag{2.109}$$

Operators subject to ETCR

$$[\Phi(t,\vec{x}),\Pi(t,\vec{y})] = [\Phi^\dagger(t,\vec{x}),\Pi^\dagger(t,\vec{y})] = i\delta^3(\vec{x}-\vec{y}). \qquad (2.110)$$

Remaining ETCR vanish.
Current operator of quantized theory reads

$$J^\mu(x) =: (i\partial^\mu\Phi^\dagger\Phi - i\Phi^\dagger\partial^\mu\Phi) :, \qquad (2.111)$$

where : : denotes normal or Wick ordering, i.e., annihilation operators appear to the right of creation operators.
Charge operator (generator of infinitesimal transformations of Hilbert space states)

$$Q = \int d^3x J^0(t,\vec{x}). \qquad (2.112)$$

Apply Eq. (2.110) to obtain equal-time commutation relations

$$\begin{aligned}
[J^0(t,\vec{x}),\Phi(t,\vec{y})] &= \delta^3(\vec{x}-\vec{y})\Phi(t,\vec{x}), \\
[J^0(t,\vec{x}),\Pi(t,\vec{y})] &= -\delta^3(\vec{x}-\vec{y})\Pi(t,\vec{x}),
\end{aligned}$$

$$
\begin{aligned}
[J^0(t,\vec{x}), \Phi^\dagger(t,\vec{y})] &= -\delta^3(\vec{x}-\vec{y})\Phi^\dagger(t,\vec{x}), \\
[J^0(t,\vec{x}), \Pi^\dagger(t,\vec{y})] &= \delta^3(\vec{x}-\vec{y})\Pi^\dagger(t,\vec{x}).
\end{aligned} \tag{2.113}
$$

(From $[A,B] = AB - BA = C$ we obtain $[A^\dagger, B^\dagger] = -C^\dagger$. Note that $J^0(x) = J^{0\dagger}(x)$).
Remark: Transition to normal ordering involves an (infinite) constant which does not contribute to commutator.
Example:

$$
\begin{aligned}
[J^0(t,\vec{x}), \Phi(t,\vec{y})] &= i[\Pi(t,\vec{x})\Phi(t,\vec{x}) - \Phi^\dagger(t,\vec{x})\Pi^\dagger(t,\vec{x}), \Phi(t,\vec{y})] \\
&= i(-i)\delta^3(\vec{x}-\vec{y})\Phi(t,\vec{x}) \\
&= \delta^3(\vec{x}-\vec{y})\Phi(t,\vec{x}).
\end{aligned}
$$

Perform space integrals in Eqs. (2.113). $\Rightarrow$

$$
\begin{aligned}
[Q, \Phi(x)] &= \Phi(x), \\
[Q, \Pi(x)] &= -\Pi(x), \\
[Q, \Phi^\dagger(x)] &= -\Phi^\dagger(x),
\end{aligned}
$$

$$[Q, \Pi^\dagger(x)] \;=\; \Pi^\dagger(x). \tag{2.114}$$

Q: Why are commutation relations involving charge operators so important?
A: They specify how operators transform.

Motivation: Consider operator A and basis $\{|k\rangle\}$.
Perform infinitesimal, unitary transformation of states

$$\begin{aligned} |k'\rangle \;&=\; (1 + i\epsilon_a Q^a)|k\rangle, \\ A' \;&=\; A + \epsilon_a \delta A_a \end{aligned}$$

such that

$$\langle i'|A'|j'\rangle \stackrel{!}{=} \langle i|A|j\rangle \quad \forall \quad i, j.$$

We obtain

$$\begin{aligned} (1 - i\epsilon_a Q^a)(A + \epsilon_b \delta A_b)(1 + i\epsilon_c Q^c) \;&=\; A, \\ \Leftrightarrow \quad A + \epsilon_b \delta A_b \;&=\; (1 + i\epsilon_a Q^a)A(1 - i\epsilon_c Q^c), \end{aligned}$$

$$\Leftrightarrow \quad \delta A_a = i[Q^a, A].$$

Physical interpretation of Eqs. (2.114):
Take eigenstate $|\alpha\rangle$ of Q with eigenvalue q_α and consider action of $\Phi(x)$ on that state,

$$Q\left(\Phi(x)|\alpha\rangle\right) = \left([Q, \Phi(x)] + \Phi(x)Q\right)|\alpha\rangle = (1 + q_\alpha)\left(\Phi(x)|\alpha\rangle\right).$$

Conclusion: Operators $\Phi(x)$ and $\Pi^\dagger(x)$ [$\Phi^\dagger(x)$ and $\Pi(x)$] increase (decrease) the Noether charge of a system by one unit.

Example 2: U(1) for fermions

Start from Lagrangian of free theory describing electrons

$$\mathcal{L}_0 = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi. \tag{2.115}$$

$\mathcal{L}_0$ is invariant under *global* U(1) transformation

$$\begin{aligned} \Psi(x) &\mapsto \Psi'(x) = e^{-i\alpha}\Psi(x), \\ \bar{\Psi}(x) &\mapsto \bar{\Psi}'(x) = \bar{\Psi}(x)e^{i\alpha}, \end{aligned}$$

where $\alpha \in [0, 2\pi]$:

$$\bar{\Psi}\Psi \;\mapsto\; \bar{\Psi}\underbrace{e^{i\alpha}e^{-i\alpha}}_{1}\Psi = \bar{\Psi}\Psi,$$

$$\bar{\Psi}\gamma_\mu\partial^\mu\Psi \;\mapsto\; \bar{\Psi}e^{i\alpha}\gamma_\mu\partial^\mu e^{-i\alpha}\Psi = \bar{\Psi}e^{i\alpha}e^{-i\alpha}\gamma_\mu\partial^\mu\Psi = \bar{\Psi}\gamma_\mu\partial^\mu\Psi.$$

Consider infinitesimal transformation

$$e^{-i\alpha} \to 1 - i\epsilon$$

and substitute $\epsilon \to \epsilon(x)$

$$\delta\mathcal{L}_0 = -i\partial_\mu\epsilon(x)i\bar{\Psi}(x)\gamma^\mu\Psi(x).$$

$\Rightarrow$

$$J^\mu = \frac{\partial\delta\mathcal{L}_0}{\partial_\mu\epsilon} = \bar{\Psi}\gamma^\mu\Psi, \tag{2.116}$$

with charge operator

$$Q = \int d^3x : \Psi^\dagger(t,\vec{x})\Psi(t,\vec{x}) : \;. \tag{2.117}$$

Assignment 4, 4. (a)

$$Q = \sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^3 2E(\vec{p})} \left[b_r^\dagger(\vec{p}) b_r(\vec{p}) - d_r^\dagger(\vec{p}) d_r(\vec{p}) \right] .$$

Minus sign for antiparticle contribution is related to normal-ordering procedure. Note that

$$Q|0\rangle = 0,$$

because

$$\begin{aligned} b_r(\vec{p})|0\rangle &= 0, \\ d_r(\vec{p})|0\rangle &= 0. \end{aligned}$$

Consider one-particle states:

$$\begin{aligned} Q|e^-(\vec{p},r)\rangle &= Q\, b_r^\dagger(\vec{p})|0\rangle = ([Q, b_r^\dagger(\vec{p})] + b_r^\dagger(\vec{p})Q)|0\rangle = +1|e^-(\vec{p},r)\rangle, \\ Q|e^+(\vec{p},r)\rangle &= Q\, d_r^\dagger(\vec{p})|0\rangle = -1|e^+(\vec{p},r)\rangle. \end{aligned}$$

We made use of

$$\begin{aligned}
[Q, b_r^\dagger(\vec{p})] &= b_r^\dagger(\vec{p}), \\
[Q, d_r^\dagger(\vec{p})] &= -d_r^\dagger(\vec{p}).
\end{aligned}$$

Verification: Express commutator in terms of anticommutator (fermions)

$$[ab, c] = abc - cab = abc + acb - acb - cab = a\{b, c\} - \{a, c\}b.$$

$$\begin{aligned}
[b_s^\dagger(\vec{q})b_s(\vec{q}), b_r^\dagger(\vec{p})] &= b_s^\dagger(\vec{q})\{b_s(\vec{q}), b_r^\dagger(\vec{p})\} - \underbrace{\{b_s^\dagger(\vec{q}), b_r^\dagger(\vec{p})\}}_{0} b_s(\vec{q}) \\
&= (2\pi)^3 2E(\vec{q})\delta^3(\vec{q} - \vec{p})\delta_{sr} b_s^\dagger(\vec{q}), \\
[d_s^\dagger(\vec{q})d_s(\vec{q}), b_r^\dagger(\vec{p})] &= 0.
\end{aligned}$$

Substitute into expression for Q, perform integration $\Rightarrow$ result.
Second calculation analogous. Minus sign originates in minus sign in Q.

Interpretation: Q is charge operator in units of $-e$, $e > 0$.
Or: Q is electron number operator.

Chapter 3

Quantum Chromodynamics and Chiral Symmetry

3.1 Some Remarks on SU(3)

Role of SU(3) in the context of the strong interactions:

1. Gauge group of QCD;
2. flavor SU(3) is approximately realized as a global symmetry of the hadron spectrum (Eightfold Way);

3. direct product $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ is the chiral-symmetry group of QCD for vanishing u-, d-, and s-quark masses.

Basic properties of SU(3) and its Lie algebra su(3)

Definition:

$$\mathrm{SU}(3) \equiv \text{set of all unitary, unimodular, } 3\times 3 \text{ matrices } U:$$
$$U^\dagger U = 1,\ \det(U) = 1.$$

Elements of SU(3) are conveniently written in terms of exponential representation

$$U(\Theta) = \exp\left(-i\sum_{a=1}^{8}\Theta_a\frac{\lambda_a}{2}\right), \tag{3.1}$$

Θ_a: real numbers;
λ_a: Gell-Mann matrices satisfying

$$\frac{\lambda_a}{2} = i\frac{\partial U}{\partial \Theta_a}(0, \cdots, 0), \tag{3.2}$$

$$\lambda_a = \lambda_a^\dagger, \tag{3.3}$$

$$\mathrm{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}, \tag{3.4}$$

$$\mathrm{Tr}(\lambda_a) = 0. \tag{3.5}$$

Eq. (3.3) $\Rightarrow U^\dagger = U^{-1}$;
$\det[\exp(C)] = \exp[\mathrm{Tr}(C)]$ + Eq. (3.5) $\Rightarrow$ det(U)=1.

Explicit representation

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \sqrt{\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{3.6}$$

Definition:

Lie algebra su(3) of SU(3) $\equiv$ set of all complex, traceless, skew-Hermitian, 3×3 matrices.

Set $\{i\lambda_a\}$ constitutes basis of su(3).
Lie product defined in terms of ordinary matrix multiplication as commutator of two elements of su(3).
Lie properties of anti-commutativity

$$[A, B] = -[B, A] \tag{3.7}$$

and Jacobi identity

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0. \qquad (3.8)$$

Structure of SU(3) is encoded in commutation relations of Gell-Mann matrices,

$$\left[\frac{\lambda_a}{2},\frac{\lambda_b}{2}\right]=if_{abc}\frac{\lambda_c}{2}. \qquad (3.9)$$

Structure constants f_{abc}: real, totally symmetric. Given by (Assignment 5, 1. (a))

$$f_{abc}=\frac{1}{4i}\mathrm{Tr}([\lambda_a,\lambda_b]\lambda_c). \qquad (3.10)$$

abc	123	147	156	246	257	345	367	458	678
f_{abc}	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{3}$

Anticommutation relations

$$\{\lambda_a,\lambda_b\}=\frac{4}{3}\delta_{ab}+2d_{abc}\lambda_c. \qquad (3.11)$$

d_{abc} : real, totally symmetric. (Assignment 5, 1. (c))

$$d_{abc} = \frac{1}{4}\mathrm{Tr}(\{\lambda_a, \lambda_b\}\lambda_c), \tag{3.12}$$

abc	118	146	157	228	247	256	338	344
d_{abc}	$\frac{1}{\sqrt{3}}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{2}$
abc	355	366	377	448	558	668	778	888
d_{abc}	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$

Often useful:

$$\lambda_0 = \sqrt{2/3}\,\mathrm{diag}(1, 1, 1).$$

Eqs. (3.3) and (3.4) satisfied by the nine matrices λ_a. $\{i\lambda_a | a = 0, \cdots, 8\}$ basis of Lie algebra u(3) of U(3), i.e., the set of all complex, skew-Hermitian, 3×3 matrices.

- Many useful properties of the Gell-Mann matrices can be found in Sect. 8 of CORE (Compendium of relations) by V. I. Borodulin, R. N. Rogalyov, and S. R. Slabospitsky, hep-ph/9507456.

Arbitrary 3×3 matrix M can be written as

$$M = \sum_{a=0}^{8} \lambda_a M_a, \tag{3.13}$$

where M_a are complex numbers given by

$$M_a = \frac{1}{2}\mathrm{Tr}(\lambda_a M).$$

3.2 Local Symmetries and the QCD Lagrangian

Gauge principle: Tremendously successful method in elementary particle physics to generate interactions between matter fields through the exchange of (massless) gauge bosons.

3.2.1 QED

Quantum electrodynamics (QED) obtained from promoting global U(1) symmetry of free-electron Lagrangian,

$$\Psi \mapsto \exp(i\Theta)\Psi : \mathcal{L}_{\text{free}} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi \mapsto \mathcal{L}_{\text{free}}, \tag{3.14}$$

to local symmetry.

Q: What does that mean?

Parameter $0 \leq \Theta \leq 2\pi$ describing element of U(1) is allowed to vary smoothly in space-time, $\Theta \to \Theta(x)$ (sometimes referred to as gauging the U(1) group).
Requirement: Keep invariance of Lagrangian under local transformations.
Resolution: Introduce four-potential $\mathcal{A}_\mu$ into theory which transforms under gauge transformation $\mathcal{A}_\mu \mapsto \mathcal{A}_\mu + \partial_\mu \Theta / e$ ("gauging Lagrangian

with respect to U(1)"):

$$\mathcal{L}_{\text{QED}} = \bar{\Psi}[i\gamma^\mu(\partial_\mu - ie\mathcal{A}_\mu) - m]\Psi - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}, \tag{3.15}$$

where $e > 0$; $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$: field strength tensor (see Sect. 2.2). Introduce covariant derivative of Ψ,

$$D_\mu \Psi \equiv (\partial_\mu - ie\mathcal{A}_\mu)\Psi.$$

Criterion: Under so-called gauge transformation of the second kind

$$\Psi(x) \mapsto \exp[i\Theta(x)]\Psi(x), \quad \mathcal{A}_\mu(x) \mapsto \mathcal{A}_\mu(x) + \partial_\mu \Theta(x)/e, \tag{3.16}$$

$D_\mu \Psi$ transforms in the same way as Ψ itself:

$$\begin{aligned} D_\mu \Psi(x) &\mapsto D'_\mu \Psi'(x) \\ &= [\partial_\mu - ie\mathcal{A}_\mu(x) - i\partial_\mu \Theta(x)]e^{i\Theta(x)}\Psi(x) \\ &= e^{i\Theta(x)}[\partial_\mu + i\partial_\mu \Theta(x) - ie\mathcal{A}_\mu(x) - i\partial_\mu \Theta(x)]\Psi(x) \\ &= e^{i\Theta(x)}[\partial_\mu - ie\mathcal{A}_\mu(x)]\Psi(x). \end{aligned} \tag{3.17}$$

Remarks:

1. $\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$ term makes gauge potential a dynamical degree of freedom as opposed to a pure external field.

2. Mass term is excluded, since it would violate gauge invariance:

$$\frac{1}{2}M^2\mathcal{A}_\mu\mathcal{A}^\mu \mapsto \frac{1}{2}M^2(\mathcal{A}_\mu\mathcal{A}^\mu + \frac{2}{e}\partial_\mu\Theta\mathcal{A}^\mu + \frac{1}{e^2}\partial_\mu\Theta\partial^\mu\Theta) \neq \frac{1}{2}M^2\mathcal{A}_\mu\mathcal{A}^\mu.$$

   Gauge principle requires massless gauge bosons. (Masses of gauge fields can be induced through a spontaneous breakdown of the gauge symmetry.)

3. Identify $\mathcal{A}_\mu$ with electromagnetic four-potential and $\mathcal{F}_{\mu\nu}$ with field strength tensor containing electric and magnetic fields.

4. Gauge principle has (naturally) generated the interaction of the electromagnetic field with matter.

3.2.2 The QCD Lagrangian

Quantum chromodynamics (QCD) is the gauge theory of the strong interactions with color SU(3) as the underlying gauge group.

Ingredients

The matter fields of QCD are the so-called quarks which are spin-1/2 fermions, with six different flavors in addition to their three possible colors. (Masses from PDG: Review of Particle Physics, 2006).

flavor	u	d	s
charge [e]	$2/3$	$-1/3$	$-1/3$
mass [MeV]	$1.5 - 3.0$	$3 - 7$	95 ± 25
flavor	c	b	t
charge [e]	$2/3$	$-1/3$	$2/3$
mass [GeV]	1.25 ± 0.09	4.20 ± 0.07	174.2 ± 3.3

Quark field components

$$q_{f,A,\alpha}$$

$f = 1, 2, 3, 4, 5, 6$: flavor index (u, d, s, c, b, t)
$A = 1, 2, 3$: color index (red, green, blue)
$\alpha = 1, 2, 3, 4$: Dirac spinor index

Consider "free" quark Lagrangian without interaction:

$$\mathcal{L}_{0,\text{quarks}} = \sum_{f=1}^{6} \sum_{A=1}^{3} \sum_{\alpha,\alpha'=1}^{4} \bar{q}_{f,A,\alpha} (\gamma^{\mu}_{\alpha\alpha'} i\partial_{\mu} - m_f \delta_{\alpha\alpha'}) q_{f,A,\alpha'}. \quad (3.18)$$

Sum of $6 \times 3 = 18$ free fermion Lagrangians.
For each quark flavor f introduce color triplet

$$q_f = \begin{pmatrix} q_{f,1} \\ q_{f,2} \\ q_{f,3} \end{pmatrix}. \quad (3.19)$$

$\mathcal{L}_{0,\text{quarks}}$ is invariant under **global** SU(3) transformations of the q_f,

$$q_f \mapsto q'_f = U(\Theta) q_f = \exp\left[-i \sum_{a=1}^{8} \Theta_a \frac{\lambda_a^c}{2}\right] q_f. \tag{3.20}$$

Superscript c denotes Gell-Mann matrices acting in color space (usually omitted).

(In principle, each flavor invariant under separate transformations.)

Apply gauge principle with respect to the group SU(3) (*all* q_f tansformed by the same SU(3) matrix):

1. We need 8 gauge potentials to compensate for

$$\delta\mathcal{L}_{0,\text{quarks}} = i \sum_{f=1}^{8} \bar{q}_f \gamma^\mu \underbrace{U^\dagger\left(\Theta(x)\right) \partial_\mu U\left(\Theta(x)\right)}_{\text{acts in color space}} q_f.$$

2. Non-abelian nature of SU(3) $\Rightarrow$ field strength tensor more complicated.

QCD Lagrangian

$$\mathcal{L}_{\mathrm{QCD}} = \sum_{f,f'=1}^{6} \sum_{A,A'=1}^{3} \sum_{\alpha,\alpha'=1}^{4} \bar{q}_{f,A,\alpha} \Big[(\gamma^{\mu}_{\alpha\alpha'} i\partial_{\mu} - m_f \delta_{\alpha\alpha'}) \delta_{AA'} \underbrace{- g_3 \sum_{a=1}^{8} \mathcal{A}_{a\mu} \frac{\lambda^{c}_{a,AA'}}{2} \gamma^{\mu}_{\alpha\alpha'}}_{\text{from gauge principle}} \Big] \delta_{ff'} q_{f',A',\alpha'} - \sum_{a=1}^{8} \frac{1}{4} \mathcal{G}_{a\mu\nu} \mathcal{G}^{\mu\nu}_{a}$$

Short version

$$\mathcal{L}_{\mathrm{QCD}} = \sum_{f=\substack{u,d,s,\\ c,b,t}} \bar{q}_f (i\not{D} - m_f) q_f - \frac{1}{4} \mathcal{G}_{a\mu\nu} \mathcal{G}^{\mu\nu}_{a}. \tag{3.21}$$

Extremely short version

$$\mathcal{L}_{\mathrm{QCD}} = \bar{q}(i\not{D} - \mathcal{M}) q - \frac{1}{2} \mathrm{Tr}_c (\mathcal{G}_{\mu\nu} \mathcal{G}^{\mu\nu}).$$

$\mathcal{L}_{\text{QCD}}$ is invariant under the gauge transformations

$$q_f \;\mapsto\; q'_f = \exp\left[-i\sum_{a=1}^{8}\Theta_a(x)\frac{\lambda_a^c}{2}\right]q_f = U[\Theta(x)]q_f,$$

$$\mathcal{A}_\mu \equiv \mathcal{A}_{a\mu}\frac{\lambda_a^c}{2} \;\mapsto\; U\mathcal{A}_\mu U^\dagger + \frac{i}{g_3}\partial_\mu U U^\dagger.$$

Covariant derivative of the quark fields

$$D_\mu q_f \equiv (\partial_\mu + ig_3\mathcal{A}_\mu)q_f \overset{\text{Exercise:}}{\mapsto} (D_\mu q_f)' = D'_\mu q'_f = UD_\mu q_f$$

transforms as quark fields.
Field strengths transform as

$$\mathcal{G}_{\mu\nu} \equiv \mathcal{G}_{a\mu\nu}\frac{\lambda_a^c}{2} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + ig_3[\mathcal{A}_\mu,\mathcal{A}_\nu] \overset{\text{Exercise:}}{\mapsto} U\mathcal{G}_{\mu\nu}U^\dagger.$$

Equivalent (Gell-Mann matrices!)

$$\mathcal{G}_{a\mu\nu} = \partial_\mu\mathcal{A}_{a\nu} - \partial_\nu\mathcal{A}_{a\mu} - g_3 f_{abc}\mathcal{A}_{b\mu}\mathcal{A}_{c\nu}.$$

Exercise: $\mathcal{L}_{\rm QCD}$ invariant under local SU(3).

Note: Squared field strength tensor gives rise to gauge-field self interaction vertices with three and four gauge fields of strength g_3 and g_3^2, respectively. Characteristic of non-Abelian gauge theories, absent in Abelian gauge theories.

Gauge invariance also allows for (quark masses originate from electroweak symmetry breaking)

$$\begin{aligned}\mathcal{L}_\theta &= -\frac{g_3^2\bar{\theta}}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}\sum_{a=1}^{8}\mathcal{G}_{a\mu\nu}\mathcal{G}_{a\rho\sigma} \\ &= -\frac{g_3^2\bar{\theta}}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}_c(\mathcal{G}_{\mu\nu}\mathcal{G}_{\rho\sigma}),\end{aligned}$$

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \{\mu,\nu,\rho,\sigma\} \text{ even permutation of } \{0,1,2,3\} \\ -1 & \text{if } \{\mu,\nu,\rho,\sigma\} \text{ odd permutation of } \{0,1,2,3\} \\ 0 & \text{otherwise} \end{cases}$$

So-called θ term implies explicit P and CP violation in the strong interactions:

$$\begin{aligned} P: \quad & \mathcal{G}_{a\mu\nu}(t,\vec{x}) \mapsto \mathcal{G}_a^{\mu\nu}(t,-\vec{x}), \\ & \epsilon^{\mu\nu\rho\sigma} = -\epsilon_{\mu\nu\rho\sigma}, \\ \Rightarrow \quad & \mathcal{L}_\theta(t,\vec{x}) \mapsto -\mathcal{L}_\theta(t,-\vec{x}). \end{aligned}$$

$\Rightarrow$

1. electric dipole moment of the neutron; empirical information: very small.
2. $\eta \to 4\pi^0$ $(\to 8\gamma)$ (not (yet) observed).

3.3 Accidental, Global Symmetries of $\mathcal{L}_{\rm QCD}$

3.3.1 Light and Heavy Quarks

The pion is special!

quark content	mesons
$u\bar{d}$	π^+, ρ^+
$(u\bar{u} - d\bar{d})/\sqrt{2}$	π^0, ρ^0
$d\bar{u}$	π^-, ρ^-

$$\begin{aligned} M_{\pi^+} = 140\,\mathrm{MeV} &\ll M_\rho = 776\,\mathrm{MeV}, \\ M_\pi &\ll m_p = 938\,\mathrm{MeV}. \end{aligned}$$

$$M_{\pi^+} < M_{K^+} = 494\,\mathrm{MeV} \ll M_{\underbrace{D^+}_{c\bar{d}}} = 1869\,\mathrm{MeV}.$$

$$\begin{pmatrix} m_u = (1.5 - 3.0)\,\text{MeV} \\ m_d = (3 - 7)\,\text{MeV} \\ m_s = (95 \pm 25)\,\text{MeV} \end{pmatrix} \ll \Lambda_\chi \approx 1\,\text{GeV} \leq \begin{pmatrix} m_c = (1.25 \pm 0.09)\,\text{GeV} \\ m_b = (4.20 \pm 0.07)\,\text{GeV} \\ m_t = (174.2 \pm 3.3)\,\text{GeV} \end{pmatrix}$$

Motivation

$$m_p \gg 2m_u + m_d$$

Consider light-flavor quarks in so-called chiral limit $m_u, m_d, m_s \to 0$ as starting point in discussion of low-energy QCD:

$$\mathcal{L}^0_{\text{QCD}} = \sum_{l=u,d,s} \bar{q}_l i \not{D}\, q_l - \frac{1}{4}\mathcal{G}_{\mu\nu,a}\mathcal{G}^{\mu\nu}_a. \tag{3.22}$$

3.3.2 Left-Handed and Right-Handed Quark Fields

Recall: Starting point of EFT are underlying symmetries.

Q: **What are the global symmetries of $\mathcal{L}^0_{\text{QCD}}$?**

Chirality matrix

$$\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5^\dagger, \quad \{\gamma^\mu, \gamma_5\} = 0, \quad \gamma_5^2 = 1. \tag{3.23}$$

Projection operators

$$P_L = \frac{1}{2}(1-\gamma_5) = P_L^\dagger, \quad P_R = \frac{1}{2}(1+\gamma_5) = P_R^\dagger. \tag{3.24}$$

Exercise: Properties

$$P_L + P_R = 1,$$
$$P_L^2 = P_L, \quad P_R^2 = P_R,$$
$$P_L P_R = P_R P_L = 0.$$

Left- and right-handed quark fields q_L and q_R

$$q_L = P_L q, \quad q_R = P_R q \tag{3.25}$$

Explanation of terminology: Consider positive-energy spinor

$$u^{(r)}(\vec{p}) = \sqrt{E+m}\begin{pmatrix} \chi_r \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_r \end{pmatrix}.$$

Take extreme-relativistic limit $E \gg m$ with spin projection in positive/negative momentum direction, i. e.,

$$\vec{\sigma} \cdot \hat{p} \chi_{\pm} = \pm \chi_{\pm}.$$

$\Rightarrow$

$$u_{\pm}(\vec{p}) \stackrel{E \gg m}{=} \sqrt{E} \begin{pmatrix} \chi_{\pm} \\ \pm \chi_{\pm} \end{pmatrix}.$$

Make use of

$$P_R = \frac{1}{2} \begin{pmatrix} 1_{2\times2} & 1_{2\times2} \\ 1_{2\times2} & 1_{2\times2} \end{pmatrix} \quad \text{and} \quad P_L = \frac{1}{2} \begin{pmatrix} 1_{2\times2} & -1_{2\times2} \\ -1_{2\times2} & 1_{2\times2} \end{pmatrix}$$

to obtain

$$\begin{aligned} P_R u_+ &= \frac{1}{2} \begin{pmatrix} 1_{2\times2} & 1_{2\times2} \\ 1_{2\times2} & 1_{2\times2} \end{pmatrix} \sqrt{E} \begin{pmatrix} \chi_+ \\ \chi_+ \end{pmatrix} = \sqrt{E} \begin{pmatrix} \chi_+ \\ \chi_+ \end{pmatrix} = u_+, \\ P_L u_+ &= \frac{1}{2} \begin{pmatrix} 1_{2\times2} & -1_{2\times2} \\ -1_{2\times2} & 1_{2\times2} \end{pmatrix} \sqrt{E} \begin{pmatrix} \chi_+ \\ \chi_+ \end{pmatrix} = 0, \end{aligned}$$

$$
\begin{aligned}
P_R u_- &= \frac{1}{2}\begin{pmatrix} 1_{2\times 2} & 1_{2\times 2} \\ 1_{2\times 2} & 1_{2\times 2} \end{pmatrix} \sqrt{E} \begin{pmatrix} \chi_- \\ -\chi_- \end{pmatrix} = 0, \\
P_L u_- &= \frac{1}{2}\begin{pmatrix} 1_{2\times 2} & -1_{2\times 2} \\ -1_{2\times 2} & 1_{\times 2} \end{pmatrix} \sqrt{E} \begin{pmatrix} \chi_- \\ -\chi_- \end{pmatrix} = u_- .
\end{aligned}
$$

In the extreme relativistic (or in the zero-mass limit), the operators P_R and P_L project to the positive and negative helicity eigenstates. In this limit chirality equals helicity.

Goal: Analyze the symmetry of the QCD Lagrangian with respect to independent global transformations of the left- and right-handed fields. Make use of (Exercise)

$$
\bar{q}\Gamma_i q = \begin{cases} \bar{q}_L\Gamma_1 q_L + \bar{q}_R\Gamma_1 q_R & \text{for } \Gamma_1 \in \{\gamma^\mu, \gamma^\mu\gamma_5\} \\ \bar{q}_R\Gamma_2 q_L + \bar{q}_L\Gamma_2 q_R & \text{for } \Gamma_2 \in \{\mathbb{1}, \gamma_5, \sigma^{\mu\nu}\} \end{cases}, \tag{3.26}
$$

where

$$
\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]
$$

and

$$\bar{q}_R = q_R^\dagger \gamma_0 = q^\dagger P_R^\dagger \gamma_0 = q^\dagger P_R \gamma_0 = q^\dagger \gamma_0 P_L = \bar{q} P_L \quad \text{und} \quad \bar{q}_L = \bar{q} P_R.$$

QCD Lagrangian in the chiral limit

$$\mathcal{L}^0_{\text{QCD}} = \sum_{l=u,d,s} (\bar{q}_{L,l} i \not{D}\, q_{L,l} + \bar{q}_{R,l} i \not{D}\, q_{R,l}) - \frac{1}{4} \mathcal{G}_{a\mu\nu} \mathcal{G}_a^{\mu\nu} \tag{3.27}$$

invariant under (covariant derivative flavor independent!)

$$\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \mapsto U_L \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} = \exp\left(-i \sum_{a=1}^{8} \Theta_a^L \frac{\lambda_a^f}{2}\right) e^{-i\Theta^L} \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix},$$
$$\begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \mapsto U_R \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} = \exp\left(-i \sum_{a=1}^{8} \Theta_a^R \frac{\lambda_a^f}{2}\right) e^{-i\Theta^R} \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix}. \tag{3.28}$$

U_L and U_R: independent unitary 3×3 matrices.
Superscript f denotes Gell-Mann matrices acting in flavor space (will be omitted from now on).

$\mathcal{L}^0_{\mathrm{QCD}}$ has a classical *global* $\mathrm{U(3)}_L \times \mathrm{U(3)}_R$ symmetry.

Applying Noether's theorem from such an invariance one would expect a total of $2 \times (8+1) = 18$ conserved currents

3.3.3 Global Symmetry Currents of the Light-Quark Sector

Consider infinitesimal, **local** tranformations (Gell-Mann-Levý trick)

$$q_L \mapsto \left(1 - i\sum_{a=1}^{8} \epsilon_a^L(x)\frac{\lambda_a}{2} - i\epsilon^L(x)\right) q_L,$$

$$q_R \;\mapsto\; \left(1 - i\sum_{a=1}^{8}\epsilon_a^R(x)\frac{\lambda_a}{2} - i\epsilon^R(x)\right) q_R. \tag{3.29}$$

Variation (sign from $i \times (-i) = 1$)

$$\delta\mathcal{L}^0_{\mathrm{QCD}} = \bar{q}_L \left(\sum_{a=1}^{8}\partial_\mu\epsilon_a^L\frac{\lambda_a}{2} + \partial_\mu\epsilon^L\right)\gamma^\mu q_L + (L \to R). \tag{3.30}$$

Currents

$$\begin{aligned} L_a^\mu \;&=\; \frac{\partial\delta\mathcal{L}^0_{\mathrm{QCD}}}{\partial\partial_\mu\epsilon_a^L} = \bar{q}_L\gamma^\mu\frac{\lambda_a}{2}q_L, \quad \partial_\mu L_a^\mu = \frac{\partial\delta\mathcal{L}^0_{\mathrm{QCD}}}{\partial\epsilon_a^L} = 0, \\ L^\mu \;&=\; \frac{\partial\delta\mathcal{L}^0_{\mathrm{QCD}}}{\partial\partial_\mu\epsilon^L} = \bar{q}_L\gamma^\mu q_L, \quad \partial_\mu L^\mu = \frac{\partial\delta\mathcal{L}^0_{\mathrm{QCD}}}{\partial\epsilon^L} = 0, \end{aligned} \tag{3.31}$$

\+ analogous expressions for R_a^μ and R^μ.

Make use of

$$P_L\gamma^\mu P_R \pm P_R\gamma^\mu P_L = \gamma^\mu(P_R^2 \pm P_L^2) = \gamma^\mu(P_R \pm P_L) = \begin{cases} \gamma^\mu \\ \gamma^\mu\gamma_5 \end{cases}$$

$\Rightarrow$ linear combinations

$$V_a^\mu = R_a^\mu + L_a^\mu = \bar{q}\gamma^\mu \frac{\lambda_a}{2} q, \tag{3.32}$$

$$A_a^\mu = R_a^\mu - L_a^\mu = \bar{q}\gamma^\mu\gamma_5 \frac{\lambda_a}{2} q. \tag{3.33}$$

Transform under parity as vector and axial-vector current densities, respectively,

$$P : V_a^\mu(t,\vec{x}) \mapsto V_{a\mu}(t,-\vec{x}), \tag{3.34}$$

$$P : A_a^\mu(t,\vec{x}) \mapsto -A_{a\mu}(t,-\vec{x}), \tag{3.35}$$

because

$$P : q(t,\vec{x}) \mapsto \gamma_0 q(t,-\vec{x}), \quad \gamma_0\gamma^\mu\gamma_0 = \gamma_\mu, \quad \gamma_0\gamma^\mu\gamma_5\gamma_0 = -\gamma_\mu\gamma_5.$$

Conserved singlet vector current (from transformation of all left-handed and right-handed quark fields by the *same* phase)

$$V^{\mu} = R^{\mu} + L^{\mu} = \bar{q}\gamma^{\mu}q, \quad \partial_{\mu}V^{\mu} = 0. \tag{3.36}$$

Singlet axial-vector current (from transformation of all left-handed quark fields with one phase and all right-handed with the *opposite* phase)

$$A^{\mu} = R^{\mu} - L^{\mu} = \bar{q}\gamma^{\mu}\gamma_5 q. \tag{3.37}$$

This symmetry is not preserved by quantization and there will be extra terms, referred to as anomalies, resulting in

$$\partial_{\mu}A^{\mu} = \frac{3g^2}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathcal{G}_a^{\mu\nu}\mathcal{G}_a^{\rho\sigma}, \quad \epsilon_{0123} = 1.$$

Factor 3 originates from the number of flavors.

3.3.4 The Chiral Algebra

Define "charge operators" as space integrals of charge densities ($a = 1, \cdots, 8$)

$$Q_{aL}(t) = \int \mathrm{d}^3x\, q_L^\dagger(t,\vec{x})\frac{\lambda_a}{2}q_L(t,\vec{x}), \tag{3.38}$$

$$Q_{aR}(t) = \int \mathrm{d}^3x\, q_R^\dagger(t,\vec{x})\frac{\lambda_a}{2}q_R(t,\vec{x}), \tag{3.39}$$

$$Q_V(t) = \int \mathrm{d}^3x\, \underbrace{\left[q_L^\dagger(t,\vec{x})q_L(t,\vec{x}) + q_R^\dagger(t,\vec{x})q_R(t,\vec{x})\right]}_{q^\dagger q}. \tag{3.40}$$

QCD Hamilton operator in the chiral limit, H^0_{QCD}, exhibits global $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ symmetry:

$$[Q_{aL}, H^0_{\mathrm{QCD}}] = [Q_{aR}, H^0_{\mathrm{QCD}}] = [Q_V, H^0_{\mathrm{QCD}}] = 0. \tag{3.41}$$

Q: What are the commutation relations among the charges?
A: Lie algebra of $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$:

$$[Q_{aL}, Q_{bL}] = i f_{abc} Q_{cL}, \tag{3.42}$$
$$[Q_{aR}, Q_{bR}] = i f_{abc} Q_{cR}, \tag{3.43}$$
$$[Q_{aL}, Q_{bR}] = 0, \tag{3.44}$$
$$[Q_{aL}, Q_V] = [Q_{aR}, Q_V] = 0. \tag{3.45}$$

Q: How does one verify these commutation relations?

1. Anti-commutation relations of Fermi fields

$$\begin{aligned}
\{q_{f,A,\alpha}(t,\vec{x}), q^\dagger_{f',A',\alpha'}(t,\vec{y})\} &= \delta^3(\vec{x}-\vec{y})\delta_{ff'}\delta_{AA'}\delta_{\alpha\alpha'}, \\
\{q_{f,A,\alpha}(t,\vec{x}), q_{f',A',\alpha'}(t,\vec{y})\} &= 0, \\
\{q^\dagger_{f,A,\alpha}(t,\vec{x}), q^\dagger_{f',A',\alpha'}(t,\vec{y})\} &= 0.
\end{aligned}$$

2. Exercise: $[ab, cd] = a\{b,c\}d - ac\{b,d\} + \{a,c\}db - c\{a,d\}b.$

3. Let F_i, C_i, and Γ_i be 3×3 flavor matrices, 3×3 color matrices, 4×4 Dirac matrices, respectively:

$$
\begin{aligned}
&[q^\dagger(t,\vec{x})F_1C_1\Gamma_1 q(t,\vec{x}), q^\dagger(t,\vec{y})F_2C_2\Gamma_2 q(t,\vec{y})] \\
&= [\underbrace{q^\dagger_{a,A,\alpha}(t,\vec{x})}_{q^\dagger_i(t,\vec{x})}\underbrace{F_{1aa'}C_{1AA'}\Gamma_{1\alpha\alpha'}}_{M_{1ii'}}q_{a',A',\alpha'}(t,\vec{x}), q^\dagger_{b,B,\beta}(t,\vec{y})F_{2bb'}C_{2BB'}\Gamma_{2\beta\beta'}q_{b',B',\beta'}(t,\vec{y})] \\
&= M_{1ii'}M_{2jj'}[q^\dagger_i(t,\vec{x})q_{i'}(t,\vec{x}), q^\dagger_j(t,\vec{y})q_{j'}(t,\vec{y})] \\
&= M_{1ii'}M_{2jj'}(q^\dagger_i(t,\vec{x})q_{j'}(t,\vec{y})\delta^3(\vec{x}-\vec{y})\delta_{i'j} - q^\dagger_j(t,\vec{y})q_{i'}(t,\vec{x})\delta^3(\vec{x}-\vec{y})\delta_{ij'}) \\
&= \delta^3(\vec{x}-\vec{y})(q^\dagger_i(t,\vec{x})M_{1ij}M_{2jj'}q_{j'}(t,\vec{y}) - q^\dagger_j(t,\vec{y})M_{2ji}M_{1ii'}q_{i'}(t,\vec{x})) \\
&= \delta^3(\vec{x}-\vec{y})(q^\dagger(t,\vec{x})M_1M_2q(t,\vec{y}) - q^\dagger(t,\vec{y})M_2M_1q(t,\vec{x})) \\
&= \delta^3(\vec{x}-\vec{y})(q^\dagger(t,\vec{x})F_1F_2C_1C_2\Gamma_1\Gamma_2q(t,\vec{y}) - q^\dagger(t,\vec{y})F_2F_1C_2C_1\Gamma_2\Gamma_1q(t,\vec{x})). \quad (3.46)
\end{aligned}
$$

4. Insert appropriate projection operators

5. Integrate with respect to $\vec{x}$ and $\vec{y}$

Example (recall $P_L^\dagger = P_L$ and $P_L^2 = P_L$)

$$\begin{aligned}
[Q_{aL}, Q_{bL}] &= \int \mathrm{d}^3x\mathrm{d}^3y [q^\dagger(t,\vec{x})P_L^\dagger\frac{\lambda_a}{2}P_L q(t,\vec{x}), q^\dagger(t,\vec{y})P_L^\dagger\frac{\lambda_b}{2}P_L q(t,\vec{y})] \\
&= \int \mathrm{d}^3x\mathrm{d}^3y \delta^3(\vec{x}-\vec{y}) q^\dagger(t,\vec{x}) \underbrace{P_L^\dagger P_L P_L^\dagger P_L}_{P_L} \frac{\lambda_a}{2}\frac{\lambda_b}{2} q(t,\vec{y}) \\
&\quad - \int \mathrm{d}^3x\mathrm{d}^3y \delta^3(\vec{x}-\vec{y}) q^\dagger(t,\vec{y}) P_L \frac{\lambda_b}{2}\frac{\lambda_a}{2} q(t,\vec{x}) \\
&= i f_{abc} \int \mathrm{d}^3x q^\dagger(t,\vec{x}) \frac{\lambda_c}{2} P_L q(t,\vec{x}) = i f_{abc} Q_{cL}.
\end{aligned}$$

3.3.5 Chiral Symmetry Breaking Due to Quark Masses

So far: Idealized world.

Finite u-, d-, and s-quark masses $\Rightarrow$ explicit divergences of the symmetry currents.

Quark-mass matrix

$$\mathcal{M} = \text{diag}(m_u, m_d, m_s).$$

Quark-mass term $\sim \mathbb{1}_{4\times 4} \Rightarrow$ mixes left- and right-handed fields

$$\mathcal{L}_{\mathcal{M}} = -\bar{q}\mathcal{M}q = -(\bar{q}_R\mathcal{M}q_L + \bar{q}_L\mathcal{M}q_R). \tag{3.47}$$

Transformation of left-handed fields

$$q_L \mapsto \left(1 - i\sum_{a=1}^{8} \epsilon_a^L \frac{\lambda_a}{2} - i\epsilon^L\right) q_L.$$

Variation $\delta\mathcal{L}_{\mathcal{M}}$

$$\begin{aligned}\delta\mathcal{L}_{\mathcal{M}} &= -\left[-i\bar{q}_R\mathcal{M}\left(\sum_{a=1}^{8}\epsilon_a^L\frac{\lambda_a}{2} + \epsilon^L\right)q_L + i\bar{q}_L\left(\sum_{a=1}^{8}\epsilon_a^L\frac{\lambda_a}{2} + \epsilon^L\right)\mathcal{M}q_R\right] \\ &= -i\left[\sum_{a=1}^{8}\epsilon_a^L\left(\bar{q}_L\frac{\lambda_a}{2}\mathcal{M}q_R - \bar{q}_R\mathcal{M}\frac{\lambda_a}{2}q_L\right) + \epsilon^L\left(\bar{q}_L\mathcal{M}q_R - \bar{q}_R\mathcal{M}q_L\right)\right].\end{aligned}$$

Divergences

$$
\begin{aligned}
\partial_\mu L^\mu_a &= \frac{\partial \delta \mathcal{L}_\mathcal{M}}{\partial \epsilon^L_a} = -i\left(\bar{q}_L \frac{\lambda_a}{2}\mathcal{M} q_R - \bar{q}_R \mathcal{M}\frac{\lambda_a}{2} q_L\right), \\
\partial_\mu L^\mu &= \frac{\partial \delta \mathcal{L}_\mathcal{M}}{\partial \epsilon^L} = -i\left(\bar{q}_L \mathcal{M} q_R - \bar{q}_R \mathcal{M} q_L\right)
\end{aligned}
\tag{3.48}
$$

+ analogous expressions for $\partial_\mu R^\mu_a$ and $\partial_\mu R^\mu$ $(R \leftrightarrow L)$. More common (linear combinations)

$$
\begin{aligned}
\partial_\mu V^\mu_a &= i\bar{q}[\mathcal{M}, \frac{\lambda_a}{2}]q, \\
\partial_\mu A^\mu_a &= i\bar{q}\{\mathcal{M}, \frac{\lambda_a}{2}\}\gamma_5 q, \\
\partial_\mu V^\mu &= 0, \\
\partial_\mu A^\mu &= 2i\bar{q}\mathcal{M}\gamma_5 q + \frac{3g^2}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathcal{G}^{\mu\nu}_a \mathcal{G}^{\rho\sigma}_a, \quad \epsilon_{0123} = 1.
\end{aligned}
\tag{3.49}
$$

Summary

- Massless quarks: 16 conserved currents L_a^μ and R_a^μ (V_a^μ and A_a^μ) + 1 conserved singlet vector current V^μ. Singlet axial-vector current A^μ has an anomaly.

- For any value of quark masses: flavor currents $\bar{u}\gamma^\mu u$, $\bar{d}\gamma^\mu d$, and $\bar{s}\gamma^\mu s$ are always conserved.

- Equal quark masses $m_u = m_d = m_s$:

  8 conserved vector currents V_a^μ ($[\lambda_a, \mathbb{1}] = 0$).
  SU(3) flavor symmetry.

  8 axial-vector currents A_a^μ are not conserved.
  Microscopic origin of the PCAC relation (partially conserved axial-vector current).

- $m_u = m_d$: isospin symmetry.

3.4 Green Functions and Chiral Ward Identities

3.4.1 Chiral Green Functions

Motivation

Recall standard chain of arguments:

Continuous symmetries ⇒ conserved currents ⇒ time-independent charge operators:

$$[Q_a, H] = 0.$$

⇒

- classify spectrum in terms of multiplets;
- transformation properties of operators:

$$[Q_a, A_b] = c_{abc} A_c.$$

Apply Wigner-Eckart theorem ⇒ relation among matrix elements of the same type (Clebsch-Gordan coefficients + reduced matrix elements); example: pion-nucleon scattering.

But: **There is more to symmetries!**

1. QFT: Objects of interest are Green functions = matrix elements of time-ordered products.
2. Pictorially: Green functions = vertices related to physical scattering amplitudes through the Lehmann-Symanzik-Zimmermann (LSZ) reduction formalism.
3. If we know all Green functions, we have completely solved the QFT.
4. Symmetries provide strong constraints for
   (a) transformation behavior of scattering amplitudes;
   (b) Green functions, in particular, relations among different Green functions (so-called Ward identities).

Famous example: Ward identity of QED associated with U(1) gauge invariance,

$$(p'_\mu - p_\mu)\Gamma^\mu(p', p) = {S'_F}^{-1}(p') - {S'_F}^{-1}(p). \tag{3.50}$$

Relates electromagnetic vertex of an electron (3-point Green function) to inverse propagator (2-point Green function).
Illustration in lowest-order perturbation theory:

$$\begin{aligned}\Gamma^\mu(p', p) &= \gamma^\mu, \\ S'_F(p) &= \frac{1}{\not p - m},\end{aligned}$$

$$(p'_\mu - p_\mu)\gamma^\mu = \not p' - \not p = (\not p' - m) - (\not p - m) = {S'_F}^{-1}(p') - {S'_F}^{-1}(p).$$

Symmetry currents relevant to $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$:

$$V_a^\mu = R_a^\mu + L_a^\mu = \bar{q}\gamma^\mu \frac{\lambda_a}{2} q, \tag{3.51}$$

$$V^\mu = \bar{q}\gamma^\mu q, \tag{3.52}$$

$$A_a^\mu = R_a^\mu - L_a^\mu = \bar{q}\gamma^\mu\gamma_5\frac{\lambda_a}{2}q, \tag{3.53}$$

+ scalar and pseudoscalar densities (see divergences of currents)

$$S_a(x) = \bar{q}(x)\lambda_a q(x), \tag{3.54}$$

$$P_a(x) = i\bar{q}(x)\gamma_5\lambda_a q(x), \tag{3.55}$$

where $a = 0, \cdots, 8$.

Some examples of Green functions

"Vacuum" sector

$\langle 0\|T[A_a^\mu(x)P_b(y)]\|0\rangle$	pion decay
$\langle 0\|T[P_a(x)J^\mu(y)P_c(z)]\|0\rangle$	pion electromagnetic form factor
$\langle 0\|T[P_a(w)P_b(x)P_c(y)P_d(z)]\|0\rangle$	pion-pion scattering

One-nucleon sector

$\langle N|J^\mu(x)|N\rangle$ nucleon electromagnetic form factors

$$\begin{array}{rl} \langle N|A_a^\mu(x)|N\rangle & \text{axial form factor + induced pseudoscalar form factor} \\ \langle N|T[J^\mu(x)J^\nu(y)]|N\rangle & \text{Compton scattering} \\ \langle N|T[J^\mu(x)P_a(y)]|N\rangle & \text{pion electroproduction} \end{array}$$

A **chiral Ward identity** relates the divergence of a Green function containing at least one factor of V_a^μ or A_a^μ [see Eqs. (3.51) and (3.53)] to some linear combination of other Green functions.

Q: Why chiral?
A: V_a^μ and A_a^μ contain L_a^μ and R_a^μ.

Simple example (The time ordering of n points $x_1, \cdots, x_n$ gives rise to $n!$ distinct orderings, each involving products of $n-1$ theta functions):

$$\begin{aligned} G^\mu_{AP\,ab}(x,y) &= \langle 0|T[A_a^\mu(x)P_b(y)]|0\rangle \\ &= \Theta(x_0-y_0)\langle 0|A_a^\mu(x)P_b(y)|0\rangle + \Theta(y_0-x_0)\langle 0|P_b(y)A_a^\mu(x)|0\rangle. \end{aligned} \tag{3.56}$$

Evaluate divergence

$$
\begin{aligned}
&\partial_\mu^x G^\mu_{AP\,ab}(x,y)\\
&= \partial_\mu^x[\Theta(x_0-y_0)\langle 0|A_a^\mu(x)P_b(y)|0\rangle + \Theta(y_0-x_0)\langle 0|P_b(y)A_a^\mu(x)|0\rangle],
\end{aligned}
$$

make use of

$$
\partial_\mu^x\Theta(x_0-y_0) = \delta(x_0-y_0)g_{0\mu} = -\partial_\mu^x\Theta(y_0-x_0),
$$

$$
\begin{aligned}
\cdots &= \delta(x_0-y_0)\langle 0|A_a^0(x)P_b(y)|0\rangle - \delta(x_0-y_0)\langle 0|P_b(y)A_a^0(x)|0\rangle\\
&\quad +\Theta(x_0-y_0)\langle 0|\partial_\mu^x A_a^\mu(x)P_b(y)|0\rangle + \Theta(y_0-x_0)\langle 0|P_b(y)\partial_\mu^x A_a^\mu(x)|0\rangle\\
&= \delta(x_0-y_0)\langle 0|[A_a^0(x),P_b(y)]|0\rangle + \langle 0|T[\partial_\mu^x A_a^\mu(x)P_b(y)]|0\rangle.
\end{aligned}
$$

Main features of (chiral) Ward identities:

1. Differentiation of the theta functions $\Rightarrow$ Equal-time commutators between a charge density and the remaining quadratic forms $\Rightarrow$ Reflection of underlying symmetry. Generation of $\delta^4(x-y)$, reduction by one power of quark bilinears [see Eq. (3.46)].

2. Divergence of the current operator in question.

- Perfect symmetry $\Rightarrow$ such terms vanish
  Example: Electromagnetic case with its U(1) symmetry.
- Approximate symmetry $\Rightarrow$ additional term involving the symmetry breaking appears.
  Soft breaking: treat divergence as a perturbation.

Generalization to $(n+1)$-point Green function is symbolically of the form

$$\begin{aligned}
&\partial_\mu^x \langle 0|T\{J^\mu(x)A_1(x_1)\cdots A_n(x_n)\}|0\rangle = \\
&\quad \langle 0|T\{[\partial_\mu^x J^\mu(x)]A_1(x_1)\cdots A_n(x_n)\}|0\rangle \\
&\quad +\delta(x^0-x_1^0)\langle 0|T\{[J_0(x),A_1(x_1)]A_2(x_2)\cdots A_n(x_n)\}|0\rangle \\
&\quad +\delta(x^0-x_2^0)\langle 0|T\{A_1(x_1)[J_0(x),A_2(x_2)]\cdots A_n(x_n)\}|0\rangle \\
&\quad +\cdots+\delta(x^0-x_n^0)\langle 0|T\{A_1(x_1)\cdots[J_0(x),A_n(x_n)]\}|0\rangle, \quad (3.57)
\end{aligned}$$

where J^μ stands generically for any of the Noether currents.

3.4.2 QCD in the Presence of External Fields and the Generating Functional

So far: Explicitly work out the chiral Ward identity you are interested in.

Q: Is it possible to somehow obtain **all** chiral Ward identities from a single expression?
A: Yes (without proof)

1. Introduce into the Lagrangian of QCD the couplings of the

   (a) nine vector currents,

   (b) eight axial-vector currents,

   (c) nine scalar quark densities,

   (d) nine pseudoscalar quark densities

   to external c-number fields $v^\mu(x)$, $v^\mu_{(s)}$, $a^\mu(x)$, $s(x)$, and $p(x)$:

$$\mathcal{L} = \mathcal{L}^0_{\rm QCD} + \mathcal{L}_{\rm ext},$$

$$\mathcal{L}_{\text{ext}} = \bar{q}\gamma_\mu\left(v^\mu + \frac{1}{3}v^\mu_{(s)} + \gamma_5 a^\mu\right)q - \bar{q}(s - i\gamma_5 p)q.$$

Parameterization

$$v^\mu = \sum_{a=1}^{8} \frac{\lambda_a}{2} v_a^\mu, \quad a^\mu = \sum_{a=1}^{8} \frac{\lambda_a}{2} a_a^\mu, \quad s = \sum_{a=0}^{8} \lambda_a s_a, \quad p = \sum_{a=0}^{8} \lambda_a p_a.$$

2. Combine all Green functions in a generating functional

$$\exp(iZ[v, a, s, p]) = \langle 0|T \exp\left[i \int \mathrm{d}^4x \mathcal{L}_{\text{ext}}(x)\right]|0\rangle_0.$$

Note: Quark field operators q in $\mathcal{L}_{\text{ext}}$ and ground state $|0\rangle$ refer to the chiral limit, indicated by subscript 0.

3. Obtain Green function through a functional derivative with respect to the external fields.

Some remarks on functional derivatives (as we need them):

Functional derivatives are natural generalizations of classical partial derivatives to infinite dimensions.

Let $\mathcal{F}$ denote a set of functions, e.g., $\mathcal{F} = C^\infty(\mathbb{R}^n, \mathbb{R})$ (infinitely differentiable functions).
Functional F: Mapping from $\mathcal{F}$ to $\mathbb{R}$ or $\mathbb{C}$, i.e., function $\mapsto$ real or complex number.

Typical example: Integral of the type

$$F[f] = \int \mathrm{d}^n x g(f(x)),$$

with g integrable $C^\infty(\mathbb{R}, \mathbb{R})$ function.

- Often used convention: Arguments of functionals are written in square brackets.
- Let f be function of two sets of variables, collectively denoted by

x and y. $F[f(y)]$ denotes functional which depends on values of f for all x at fixed y.

Functional derivative

Consider Dirac's delta function

$$\delta_{\vec{y}} : \begin{cases} \mathbb{R}^n \to \mathbb{R}, \\ \vec{x} \mapsto \delta_{\vec{y}}(\vec{x}\,) = \delta^n(\vec{x} - \vec{y}\,). \end{cases}$$

Introduce functional derivative as

$$\frac{\delta F[f]}{\delta f(\vec{y}\,)} := \lim_{\epsilon \to 0} \frac{F[f + \epsilon \delta_{\vec{y}}] - F[f]}{\epsilon}. \tag{3.58}$$

Analogy to partial derivative of ordinary function

$$\frac{\partial f(\vec{x}\,)}{\partial x_i} := \lim_{\epsilon \to 0} \frac{f(\vec{x} + \epsilon \hat{e}_i) - f(\vec{x}\,)}{\epsilon}.$$

Basic properties:

$$\begin{aligned}\text{Linearity:}\quad & \frac{\delta}{\delta f(\vec{x})}\left(\alpha_1 F_1[f] + \alpha_2 F_2[f]\right) = \alpha_1 \frac{\delta F_1[f]}{\delta f(\vec{x})} + \alpha_2 \frac{\delta F_2[f]}{\delta f(\vec{x})},\\ \text{product rule:}\quad & \frac{\delta}{\delta f(\vec{x})}(F_1[f]F_2[f]) = \frac{\delta F_1[f]}{\delta f(\vec{x})} F_2[f] + F_1[f] \frac{\delta F_2[f]}{\delta f(\vec{x})},\\ \text{chain rule:}\quad & \frac{\delta}{\delta f(\vec{x})} F[g(f)] = g'(f(\vec{x})) \frac{\delta F}{\delta h(\vec{x})}[h = g(f)].\end{aligned}$$

Important rule (functional derivative of a funtion):

$$\frac{\delta f(\vec{y})}{\delta f(\vec{x})} = \delta^n(\vec{y} - \vec{x}),$$

because: Define $f(\vec{y})$ as functional

$$f(\vec{y}) = F_{\vec{y}}[f] = \int \mathrm{d}^n x \delta^n(\vec{y} - \vec{x}) f(\vec{x})$$

and consider functional derivative

$$
\begin{aligned}
\frac{\delta F_{\vec{y}}[f]}{\delta f(\vec{x}\,)} &= \lim_{\epsilon\to 0} \frac{F_{\vec{y}}[f+\epsilon\delta_{\vec{x}}] - F_{\vec{y}}[f]}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{\int \mathrm{d}^n x\,'\delta^n(\vec{y}-\vec{x}\,')(f(\vec{x}\,') + \epsilon\delta^n(\vec{x}\,'-\vec{x}\,)) - \int \mathrm{d}^n x'\delta^n(\vec{y}-\vec{x}\,')f(\vec{x}\,')}{\epsilon} \\
&= \int \mathrm{d}^n x\,'\delta^n(\vec{y}-\vec{x}\,')\delta^n(\vec{x}\,'-\vec{x}\,) = \delta^n(\vec{y}\,-\vec{x}\,).
\end{aligned}
$$

Analogously:

$$
\frac{\delta g(f(\vec{y}\,))}{\delta f(\vec{x}\,)} = \delta^n(\vec{y}-\vec{x}\,)g'(f(\vec{y}\,))
$$

and

$$
\frac{\delta^k g(f(\vec{y}\,))}{\delta f(\vec{x}_k)\cdots\delta f(\vec{x}_1)} = \delta^n(\vec{y}-\vec{x}_k)\cdots\delta^n(\vec{y}-\vec{x}_1)g^{(k)}(f(\vec{y}\,)).
$$

4. Pedagogical illustration

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_0(\phi) + \mathcal{L}_{\rm ext},\\ \mathcal{L}_{\rm ext} &= j(x)\phi(x)\end{aligned}$$

Generating functional for Green functions of the type

$$G(x_1, \cdots, x_n) = \langle 0|T[\phi(x_1)\cdots\phi(x_n)]|0\rangle$$

$$\begin{aligned}\exp(iZ[j]) &= \langle 0|T\exp\left[i\int \mathrm{d}^4x\mathcal{L}_{\rm ext}(x)\right]|0\rangle\\ &= 1 + i\int \mathrm{d}^4xj(x)\langle 0|\phi(x)|0\rangle\\ &\quad + \sum_{k=2}^{\infty}\frac{i^k}{k!}\int \mathrm{d}^4x_1\cdots\mathrm{d}^4x_kj(x_1)\cdots j(x_k)\langle 0|T[\phi(x_1)\cdots\phi(x_k)]|0\rangle\\ &= \cdots + \frac{i^2}{2}\int \mathrm{d}^4x_1\mathrm{d}^4x_2j(x_1)j(x_2)\langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle + \cdots.\end{aligned}$$

Remark: In many textbooks you will find the nomenclature $Z[j]$ for our $\exp(iZ[j])$ and $W[j]$ for our $Z[j]$. We follow Gasser and Leutwyler.

Example

$$\begin{aligned} G(x_1,x_2) &= \langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle \\ &= (-i)^2 \frac{\delta^2 \exp(iZ[j])}{\delta j(x_1)\delta j(x_2)}\bigg|_{j=0}. \end{aligned}$$

Powers and sort of functional derivatives must match:

$$1, \quad i\int \mathrm{d}^4 x j(x)\langle 0|\phi(x)|0\rangle : \quad \text{too few terms}$$

$$\frac{i^k}{k!}\int \mathrm{d}^4 x_1 \cdots \mathrm{d}^4 x_k j(x_1)\cdots j(x_k)\langle 0|\phi(x_1)\cdots\phi(x_k)|0\rangle, k\geq 3 :$$

too many terms, because j is set equal to 0 at the end.

Analogy: Consider the series $(a_0, a_1, a_2, \cdots)$. Define generating function f as

$$f(x) = a_0 + a_1 x + \frac{1}{2} a_2 x^2 + \frac{1}{3!} a_3 x^3 + \cdots .$$

The element a_n is obtained as

$$a_n = \frac{\mathrm{d}^n f}{\mathrm{d}\, x^n}(x = 0).$$

Exercise: Make use of

$$\frac{\delta j(x)}{\delta j(y)} = \delta^4(x - y)$$

to obtain

$$\frac{\delta^2}{\delta j(x_1) \delta j(x_2)} \frac{1}{2} \int \mathrm{d}^4 x \mathrm{d}^4 y j(x) j(y) \langle 0 | T[\phi(x)\phi(y)] | 0 \rangle = \langle 0 | T[\phi(x_1)\phi(x_2)] | 0 \rangle .$$

5. Back to QCD

Recall

$$\mathcal{L} = \mathcal{L}^0_{\text{QCD}} + \mathcal{L}_{\text{ext}},$$
$$\exp(iZ[v,a,s,p]) = \langle 0|T\exp\left[i\int \mathrm{d}^4x\mathcal{L}_{\text{ext}}(x)\right]|0\rangle_0,$$

where

$$\mathcal{L}_{\text{ext}} = \underbrace{\sum_{a=1}^{8} v_a^\mu \bar{q}\gamma_\mu \frac{\lambda_a}{2} q}_{\bar{q}v^\mu\gamma_u q} + v_{(s)}^\mu \frac{1}{3}\bar{q}\gamma_\mu q + \sum_{a=1}^{8} a_a^\mu \bar{q}\gamma_\mu\gamma_5 \frac{\lambda_a}{2} q$$
$$- \sum_{a=0}^{8} s_a \bar{q}\lambda_a q + \sum_{a=0}^{8} p_a i\bar{q}\gamma_5\lambda_a q,$$

i.e. we have 35 real functions v_a^μ, $v_{(s)}^\mu$, a_a^μ, s_a, and p_a, collectively denoted by $[v,a,s,p]$.

Remark: The subscript 0 reminds us that both quark fields and ground state are considered in the chiral limit.

For example

$$V_a^\mu(x) = \bar{q}(x)\gamma^\mu \frac{\lambda_a}{2} q(x) = \frac{\delta}{\delta v_{a\,\mu}(x)} \int \mathrm{d}^4 y \mathcal{L}_{\text{ext}}(y).$$

- Scalar quark condensate in the chiral limit, $\langle 0|\bar{u}u|0\rangle_0$,

$$\langle 0|\bar{u}(x)u(x)|0\rangle_0 = \frac{i}{2}\left[\sqrt{\frac{2}{3}}\frac{\delta}{\delta s_0(x)} + \frac{\delta}{\delta s_3(x)} + \frac{1}{\sqrt{3}}\frac{\delta}{\delta s_8(x)}\right] \exp(iZ[v,a,s,p])\Big|_{v=a=s=p=0}$$

Note: Express $\bar{u}u$ as $\bar{q}Mq$ with appropriate matrix M, namely

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Make use of

$$M = \sum_{a=0}^{8} M_a \lambda_a, \quad M_a = \frac{1}{2}\mathrm{Tr}(\lambda_a M),$$

$\Rightarrow$

$$\bar{u}u = \frac{1}{2}\sqrt{\frac{2}{3}}\bar{q}\lambda_0 q + \frac{1}{2}\bar{q}\lambda_3 q + \frac{1}{2}\frac{1}{\sqrt{3}}\bar{q}\lambda_8 q.$$

- We can even obtain Green functions of the "real world," where quark fields and ground state are those with finite quark masses.
- Two-point function of two axial-vector currents of the "real world," i.e., for $s = \mathrm{diag}(m_u, m_d, m_s)$, and the "true vacuum" $|0\rangle$,

$$\langle 0|T[A_a^\mu(x) A_b^\nu(0)]|0\rangle = \\ (-i)^2 \frac{\delta^2}{\delta a_{a\,\mu}(x)\delta a_{b\,\nu}(0)} \exp(iZ[v,a,s,p])\Big|_{v=a=p=0, s=\mathrm{diag}(m_u,m_d,m_s)}.$$

Note: Left-hand side involves quark fields and ground state of "real world," right-hand side is generating functional defined in terms quark fields and ground state of the chiral limit.

6. Q: But where is QCD?

   A: In $|0\rangle$ and q (solutions to EOM).

   (The actual value of the generating functional for a given configuration of external fields v, a, s, and p reflects the dynamics generated by the QCD Lagrangian.)

7. Q: But where is the (infinite) set of *all* chiral Ward identities?

   A: Ward identities obeyed by the Green functions are equivalent to an invariance of the generating functional under a **local** transformation of the external fields

8. The use of local transformations allows one to also consider divergences of Green functions.

9. Q: What do we require of the external fields?

A: We want $\mathcal{L}$ to be a Hermitian Lorentz scalar, to be even under P, C, and T, and to be invariant under *local* chiral transformations.

What does that imply for the external fields?

- **Parity**

Transformation behavior of quark fields:

$$q_f(t,\vec{x}) \stackrel{P}{\mapsto} \gamma^0 q_f(t,-\vec{x}).$$

Properties of the Dirac matrices Γ:

Γ	1	γ^μ	$\sigma^{\mu\nu}$	γ_5	$\gamma^\mu\gamma_5$
$\gamma_0\Gamma\gamma_0$	1	γ_μ	$\sigma_{\mu\nu}$	$-\gamma_5$	$-\gamma_\mu\gamma_5$

Requirement of parity conservation:

$$\mathcal{L}(t,\vec{x}) \stackrel{P}{\mapsto} \mathcal{L}(t,-\vec{x}).$$

$\Rightarrow$

$$v^\mu \overset{P}{\mapsto} v_\mu, \quad v^\mu_{(s)} \overset{P}{\mapsto} v_\mu^{(s)}, \quad a^\mu \overset{P}{\mapsto} -a_\mu, \quad s \overset{P}{\mapsto} s, \quad p \overset{P}{\mapsto} -p.$$

(Change of arguments from $(t,\vec{x})$ to $(t,-\vec{x})$ implied.)

Example:

$$\bar{q}(t,\vec{x})\gamma^\mu v_\mu(t,\vec{x})q(t,\vec{x}) \overset{P}{\mapsto} \bar{q}(t,-\vec{x})\gamma^0\gamma^\mu\tilde{v}_\mu(t,-\vec{x})\gamma^0 q(t,-\vec{x})$$

Tilde denotes the transformed external field.

Make use of table, i.e., $\gamma^0\gamma^\mu\gamma^0 = \gamma_\mu$,

$$\cdots = \bar{q}(t,-\vec{x})\gamma_\mu\tilde{v}_\mu(t,-\vec{x})q(t,-\vec{x}) \overset{!}{=} \bar{q}(t,-\vec{x})\gamma_\mu v^\mu(t,-\vec{x})q(t,-\vec{x}).$$

We thus obtain

$$v_\mu(t,\vec{x}) \overset{P}{\mapsto} v^\mu(t,-\vec{x}).$$

• **Charge conjugation**

Transformation behavior of the quark fields

$$q_{\alpha,f} \overset{C}{\mapsto} C_{\alpha\beta}\bar{q}_{\beta,f}, \quad \bar{q}_{\alpha,f} \overset{C}{\mapsto} -q_{\beta,f}C^{-1}_{\beta\alpha},$$

α and β: Dirac spinor indices,

f: flavor index

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = -C^{-1} = -C^{\dagger} = -C^T$$

usual charge conjugation matrix.

Properties of the Dirac matrices Γ:

Γ	1	γ^μ	$\sigma^{\mu\nu}$	γ_5	$\gamma^\mu\gamma_5$
$-C\Gamma^T C$	1	$-\gamma^\mu$	$-\sigma^{\mu\nu}$	γ_5	$\gamma^\mu\gamma_5$

Using

$$\begin{aligned}
\bar{q}\Gamma F q &= \bar{q}_{\alpha,f}\Gamma_{\alpha\beta}F_{ff'}q_{\beta,f'} \\
&\overset{C}{\mapsto} -q_{\gamma,f}C^{-1}_{\gamma\alpha}\Gamma_{\alpha\beta}F_{ff'}C_{\beta\delta}\bar{q}_{\delta,f'} \\
&\overset{\text{Fermi statistics}}{=} \bar{q}_{\delta,f'}\underbrace{F_{ff'}}_{F^T_{f'f}}\underbrace{C^{-1}_{\gamma\alpha}\Gamma_{\alpha\beta}C_{\beta\delta}}_{(C^{-1}\Gamma C)^T_{\delta\gamma}}q_{\gamma,f} \\
&= \bar{q}F^T\underbrace{(C^{-1}\Gamma C)^T}_{C^T\Gamma^T C^{-1\,T}}q \\
&= -\bar{q}C\Gamma^T C F^T q.
\end{aligned}$$

Invariance of $\mathcal{L}_{\text{ext}}$ under charge conjugation requires the transformation properties

$$v_\mu \overset{C}{\to} -v_\mu^T, \quad v_\mu^{(s)} \overset{C}{\to} -v_\mu^{(s)T}, \quad a_\mu \overset{C}{\to} a_\mu^T, \quad s,p \overset{C}{\to} s^T, p^T,$$

transposition refers to the flavor space.

- **Time reversal**: Nothing new (because of CPT theorem).
- **Local chiral $\mathbf{SU(3)}_L \times \mathbf{SU(3)}_R \times \mathrm{U}(1)_V$ transformations**

  First step: Rewrite in terms of the left- and right-handed quark fields.

  Exercise

  We first define

$$r_\mu = v_\mu + a_\mu, \quad l_\mu = v_\mu - a_\mu.$$

1. Make use of the projection operators P_L and P_R and verify

$$\bar{q}\gamma^\mu(v_\mu + \frac{1}{3}v_\mu^{(s)} + \gamma_5 a_\mu)q =$$
$$\bar{q}_R\gamma^\mu\left(r_\mu + \frac{1}{3}v_\mu^{(s)}\right)q_R + \bar{q}_L\gamma^\mu\left(l_\mu + \frac{1}{3}v_\mu^{(s)}\right)q_L.$$

2. Also verify

$$\bar{q}(s - i\gamma_5 p)q = \bar{q}_L(s - ip)q_R + \bar{q}_R(s + ip)q_L.$$

$\Rightarrow$ QCD Lagrangian with coupling to external fields

$$\begin{aligned}\mathcal{L} &= \mathcal{L}^0_{\text{QCD}} + \bar{q}_L\gamma^\mu\left(l_\mu + \frac{1}{3}v^{(s)}_\mu\right)q_L + \bar{q}_R\gamma^\mu\left(r_\mu + \frac{1}{3}v^{(s)}_\mu\right)q_R \\ &\quad -\bar{q}_R(s+ip)q_L - \bar{q}_L(s-ip)q_R. \end{aligned} \tag{3.59}$$

Eq. (3.59) remains invariant under *local* transformations

$$\begin{aligned} q_R &\mapsto \exp\left(-i\frac{\Theta(x)}{3}\right)V_R(x)q_R, \\ q_L &\mapsto \exp\left(-i\frac{\Theta(x)}{3}\right)V_L(x)q_L, \end{aligned} \tag{3.60}$$

$V_R(x)$ and $V_L(x)$: independent space-time-dependent SU(3) matrices, provided the external fields are subject to the transformations

$$\begin{aligned} r_\mu &\mapsto V_R r_\mu V_R^\dagger - i\partial_\mu V_R V_R^\dagger, \\ l_\mu &\mapsto V_L l_\mu V_L^\dagger - i\partial_\mu V_L V_L^\dagger, \\ v^{(s)}_\mu &\mapsto v^{(s)}_\mu - \partial_\mu\Theta, \end{aligned}$$

$$
\begin{aligned}
s + ip &\mapsto V_R(s + ip)V_L^\dagger, \\
s - ip &\mapsto V_L(s - ip)V_R^\dagger.
\end{aligned} \tag{3.61}
$$

(Derivative terms in serve the same purpose as in the construction of gauge theories, i.e., they cancel analogous terms originating from the kinetic part of the quark Lagrangian. Note: External currents are coupled with "opposite" sign in comparison with our convention for gauge theories.)

- Practical implications of the local invariance

  Allows one to also discuss a coupling to external gauge fields in the transition to the EFT.

  1. Coupling of the electromagnetic field to point-like fundamental particles results from gauging a U(1) symmetry. Here, the corresponding U(1) group is to be understood as a subgroup of a local $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$.

2. Interaction of the light quarks with the charged and neutral gauge bosons of the weak interactions.

Q: What do we have to insert for the external fields to describe the **electromagnetic interaction** of quarks?

A:

$$r_\mu = l_\mu = -eQ\mathcal{A}_\mu, \quad Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} : \quad \text{quark charge matrix}$$

Verification

$$\begin{aligned} \mathcal{L}_{\text{ext}} &= -e\mathcal{A}_\mu(\bar{q}_L Q\gamma^\mu q_L + \bar{q}_R Q\gamma^\mu q_R) = -e\mathcal{A}_\mu \bar{q} Q\gamma^\mu q \\ &= -e\mathcal{A}_\mu \left(\frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d - \frac{1}{3}\bar{s}\gamma^\mu s\right) \\ &= -e\mathcal{A}_\mu J^\mu. \end{aligned}$$

"SU(2) version" of ChPT:

$$r_\mu = l_\mu = -e\frac{\tau_3}{2}\mathcal{A}_\mu, \quad v_\mu^{(s)} = -\frac{e}{2}\mathcal{A}_\mu,$$

because

$$Q = \frac{1}{6}1_{2\times 2} + \frac{\tau_3}{2}.$$

Chapter 4

Spontaneous Symmetry Breaking and the Goldstone Theorem

4.1 Spontaneous Breakdown of a Global, Continuous, Non-Abelian Symmetry

Spontaneous symmetry breaking occurs if the ground state has a lower symmetry than the Hamiltonian.
Illustration of relevant features in terms of O(3) "sigma model:"

$$\mathcal{L}(\vec{\Phi}, \partial_\mu \vec{\Phi}) = \mathcal{L}(\Phi_1, \Phi_2, \Phi_3, \partial_\mu \Phi_1, \partial_\mu \Phi_2, \partial_\mu \Phi_3)$$

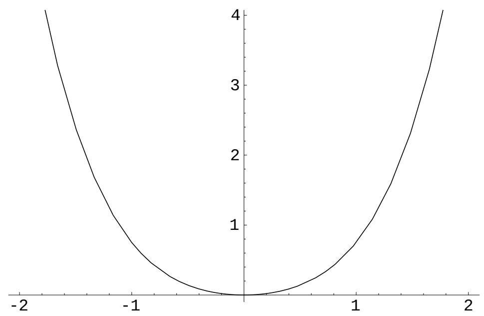
$$= \frac{1}{2}\partial_\mu \Phi_i \partial^\mu \Phi_i - \frac{m^2}{2}\Phi_i\Phi_i - \frac{\lambda}{4}(\Phi_i\Phi_i)^2. \qquad (4.1)$$

Φ_i: Real (hermitian) fields.
Hamilton density

$$\mathcal{H} = \Pi_i\dot{\Phi}_i - \mathcal{L} = \underbrace{\underbrace{\frac{1}{2}\Pi_i^2}_{\frac{1}{2}\dot{\Phi}_i^2} + \frac{1}{2}\vec{\nabla}\Phi_i \cdot \vec{\nabla}\Phi_i}_{\geq 0} + \underbrace{\frac{m^2}{2}\Phi_i\Phi_i + \frac{\lambda}{4}(\Phi_i\Phi_i)^2}_{\mathcal{V}(\vec{\Phi})}. \qquad (4.2)$$

$\lambda > 0$: $\mathcal{H}$ bounded from below.
Fields $\vec{\Phi}_{\text{min}}$ minimizing $\mathcal{H}$ must be constant and uniform and must also minimize potential since $\mathcal{V}(\vec{\Phi}(x)) \geq \mathcal{V}(\vec{\Phi}_{\text{min}})$.
Distinguish two different cases:

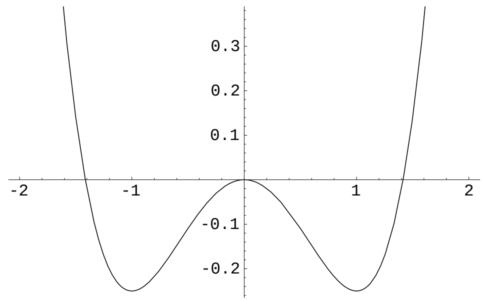
- $m^2 > 0$: **Wigner-Weyl** realization of the symmetry.



$$\mathcal{V}(x) = x^2/2 + x^4/4.$$

$\mathcal{V}$ has its minimum for $\vec{\Phi} = 0$. In the quantized theory we associate a unique ground state $|0\rangle$ with this minimum.

- $m^2 < 0$: **Nambu-Goldstone** realization.



$$\mathcal{V}(x) = -x^2/2 + x^4/4.$$

Several distinct minima.

In the following: $m^2 < 0$.

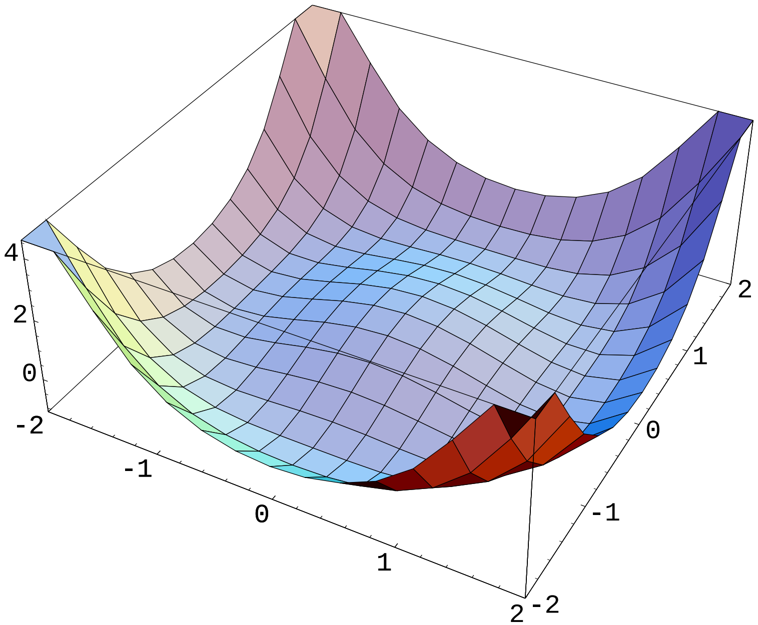
$\mathcal{L}$ invariant under a global "isospin" rotation

$$g \in \mathrm{SO}(3): \;\; \Phi_i \to \Phi_i' = D_{ij}(g)\Phi_j = (e^{-i\alpha_k T_k})_{ij}\Phi_j, \qquad (4.3)$$

$$[T_i, T_j] = i\epsilon_{ijk}T_k,$$

where

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$



Two-dimensional rotationally invariant potential:

$$\mathcal{V}(x,y) = -(x^2+y^2) + \frac{(x^2+y^2)^2}{4}$$

Exercise: Determine the minimum of the potential

$$\mathcal{V}(\Phi_1,\Phi_2,\Phi_3) = \frac{m^2}{2}\Phi_i\Phi_i + \frac{\lambda}{4}(\Phi_i\Phi_i)^2.$$

We find

$$|\vec{\Phi}_{\rm min}| = \sqrt{\frac{-m^2}{\lambda}} \equiv v, \quad |\vec{\Phi}| = \sqrt{\Phi_1^2 + \Phi_2^2 + \Phi_3^2}. \tag{4.4}$$

Note that $\vec{\Phi}_{\rm min}$ satisfies EOM:

$$\underbrace{\Box \Phi_{i\rm min}}_{0} \underbrace{+ m^2 \Phi_{i\rm min} + \lambda \vec{\Phi}^2_{\rm min} \Phi_{i\rm min}}_{\frac{\partial \mathcal{V}}{\partial \phi_i}\big|_{\vec{\Phi}_{\rm min}}} = \Phi_{i\rm min}(m^2 + \lambda \vec{\Phi}^2_{\rm min}) = 0.$$

$\vec{\Phi}_{\rm min}$ can point in any direction in isospin space.
$\Rightarrow$ non-countably infinite number of degenerate vacua.

Spontaneous symmetry breaking (hidden symmetry)

Any infinitesimal external perturbation which is not invariant under SO(3) will select a particular direction.

Appropriate orientation of the internal coordinate frame $\Rightarrow$

$$\vec{\Phi}_{\rm min} = v\hat{e}_3 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} . \tag{4.5}$$

1. $\vec{\Phi}_{\rm min}$ *not* invariant under full group $G = \mathrm{SO}(3)$.
   Rotations about the 1 and 2 axis change $\vec{\Phi}_{\rm min}$, i. e. T_1 and T_2 do not annihilate $\vec{\Phi}_{\rm min}$:

$$\begin{aligned} T_1\vec{\Phi}_{\rm min} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} v \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = v \begin{pmatrix} 0 \\ -i \\ 0 \end{pmatrix} , \\ T_2\vec{\Phi}_{\rm min} &= \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} v \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = v \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix} . \end{aligned} \tag{4.6}$$

   Set of transformations which do not leave $\vec{\Phi}_{\rm min}$ invariant does *not* form a group, because it does not contain the identity.

2. $\vec{\Phi}_{\rm min}$ invariant under subgroup H of G: rotations about the 3 axis

$$h \in H: \quad \vec{\Phi}' = D(h)\vec{\Phi} = e^{-i\alpha_3 T_3}\vec{\Phi}, \quad D(h)\vec{\Phi}_{\rm min} = \vec{\Phi}_{\rm min},$$

i. e.

$$T_3\vec{\Phi}_{\rm min} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} v \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0.$$

Exercise: Expand $\Phi_3(x) = v + \eta(x)$. $\Rightarrow$ New expression for the potential

$$\tilde{\mathcal{V}} = \frac{1}{2}(-2m^2)\eta^2 + \underbrace{\lambda v\eta(\Phi_1^2 + \Phi_2^2 + \eta^2) + \frac{\lambda}{4}(\Phi_1^2 + \Phi_2^2 + \eta^2)^2}_{\text{interaction terms}} - \underbrace{\frac{\lambda}{4}v^4}_{\text{constant}} . \tag{4.7}$$

Read off masses:

$$m_{\Phi_1}^2 = m_{\Phi_2}^2 = 0, \quad m_\eta^2 = -2m^2.$$

Model-independent feature of the above example:

1. For each of the two generators T_1 and T_2 which do not annihilate the ground state one obtains a *massless* Goldstone boson.

2. Number of Goldstone bosons is determined by the structure of the symmetry groups:

   - G symmetry group of the Lagrangian with n_G generators.
   - H subgroup with n_H generators which leaves the ground state after spontaneous symmetry breaking invariant.
   - # of Goldstone bosons: $n_G - n_H$.

3. Criterion for spontaneous symmetry breaking: Non-vanishing vacuum expectation value of some Hermitian operator, here $\langle 0|\Phi_3(0)|0\rangle = v$.

4.2 Goldstone Theorem

Different approach to Goldstone bosons.

Presupposition:

1. Some Hamilton operator with a global symmetry group $G = \mathrm{SO}(3)$.
2. $\vec{\Phi}(x) = (\Phi_1(x), \Phi_2(x), \Phi_3(x))$: Triplet of local hermitian operators transforming as a vector under G:

$$\begin{aligned} g \in G: \quad \vec{\Phi}(x) \mapsto \vec{\Phi}'(x) &= e^{i\sum_{k=1}^{3}\alpha_k Q_k}\vec{\Phi}(x)e^{-i\sum_{l=1}^{3}\alpha_l Q_l} \\ &= e^{-i\sum_{k=1}^{3}\alpha_k T_k}\vec{\Phi}(x) \neq \vec{\Phi}(x). \end{aligned} \tag{4.8}$$

   Q_i: Generators of the SO(3) transformations on the Hilbert space satisfying $[Q_i, Q_j] = i\epsilon_{ijk}Q_k$.
   $T_i = (t^i_{jk})$: Matrices of the three-dimensional representation satisfying $t^i_{jk} = -i\epsilon_{ijk}$.

3. One component of the multiplet acquires a non-vanishing vacuum expectation value:

$$\langle 0|\Phi_1(x)|0\rangle = \langle 0|\Phi_2(x)|0\rangle = 0, \quad \langle 0|\Phi_3(x)|0\rangle = v \neq 0. \tag{4.9}$$

Claim:

1. The two generators Q_1 and Q_2 do not annihilate the ground state.
2. To each such generator corresponds a massless Goldstone boson.

Proof:

1. Expand Eq. (4.8) to first order in the α_k:

$$\vec{\Phi}' = \vec{\Phi} + i\sum_{k=1}^{3}\alpha_k[Q_k,\vec{\Phi}] = (1 - i\sum_{k=1}^{3}\alpha_k T_k)\vec{\Phi} = \vec{\Phi} + \vec{\alpha}\times\vec{\Phi}.$$

Compare terms linear in the α_k:

$$i[\alpha_k Q_k, \Phi_l] = \epsilon_{lkm}\alpha_k\Phi_m.$$

α_k can be chosen independently $\Rightarrow$

$$i[Q_k, \Phi_l] = -\epsilon_{klm}\Phi_m,$$

i. e. field operators Φ_i transform as a (iso-) vector.

Analogy

$$\begin{aligned} Q_i &\rightarrow l_i, \\ \Phi_i &\rightarrow x_i, \\ i[l_k, x_l] &= -\epsilon_{klm}x_m. \end{aligned}$$

$\epsilon_{klm}\epsilon_{kln} = 2\delta_{mn} \Rightarrow$

$$-\frac{i}{2}\epsilon_{kln}[Q_k, \Phi_l] = \delta_{mn}\Phi_m = \Phi_n.$$

In particular,

$$\Phi_3 = -\frac{i}{2}([Q_1, \Phi_2] - [Q_2, \Phi_1]), \tag{4.10}$$

with cyclic permutations for the other two cases.

Consider Eq. (4.8) for $\vec{\alpha} = (0, \pi/2, 0)$,

$$\begin{aligned} e^{-i\frac{\pi}{2}T_2}\vec{\Phi} &= \begin{pmatrix} \cos(\pi/2) & 0 & \sin(\pi/2) \\ 0 & 1 & 0 \\ -\sin(\pi/2) & 0 & \cos(\pi/2) \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} = \begin{pmatrix} \Phi_3 \\ \Phi_2 \\ -\Phi_1 \end{pmatrix} \\ &= e^{i\frac{\pi}{2}Q_2} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} e^{-i\frac{\pi}{2}Q_2}. \end{aligned}$$

First row $\Rightarrow$

$$\Phi_3 = e^{i\frac{\pi}{2}Q_2}\Phi_1 e^{-i\frac{\pi}{2}Q_2}.$$

Take vacuum expectation value

$$v = \langle 0|e^{i\frac{\pi}{2}Q_2}\Phi_1 e^{-i\frac{\pi}{2}Q_2}|0\rangle.$$

Since $v \neq 0$, clearly $Q_2|0\rangle \neq 0$, because otherwise the exponential operator could be replaced by unity and the right-hand side would vanish ($\langle 0|\Phi_1|0\rangle = 0$).

Analogously $Q_1|0\rangle \neq 0$.

Remarks:

(a) "States" $Q_{1(2)}|0\rangle$ cannot be normalized. Rigorous derivation:

$$\int \mathrm{d}^3x \langle 0|[J_b^0(t,\vec{x}), \Phi_c(0)]|0\rangle,$$

determine commutator *before* evaluating the integral.

(b) Some derivations of Goldstone's theorem right away start by assuming $Q_{1(2)}|0\rangle \neq 0$. However, in QCD it is advantageous to establish the connection between the existence of Goldstone bosons and a non-vanishing vacuum expectation value.

2. Existence of Goldstone bosons.

$$0 \neq v = \langle 0|\Phi_3(0)|0\rangle = -\frac{i}{2}\langle 0|\left([Q_1, \Phi_2(0)] - [Q_2, \Phi_1(0)]\right)|0\rangle \equiv -\frac{i}{2}(A-B).$$

Show $A = -B$. Perform rotation of fields and generators by $\pi/2$ about the 3 axis ($\vec{\alpha} = (0, 0, \pi/2)$):

$$e^{-i\frac{\pi}{2}T_3}\vec{\Phi} = \begin{pmatrix} -\Phi_2 \\ \Phi_1 \\ \Phi_3 \end{pmatrix} = e^{i\frac{\pi}{2}Q_3} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} e^{-i\frac{\pi}{2}Q_3},$$

and

$$\begin{pmatrix} -Q_2 \\ Q_1 \\ Q_3 \end{pmatrix} = e^{i\frac{\pi}{2}Q_3} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} e^{-i\frac{\pi}{2}Q_3}.$$

$\Rightarrow$

$$\begin{aligned} B = \langle 0|[Q_2, \Phi_1(0)]|0\rangle &= \langle 0|(e^{i\frac{\pi}{2}Q_3}(-Q_1)\underbrace{e^{-i\frac{\pi}{2}Q_3}e^{i\frac{\pi}{2}Q_3}}_{1}\Phi_2(0)e^{-i\frac{\pi}{2}Q_3} \\ &\quad -e^{i\frac{\pi}{2}Q_3}\Phi_2(0)e^{-i\frac{\pi}{2}Q_3}e^{i\frac{\pi}{2}Q_3}(-Q_1)e^{-i\frac{\pi}{2}Q_3})|0\rangle \\ &= -\langle 0|[Q_1, \Phi_2(0)]|0\rangle = -A. \end{aligned}$$

We made use of $Q_3|0\rangle = 0$ (vacuum invariant under rotations about

the 3 axis).
$\Rightarrow v$ can also be written as

$$\begin{aligned} 0 \neq v &= \langle 0|\Phi_3(0)|0\rangle = -i\langle 0|[Q_1, \Phi_2(0)]|0\rangle \\ &= -i\int \mathrm{d}^3x \langle 0|[J_1^0(t,\vec{x}), \Phi_2(0)]|0\rangle. \end{aligned} \tag{4.11}$$

Insert complete set of states $1 = \sum_n |n\rangle\langle n|$ (abbreviation includes integral over the total momentum $\vec{p}$ and all other quantum numbers necessary to fully specify the states) into commutator

$$v = -i\sum_n\!\!\!\!\!\!\!\int \int \mathrm{d}^3x \left(\langle 0|J_1^0(t,\vec{x})|n\rangle\langle n|\Phi_2(0)|0\rangle - \langle 0|\Phi_2(0)|n\rangle\langle n|J_1^0(t,\vec{x})|0\rangle\right),$$

translational invariance, $A(x) = e^{iP\cdot x}A(0)e^{-iP\cdot x}$,

$$= -i\sum_n\!\!\!\!\!\!\!\int \int \mathrm{d}^3x \left(e^{-iP_n\cdot x}\langle 0|J_1^0(0)|n\rangle\langle n|\Phi_2(0)|0\rangle - \cdots\right)$$

$$= -i \sum\!\!\!\!\!\!\int_n (2\pi)^3 \delta^3(\vec{P}_n) \left(e^{-iE_n t} \langle 0 | J_1^0(0) | n \rangle \langle n | \Phi_2(0) | 0 \rangle \right.$$
$$\left. - e^{iE_n t} \langle 0 | \Phi_2(0) | n \rangle \langle n | J_1^0(0) | 0 \rangle \right) .$$

Integration with respect to the momentum of the inserted intermediate states $\Rightarrow$

$$= -i(2\pi)^3 {\sum_n}' \left(e^{-iE_n t} \cdots - e^{iE_n t} \cdots \right) ,$$

prime indicates that only states with $\vec{P} = 0$ need to be considered. Hermiticity of the symmetry current operators J_a^μ and the Φ_l $\Rightarrow$

$$c_n := \langle 0 | J_1^0(0) | n \rangle \langle n | \Phi_2(0) | 0 \rangle = \langle n | J_1^0(0) | 0 \rangle^* \langle 0 | \Phi_2(0) | n \rangle^*,$$

such that

$$v = -i(2\pi)^3 {\sum_n}' \left(c_n e^{-iE_n t} - c_n^* e^{iE_n t} \right) . \tag{4.12}$$

Conclusions:

(a) $v \neq 0 \Rightarrow$ there must exist states $|n\rangle$ for which both $\langle 0|J^0_{1(2)}(0)|n\rangle$ and $\langle n|\Phi_{1(2)}(0)|0\rangle$ do not vanish. Vacuum itself cannot contribute because $\langle 0|\Phi_{1(2)}(0)|0\rangle = 0$.

(b) States with $E_n > 0$ contribute (φ_n is the phase of c_n)

$$\begin{aligned}\frac{1}{i}\left(c_n e^{-iE_nt} - c_n^* e^{iE_nt}\right) &= \frac{1}{i}|c_n|\left(e^{i\varphi_n}e^{-iE_nt} - e^{-i\varphi_n}e^{iE_nt}\right)\\ &= 2|c_n|\sin(\varphi_n - E_nt).\end{aligned}$$

v is time-independent $\Rightarrow$ the sum over states with $(E_n > 0, \vec{0})$ must vanish.

(c) $\Rightarrow$ contribution from states with zero energy as well as zero momentum thus zero mass. These zero-mass states are the Goldstone bosons.

4.3 Particle Spectrum in the Presence of Spontaneous Symmetry Breaking

- Indication for spontaneous symmetry breaking: Existence of (almost) massless spin-0 particles. Properties of these Goldstone bosons are tightly connected to the properties of the generators which do not annihilate the vacuum. To each Q_a with $Q_a|0\rangle \neq 0$ corresponds a Goldstone boson.

- The multiplet structure of a theory (in the presence of spontaneous symmetry breaking) is determined by irreducible representation of the group G leaving the ground state invariant (Coleman theorem). The symmetry group of the ground state is always a symmetry group of the Hamilton operator (but not viceversa).

4.4 Explicit Symmetry Breaking: A First Look

Modify potential by adding $a\Phi_3$,

$$\mathcal{V}(\Phi_1, \Phi_2, \Phi_3) = \frac{m^2}{2}\Phi_i\Phi_i + \frac{\lambda}{4}(\Phi_i\Phi_i)^2 + a\Phi_3, \tag{4.13}$$

$m^2 < 0$, $\lambda > 0$, $a > 0$ and real fields Φ_i.
New potential has **lower symmetry**: O(2) symmetry (rotations about the 3 axis).
Conditions for the new minimum (from $\vec{\nabla}_\Phi \mathcal{V} = 0$) read

$$\Phi_1 = \Phi_2 = 0, \quad \lambda\Phi_3^3 + m^2\Phi_3 + a = 0.$$

Exercise: Solve using a perturbative ansatz

$$\langle\Phi_3\rangle = \Phi_3^{(0)} + a\Phi_3^{(1)} + \mathcal{O}(a^2).$$

Result:

$$\Phi_3^{(0)} = \pm\sqrt{-\frac{m^2}{\lambda}}, \quad \Phi_3^{(1)} = \frac{1}{2m^2}.$$

$\Phi_3^{(0)}$: Result without explicit breaking.
Expand potential with $\Phi_3 = \langle \Phi_3 \rangle + \chi \Rightarrow$

$$m_{\Phi_1}^2 = m_{\Phi_2}^2 = a\sqrt{\frac{\lambda}{-m^2}}, \quad \left(m_\chi^2 = -2m^2 + 3a\sqrt{\frac{\lambda}{-m^2}} \right).$$

Remarks:

- The Goldstone bosons have acquired a mass.
- Squared masses $\sim a$.
- Quantum corrections lead to observables which are nonanalytic in the symmetry breaking parameter a, e.g. $a \ln(a)$ (so-called chiral logarithms).
- Analogue of a in QCD: Quark masses.

4.5 Spontaneous Symmetry Breaking in QCD

4.5.1 Indications from the Hadron Spectrum

Example: $H_{\rm str}$ is isospin invariant, i.e.,

$$[H_{\rm str}, T_i] = 0, \quad [T_i, T_j] = i\epsilon_{ijk}T_k.$$

Hadrons can be classified as irreducible multiplets of isospin SU(2):

$$\begin{aligned}
T = 0: &\quad d \\
T = \frac{1}{2}: &\quad \begin{pmatrix} p \\ n \end{pmatrix}, \quad \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \quad \begin{pmatrix} \bar{K}^0 \\ K^- \end{pmatrix} \\
T = 1: &\quad \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}
\end{aligned}$$

$$T = \frac{3}{2}: \quad \begin{pmatrix} \Delta^{++} \\ \Delta^{+} \\ \Delta^{0} \\ \Delta^{-} \end{pmatrix}$$

- Q: Where does this symmetry come from?
- A: Accidental global symmetry of QCD for $m_u = m_d$.

Consider linear combinations $(a = 1, \cdots, 8)$

$$\begin{aligned} Q_{aV} &= Q_{aR} + Q_{aL} \overset{P}{\mapsto} Q_{aV}, \\ Q_{aA} &= Q_{aR} - Q_{aL} \overset{P}{\mapsto} -Q_{aA}. \end{aligned}$$

Exercise: Commutation relations

$$[Q_{aV}, Q_{bV}] = if_{abc}Q_{cV}, \quad [Q_{aV}, Q_{bA}] = if_{abc}Q_{cA}, \quad [Q_{aA}, Q_{bA}] = if_{abc}Q_{cV}. \tag{4.14}$$

Hamilton operator of QCD in chiral limit has $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry, i.e.,

$$[H^0_{\mathrm{QCD}}, Q_{aL}] = [H^0_{\mathrm{QCD}}, Q_{aR}] = 0,$$

or equivalently

$$[H^0_{\mathrm{QCD}}, Q_{aV}] = [H^0_{\mathrm{QCD}}, Q_{aA}] = 0.$$

Naive expectation: Parity doubling.
Assume $|\alpha, +\rangle$ to be eigenstate of H^0_{QCD} and parity

$$H^0_{\mathrm{QCD}}|\alpha, +\rangle = E_\alpha|\alpha, +\rangle,$$

$$P|\alpha, +\rangle = +|\alpha, +\rangle,$$

(e.g. member of the ground state baryon octet (in the chiral limit)).
Define $|\phi_{a\alpha}\rangle = Q_{aA}|\alpha, +\rangle$. $[H^0_{\mathrm{QCD}}, Q_{aA}] = 0 \Rightarrow$

$$H^0_{\mathrm{QCD}}|\phi_{a\alpha}\rangle = H^0_{\mathrm{QCD}}Q_{aA}|\alpha, +\rangle = Q_{aA}H^0_{\mathrm{QCD}}|\alpha, +\rangle = E_\alpha Q_{aA}|\alpha, +\rangle = E_\alpha|\phi_{a\alpha}\rangle,$$

$$P|\phi_{a\alpha}\rangle = PQ_{aA}P^{-1}P|\alpha, +\rangle = -Q_{aA}(+|\alpha, +\rangle) = -|\phi_{a\alpha}\rangle.$$

Naively expand $|\phi_{a\alpha}\rangle$ in terms of the members of the multiplet with negative parity,

$$|\phi_{a\alpha}\rangle = Q_{aA}|\alpha,+\rangle = |\beta,-\rangle\langle\beta,-|Q_{aA}|\alpha,+\rangle = t_{a\beta\alpha}|\beta,-\rangle.$$

Problem: Low-energy spectrum of baryons does not contain a degenerate baryon octet of negative parity.

- Q: What's wrong?
- A: We have tacitly assumed that the ground state of QCD is annihilated by Q_{aA}.

$b^\dagger_{\alpha+}$: operator creating quanta with quantum numbers of state $|\alpha,+\rangle$.
$b^\dagger_{\alpha-}$: creates degenerate quanta of opposite parity.

Expand

$$[Q_{aA}, b^\dagger_{\alpha+}] = b^\dagger_{\beta-}t_{a\beta\alpha}.$$

Usual chain of arguments

$$\begin{aligned} Q_{aA}|\alpha,+\rangle &= Q_{aA}b^\dagger_{\alpha+}|0\rangle \\ &= ([Q_{aA},b^\dagger_{\alpha+}] + b^\dagger_{\alpha+}\underbrace{Q_{aA}}_{\hookrightarrow 0})|0\rangle \\ &= t_{a\beta\alpha}b^\dagger_{\beta-}|0\rangle. \end{aligned} \tag{4.15}$$

However: Not true if ground state is *not* annihilated by Q_{aA}.

- Coleman theorem [S. Coleman, J. Math. Phys. **7**, 787 (1966)]:
  The symmetry of the ground state determines the symmetry of the spectrum (reverse argument: infer symmetry of the ground state from the symmetry of the spectrum).

$$Q_{aV}|0\rangle = Q_V|0\rangle = 0. \tag{4.16}$$

$\Rightarrow$ $\mathrm{SU}(3)_V$ multiplets + baryon number classification.

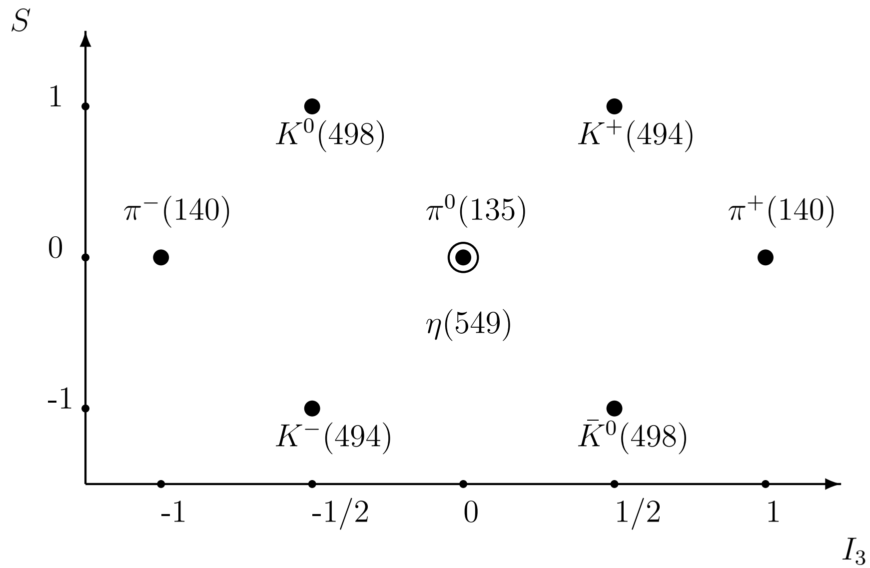


Figure 4.1: Pseudoscalar meson octet in an (I_3, S) diagram. Baryon number $B = 0$. Masses in MeV.

- Examples: Figs. 4.1 and 4.2.

- Goldstone theorem [J. Goldstone, Nuovo Cim. **19**, 154 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962)]:
  To each generator that does not annihilate the ground state exists a

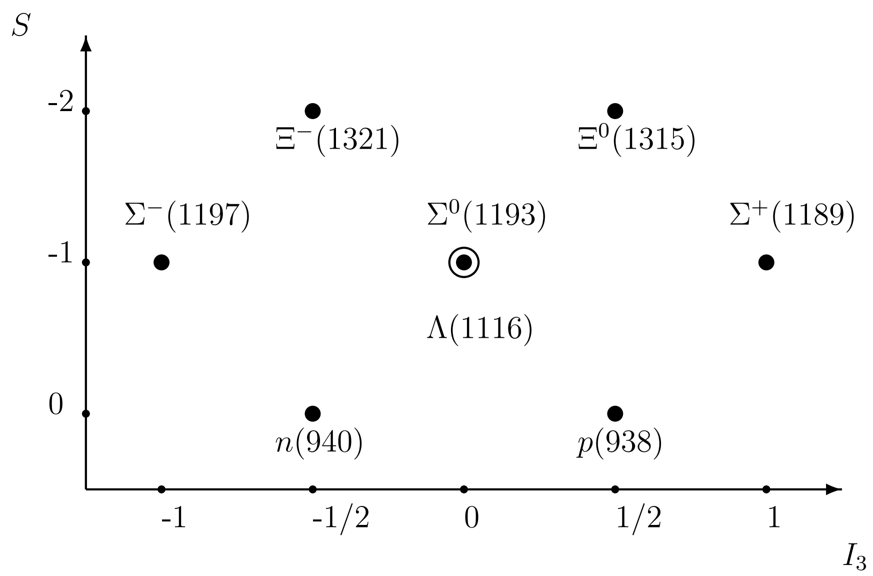


Figure 4.2: Baryon octet ($J = \frac{1}{2}$) in an (I_3, S) diagram. Masses in MeV. Baryon number $B = 1$.

massless Goldstone boson.

$$Q_{aA}|0\rangle \neq 0. \tag{4.17}$$

Symmetry properties of Goldstone boson fields are tightly connected to the generator in question:

Parity:

$$\phi_a(t,\vec{x}) \overset{P}{\mapsto} -\phi_a(t,-\vec{x}), \tag{4.18}$$

(pseudoscalars).
Transform under subgroup $H = \mathrm{SU}(3)_V$ leaving vacuum invariant as an octet [see Eq. (4.14)]:

$$[Q_{aV}, \phi_b(x)] = if_{abc}\phi_c(x). \tag{4.19}$$

- Here
  - H^0_{QCD} invariant under $G = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$
  - $|0\rangle$ invariant under

$$H = \{(V,V)\} \cong \mathrm{SU}(3)_V \quad \text{(flavor SU(3))}$$

  - idealized: 8 massless Goldstone bosons π, K, η (see Fig. 4.1).

4.5.2 The Scalar Singlet Quark Condensate

Reference:

- G. Colangelo, J. Gasser, and H. Leutwyler, Phys. Rev. Lett. **86**, 5008 (2001)

Claim: A non-vanishing scalar quark condensate in the chiral limit is a sufficient (but not a necessary) condition for a spontaneous symmetry breaking in QCD.
Outline of proof:
Recall definition

$$S_a(y) = \bar{q}(y)\lambda_a q(y), \quad a = 0, \cdots, 8, \tag{4.20}$$
$$P_a(y) = i\bar{q}(y)\gamma_5\lambda_a q(y), \quad a = 0, \cdots, 8. \tag{4.21}$$

We need ETCR of two quark operators of the form $A_i(x) = q^\dagger(x)\hat{A}_i q(x)$ (see Eq. (3.46)):

$$[A_1(t, \vec{x}\,), A_2(t, \vec{y}\,)] = \delta^3(\vec{x} - \vec{y}\,)q^\dagger(x)[\hat{A}_1, \hat{A}_2]q(x). \tag{4.22}$$

Recall definition

$$Q_{aV}(t) = \int \mathrm{d}^3x q^\dagger(t,\vec{x}\,)\frac{\lambda_a}{2}q(t,\vec{x}\,).$$

Make use of

$$\begin{aligned}
[\frac{\lambda_a}{2},\gamma_0\lambda_0] &= \frac{\lambda_a}{2}\gamma_0\lambda_0 - \gamma_0\lambda_0\frac{\lambda_a}{2} = \gamma_0[\frac{\lambda_a}{2},\lambda_0] = 0,\\
[\frac{\lambda_a}{2},\gamma_0\lambda_b] &= \gamma_0 i f_{abc}\lambda_c,
\end{aligned}$$

+ integration of Eq. (4.22) over $\vec{x}$:

$$[Q_{aV}(t), S_0(y)] = 0, \quad a = 1,\cdots,8, \tag{4.23}$$

$$[Q_{aV}(t), S_b(y)] = i\sum_{c=1}^{8} f_{abc}S_c(y), \quad a,b = 1,\cdots,8, \tag{4.24}$$

+ analogous results for pseudoscalar quark densities (Exercise).
Make use of

$$\sum_{a,b=1}^{8} f_{abc} f_{abd} = 3\delta_{cd}$$

$\Rightarrow$

$$S_a(y) = -\frac{i}{3} \sum_{b,c=1}^{8} f_{abc} [Q_{bV}(t), S_c(y)]. \tag{4.25}$$

Compare with

$$x_3 = -\frac{i}{2}([l_1, x_2] - [l_2, x_1]) = -\frac{i}{2}\epsilon_{3ij}[l_i, x_j].$$

Equation (4.25) is the analogue of Eq. (4.10) in discussion of Goldstone theorem.
Without proof [see C. Vafa and E. Witten, Nucl. Phys. **B234**, 173 (1984)]:

In the chiral limit the ground state is necessarily invariant under $\mathrm{SU}(3)_V$, i.e., $Q_{aV}|0\rangle = 0$. $\Rightarrow$

$$\langle 0|S_a(y)|0\rangle = \langle 0|S_a(0)|0\rangle \equiv \langle S_a\rangle \stackrel{\mathrm{Eq}(4.25)}{=} 0, \quad a = 1, \cdots, 8. \quad (4.26)$$

Intermediate result: Octet components of the scalar quark condensate *must* vanish in the chiral limit.
Equation (4.26) for $a = 3$:

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad S_3 = \bar{u} - \bar{d}d \quad \Rightarrow \quad \langle \bar{u}u\rangle - \langle \bar{d}d\rangle = 0,$$

and for $a = 8$

$$\lambda_8 = \sqrt{\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad \Rightarrow \quad \langle \bar{u}u\rangle + \langle \bar{d}d\rangle - 2\langle \bar{s}s\rangle = 0,$$

i.e. $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = \langle \bar{s}s \rangle$.
Similar argument does not work for the singlet condensate: 0=0.

Assumption:

$$0 \neq \langle \bar{q}q \rangle = \langle \bar{u}u + \bar{d}d + \bar{s}s \rangle = 3\langle \bar{u}u \rangle = 3\langle \bar{d}d \rangle = 3\langle \bar{s}s \rangle. \tag{4.27}$$

Make use of (no summation implied)

$$(i)^2[\gamma_5 \frac{\lambda_a}{2}, \gamma_0 \gamma_5 \lambda_a] = \lambda_a^2 \gamma_0$$

and

$$\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4^2 = \lambda_5^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\lambda_6^2 = \lambda_7^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\lambda_8^2 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$\Rightarrow$

$$i[Q_{aA}(t), P_a(y)] = \begin{cases} \bar{u}u + \bar{d}d, & a = 1, 2, 3 \\ \bar{u}u + \bar{s}s, & a = 4, 5 \\ \bar{d}d + \bar{s}s, & a = 6, 7 \\ \frac{1}{3}(\bar{u}u + \bar{d}d + 4\bar{s}s), & a = 8 \end{cases} \tag{4.28}$$

(y dependence suppressed on rhs)
Evaluate Eq. (4.28) between $\mathrm{SU}(3)_V$-invariant ground state:

$$\langle 0|i[Q_{aA}(t), P_a(y)]|0\rangle \overset{\text{Eq. (4.27)}}{=} \frac{2}{3}\langle \bar{q}q\rangle, \quad a = 1, \cdots, 8, \tag{4.29}$$

(translational invariance).
Insert complete set of states into commutator. Note $\langle \bar{q}q \rangle \neq 0$. Chain of arguments now as in Sec. 4.2 (Goldstone theorem): Both pseudoscalar density $P_a(y)$ and axial charge operator Q_{aA} must have non-vanishing matrix element between vacuum and massless one-particle states $|\phi_a\rangle$!
Lorentz covariance

$$\langle 0|A_a^\mu(0)|\phi_b(p)\rangle = ip^\mu F_0 \delta_{ab}, \tag{4.30}$$

$F_0 \approx 93$ MeV: "decay" constant of Goldstone bosons in chiral limit.
Remarks:

- Assume $Q_{aA}|0\rangle \neq 0$. $F_0 \neq 0$ is a necessary and sufficient criterion for spontaneous chiral symmetry breaking.
- $\langle \bar{q}q \rangle$ is a sufficient (but not a necessary) condition for a spontaneous symmetry breakdown in QCD.

Summary of patterns of spontaneous symmetry breaking

	Sec. 3.3	O(N) linear sigma model	QCD
Symmetry group G of $\mathcal{L}$	O(3)	O(N)	$\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$
Number of generators n_G	3	$N(N-1)/2$	16
Symmetry group H of $\vert 0\rangle$	O(2)	O($N-1$)	$\mathrm{SU}(3)_V$
Number of generators n_H	1	$(N-1)(N-2)/2$	8
Number of Goldstone bosons $n_G - n_H$	2	$N-1$	8
Multiplet of Goldstone boson fields	$(\Phi_1(x), \Phi_2(x))$	$(\Phi_1(x), \cdots, \Phi_{N-1}(x))$	$i\bar{q}(x)\gamma_5\lambda_a q(x)$
Vacuum expectation value	$v = \langle\Phi_3\rangle$	$v = \langle\Phi_N\rangle$	$v = \langle\bar{q}q\rangle$

Chapter 5

Chiral Perturbation Theory for Mesons

5.1 Transformation Properties of the Goldstone Bosons

References:

- S. Weinberg, Phys. Rev. **166**, 1568 (1968)
- S. R. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1969)
- C. G. Callan, S. R. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2247 (1969)

- A. P. Balachandran, G. Marmo, B. S. Skagerstam, and A. Stern, *Classical Topology and Quantum States* (World Scientific, Singapore, 1991) Chap. 12.2
- H. Leutwyler, in *Perspectives in the Standard Model*, Proceedings of the 1991 Advanced Theoretical Study Institute in Elementary Particle Physics, Boulder, Colorado, 2 - 28 June, 1991, edited by R. K. Ellis, C. T. Hill, and J. D. Lykken (World Scientific, Singapore, 1992)

Purpose of this section: Transformation properties of field variables describing Goldstone bosons.
New concept: **Nonlinear realization** of a group.

5.1.1 General Considerations

Consider physical system with Hamilton operator $\hat{H}$ which is invariant under a compact Lie group G.

Assumption: Ground state $|0\rangle$ is invariant under subgroup H of $G \Rightarrow$ $n = n_G - n_H$ Goldstone bosons.
Describe each Goldstone boson by independent field ϕ_a (smooth real function on Minkowski space M^4).
Collect fields in n-component vector Φ and define vector space

$$M_1 \equiv \{\Phi : M^4 \to \mathbb{R}^n | \phi_a : M^4 \to \mathbb{R} \text{ smooth}\}. \tag{5.1}$$

Aim: Find mapping $\varphi : G \times M_1 \to M_1$ with the following properties:

$$\varphi(e, \Phi) = \Phi \ \forall \ \Phi \in M_1, \ e \text{ identity of } G, \tag{5.2}$$

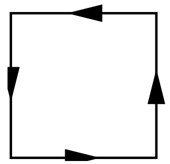
$$\varphi(g_1, \varphi(g_2, \Phi)) = \varphi(g_1 g_2, \Phi) \ \forall \ g_1, g_2 \in G, \ \forall \, \Phi \in M_1. \tag{5.3}$$

- Nomenclature: φ defines *operation* of G on M_1.
- Eq. (5.3): Group-homomorphism property.
- φ will in general, *not* define a *representation* of G, because $\varphi(g, \lambda\Phi) \neq \lambda\varphi(g, \Phi)$.

Construction:

- $\Phi = 0$: "Origin" of M_1, corresponds to ground state configuration.
- $h \in H \Rightarrow \varphi(h, 0) = 0$, H is so-called little group of $\Phi = 0$.
- Establish connection between Goldstone boson fields and set of all left cosets $\{gH | g \in G\}$ (so-called quotient G/H).
- Elements of quotient are sets of group elements.
- Cosets either completely overlap or are completely disjoint.
- Illustration:

  Symmetry group C_4 of a square with directed sides:



$$\begin{aligned} G = C_4 &= \{e, a, a^2, a^3\}, \quad a \text{ rotation by } 90^\circ, \quad a^4 = e, \\ H &= \{e, a^2\}. \end{aligned}$$

$$eH = \{e, a^2\},\ aH = \{a, a^3\},\ a^2H = \{e, a^2\},\ a^3H = \{a, a^3\}.$$

$$G/H = \{gH | g \in G\} = \{\{e, a^2\}, \{a, a^3\}\}.$$

- Under all elements of a given coset gH the origin is mapped onto the same vector in $\mathbb{R}^n$:

$$\varphi(gh, 0) = \varphi(g, \varphi(h, 0)) = \varphi(g, 0)\ \forall\ h \in H.$$

- φ is injective with respect to elements of G/H.

Consider $g, g' \in G$ with $g' \notin gH$. Need to show $\varphi(g, 0) \neq \varphi(g', 0)$.
Assume $\varphi(g, 0) = \varphi(g', 0)$:
$0 = \varphi(e, 0) = \varphi(g^{-1}g, 0) = \varphi(g^{-1}, \varphi(g, 0)) = \varphi(g^{-1}, \varphi(g', 0)) = \varphi(g^{-1}g', 0).$

Implies $g^{-1}g' \in H$ or $g' \in gH$ in contradiction to assumption. $\Rightarrow$ $\varphi(g, 0) = \varphi(g', 0)$ cannot be true. $\Rightarrow$ Mapping can be inverted on the image of $\varphi(g, 0)$.

- Conclusion: There exists *isomorphic mapping* between quotient G/H and Goldstone boson fields (given by image of 0 under all g).

Transformation behavior of Goldstone boson fields under arbitrary $g \in G$ in terms of above mapping:

- To each Φ corresponds coset $\tilde{g}H$ with appropriate $\tilde{g}$. Let $f = \tilde{g}h \in \tilde{g}H$ denote representative of this coset such that

$$\Phi = \varphi(f, 0) = \varphi(\tilde{g}h, 0).$$

Apply mapping $\varphi(g)$ to Φ:

$$\varphi(g, \Phi) = \varphi(g, \varphi(\tilde{g}h, 0)) = \varphi(g\tilde{g}h, 0) = \varphi(f', 0) = \Phi', \quad f' \in g(\tilde{g}H).$$

$$\begin{array}{ccc} \Phi & \xrightarrow{g} & \Phi' \\ \downarrow & & \uparrow \\ \tilde{g}H & \xrightarrow{g} & g\tilde{g}H \end{array}$$

Procedure uniquely determines transformation behavior of Goldstone bosons up to appropriate choice of variables parameterizing elements of quotient G/H.

5.1.2 Application to QCD

Groups relevant to application in QCD:

$$\begin{aligned} G &= \mathrm{SU}(N) \times \mathrm{SU}(N) = \{(L, R) | L \in \mathrm{SU}(N), R \in \mathrm{SU}(N)\}, \\ H &= \{(V, V) | V \in \mathrm{SU}(N)\} \cong \mathrm{SU}(N). \end{aligned}$$

Let $\tilde{g} = (\tilde{L}, \tilde{R}) \in G$. Characterize left coset

$$\tilde{g}H = \{(\tilde{L}V, \tilde{R}V) | V \in \mathrm{SU}(N)\}$$

through SU(N) matrix $U = \tilde{R}\tilde{L}^\dagger$:

$$(\tilde{L}V, \tilde{R}V) = (\tilde{L}V, \tilde{R}\tilde{L}^\dagger\tilde{L}V) = (1, \tilde{R}\tilde{L}^\dagger)\underbrace{(\tilde{L}V, \tilde{L}V)}_{\in H}, \text{ i.e. } \tilde{g}H = (1, \tilde{R}\tilde{L}^\dagger)H,$$

with convention that representative of coset is chosen such that the unit matrix stands in its first argument.

U is isomorphic to a Φ.

Transformation behavior of U under $g = (L, R) \in G$ is obtained by multiplication in left coset:

$$g\tilde{g}H = (L, R\tilde{R}\tilde{L}^\dagger)H = (1, R\tilde{R}\tilde{L}^\dagger L^\dagger)(L, L)H = (1, R(\tilde{R}\tilde{L}^\dagger)L^\dagger)H,$$

i.e.

$$U = \tilde{R}\tilde{L}^\dagger \mapsto U' = R(\tilde{R}\tilde{L}^\dagger)L^\dagger = RUL^\dagger. \tag{5.4}$$

Introduce x dependence (transition to fields) so that

$$U(x) \mapsto RU(x)L^\dagger. \tag{5.5}$$

Physically relevant cases of $N = 2$ and $N = 3$. Define

$$M_1 \equiv \begin{cases} \{\Phi : M^4 \to \mathbb{R}^3 | \phi_i : M^4 \to \mathbb{R} \text{ smooth}\} \text{ for } N = 2, \\ \{\Phi : M^4 \to \mathbb{R}^8 | \phi_i : M^4 \to \mathbb{R} \text{ smooth}\} \text{ for } N = 3. \end{cases}$$

$\tilde{\mathcal{H}}(N)$: Set of all Hermitian and traceless $N \times N$ matrices,

$$\tilde{\mathcal{H}}(N) \equiv \{A \in \mathrm{gl}(N, C) | A^\dagger = A \wedge \mathrm{Tr}(A) = 0\},$$

(real vector space under addition of matrices). Define

$$M_2 := \{\phi : M^4 \to \tilde{\mathcal{H}}(N) | \phi \text{ smooth}\}, \text{ entries are smooth functions.}$$

Relation between M_1 and M_2 for $N = 2$

$$\phi(x) = \sum_{i=1}^{3} \tau_i \phi_i(x) = \begin{pmatrix} \phi_3 & \phi_1 - i\phi_2 \\ \phi_1 + i\phi_2 & -\phi_3 \end{pmatrix} \equiv \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix},$$

τ_i: usual Pauli matrices and $\phi_i(x) = \frac{1}{2}\mathrm{Tr}[\tau_i \phi(x)]$.

Analogously for $N = 3$ (Exercise),

$$\phi(x) = \sum_{a=1}^{8} \lambda_a \phi_a(x) = \begin{pmatrix} \phi_3 + \frac{1}{\sqrt{3}}\phi_8 & \phi_1 - i\phi_2 & \phi_4 - i\phi_5 \\ \phi_1 + i\phi_2 & -\phi_3 + \frac{1}{\sqrt{3}}\phi_8 & \phi_6 - i\phi_7 \\ \phi_4 + i\phi_5 & \phi_6 + i\phi_7 & -\frac{2}{\sqrt{3}}\phi_8 \end{pmatrix}$$

$$\equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix},$$

λ_a: Gell-Mann matrices and $\phi_a(x) = \frac{1}{2}\mathrm{Tr}[\lambda_a \phi(x)]$.
Define

$$M_3 \equiv \left\{ U : M^4 \to \mathrm{SU}(N) | U(x) = \exp\left(i\frac{\phi(x)}{F_0} \right), \phi \in M_2 \right\}.$$

M_3 does not define a vector space. E.g. $\det(\lambda U) \neq 1$.
Definition of nonlinear realization of $\mathrm{SU}(N) \times \mathrm{SU}(N)$ on M_3:

Homomorphism

$$\varphi : G \times M_3 \to M_3 \quad \text{with} \quad \varphi[(L,R),U](x) \equiv RU(x)L^\dagger,$$

defines operation of G on M_3, because

1. $RUL^\dagger \in M_3$, since $U \in M_3$ and $R, L^\dagger \in \mathrm{SU}(N)$.
2. $\varphi[(\mathbb{1},\mathbb{1}),U](x) = \mathbb{1}U(x)\mathbb{1} = U(x)$.
3. Let $g_i = (L_i, R_i) \in G$ and thus $g_1 g_2 = (L_1 L_2, R_1 R_2) \in G$.

$$\begin{aligned} \varphi[g_1, \varphi[g_2, U]](x) &= \varphi[g_1, (R_2 U L_2^\dagger)](x) = R_1 R_2 U(x) L_2^\dagger L_1^\dagger, \\ \varphi[g_1 g_2, U](x) &= R_1 R_2 U(x) (L_1 L_2)^\dagger = R_1 R_2 U(x) L_2^\dagger L_1^\dagger. \end{aligned}$$

φ is called a nonlinear realization, because M_3 is *not* a vector space.
The origin $\phi(x) = 0$, i.e. $U_0 = \mathbb{1}$, denotes ground state of system.
Q: What does φ do to ground state?

A:

1. Ground state remains invariant under subgroup $H = \{(V, V)|V \in \mathrm{SU}(N)\}$ (corresponding to rotating both left- and right-handed quark fields in QCD by the same V):
$$\varphi[g = (V, V), U_0] = V U_0 V^\dagger = V V^\dagger = \mathbb{1} = U_0.$$
2. Under "axial transformations" (rotating the left-handed quarks by A and the right-handed quarks by $A^\dagger$) ground state does *not* remain invariant,
$$\varphi[g = (A, A^\dagger), U_0] = A^\dagger U_0 A^\dagger = A^\dagger A^\dagger \neq U_0.$$
Consistent with the assumed spontaneous symmetry breakdown!

Transformation behavior of $\phi(x)$ under subgroup $H = \{(V, V)|V \in \mathrm{SU}(N)\}$? Expand
$$U = 1 + i\frac{\phi}{F_0} - \frac{\phi^2}{2F_0^2} + \cdots,$$

$\Rightarrow$ Realization restricted to the subgroup H,

$$1+i\frac{\phi}{F_0}-\frac{\phi^2}{2F_0^2}+\cdots\mapsto V(1+i\frac{\phi}{F_0}-\frac{\phi^2}{2F_0^2}+\cdots)V^\dagger = 1+i\frac{V\phi V^\dagger}{F_0}-\frac{V\phi V^\dagger V\phi V^\dagger}{2F_0^2}+\cdots, \tag{5.6}$$

defines a linear representation on $M_2 \ni \phi \mapsto V\phi V^\dagger \in M_2$, because

$$(V\phi V^\dagger)^\dagger = V\phi V^\dagger, \quad \mathrm{Tr}(V\phi V^\dagger) = \mathrm{Tr}(\phi) = 0,$$
$$V_1(V_2\phi V_2^\dagger)V_1^\dagger = (V_1V_2)\phi(V_1V_2)^\dagger.$$

Example: $N = 3$

1. Parameterize $V = \exp\left(-i\Theta_a^V\frac{\lambda_a}{2}\right)$ and compare both sides of Eq. (5.6),

$$\phi = \lambda_b\phi_b \overset{h\in \mathrm{SU}(3)_V}{\mapsto} V\phi V^\dagger = \phi - i\Theta_a^V \underbrace{[\frac{\lambda_a}{2},\phi_b\lambda_b]}_{\phi_b i f_{abc}\lambda_c}+\cdots$$
$$= \phi + f_{abc}\Theta_a^V\phi_b\lambda_c+\cdots. \tag{5.7}$$

Corresponds to adjoint representation.
In SU(3) the fields ϕ_a transform as an octet. Consistent with transformation behavior quoted in Eq. (4.19):

$$\begin{aligned} e^{i\Theta_a^V Q_V^a}\lambda_b\phi_b e^{-i\Theta_a^V Q_V^a} &= \lambda_b\phi_b + i\Theta_a^V\lambda_b \underbrace{[Q_V^a,\phi_b]}_{if_{abc}\phi_c} + \cdots \\ &= \phi + f_{abc}\Theta_a^V\phi_b\lambda_c + \cdots . \end{aligned} \quad (5.8)$$

2. For group elements $(A, A^\dagger)$ of G proceed analogously. $\Rightarrow$ Fields ϕ_a do *not* have simple transformation behavior. Commutation relations of fields with *axial* charges are complicated nonlinear functions of fields.

5.2 The Lowest-Order Effective Lagrangian

References:

- J. Gasser and H. Leutwyler, Annals Phys. **158**, 142 (1984)
- J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465 (1985)
- H. Georgi, *Weak Interactions and Modern Particle Theory* (Benjamin/Cummings, Menlo Park, 1984)

Goal: Construction of the most general theory describing the dynamics of the Goldstone bosons associated with the spontaneous symmetry breakdown in QCD.
Requirements:

1. In chiral limit $\mathcal{L}_{\text{eff}}$ is invariant under $\text{SU}(3)_L \times \text{SU}(3)_R \times \text{U}(1)_V$.
2. Theory contains exactly eight pseudoscalar degrees of freedom transforming as octet under subgroup $H = \text{SU}(3)_V$.

3. Because of ssb, ground state is only invariant under $\mathrm{SU}(3)_V \times \mathrm{U}(1)_V$.

Sec. 5.1.2 $\Rightarrow$ Collect dynamical variables in SU(3) matrix $U(x)$,

$$
\begin{aligned}
U(x) &= \exp\left(i\frac{\phi(x)}{F_0}\right), \\
\phi(x) &= \sum_{a=1}^{8} \lambda_a \phi_a(x) \equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}. \quad (5.9)
\end{aligned}
$$

Most general, chirally invariant, effective Lagrangian density with minimal number of derivatives:

$$
\mathcal{L}_{\mathrm{eff}} = \frac{F_0^2}{4}\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right), \quad (5.10)
$$

$F_0 \approx 93$ MeV is free parameter (related to pion decay $\pi^+ \to \mu^+\nu_\mu$).

- $\mathcal{L}_{\mathrm{eff}}$ is invariant under *global* $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ transformations:

$$
U \mapsto RUL^\dagger,
$$

$$\begin{aligned}
\partial_\mu U &\mapsto \partial_\mu(RUL^\dagger) = \underbrace{\partial_\mu R}_{0} UL^\dagger + R\partial_\mu UL^\dagger + RU \underbrace{\partial_\mu L^\dagger}_{0} = R\partial_\mu UL^\dagger, \\
U^\dagger &\mapsto LU^\dagger R^\dagger, \\
\partial_\mu U^\dagger &\mapsto L\partial_\mu U^\dagger R^\dagger,
\end{aligned}$$

because

$$\mathcal{L}_{\text{eff}} \mapsto \frac{F_0^2}{4}\text{Tr}(R\partial_\mu U \underbrace{L^\dagger L}_{1} \partial^\mu U^\dagger R^\dagger) \stackrel{*}{=} \frac{F_0^2}{4}\text{Tr}(\underbrace{R^\dagger R}_{1} \partial_\mu U \partial^\mu U^\dagger) = \mathcal{L}_{\text{eff}}.$$

$*$: trace property $\text{Tr}(AB) = \text{Tr}(BA)$.

- Global $\text{U}(1)_V$ invariance is trivially satisfied: Goldstone bosons have baryon number zero $\Rightarrow \phi \mapsto \phi$ under $\text{U}(1)_V \Rightarrow U \mapsto U$.
- Consider substitution $\phi_a(t,\vec{x}) \mapsto -\phi_a(t,\vec{x})$ or, equivalently, $U(t,\vec{x}) \mapsto U^\dagger(t,\vec{x})$: Test whether expression is of so-called even or odd *intrinsic* parity, i.e., even or odd in the number of Goldstone boson fields.

$\mathcal{L}_{\text{eff}}$ of Eq. (5.10) is even (Exercise).
Note: Goldstone bosons are pseudoscalars $\Rightarrow$ true parity transformation given by $\phi_a(t,\vec{x}) \mapsto -\phi_a(t,-\vec{x})$ or, equivalently, $U(t,\vec{x}) \mapsto U^\dagger(t,-\vec{x})$.

- Purpose of multiplicative constant $F_0^2/4$: Generates standard form of kinetic term $\frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a$. Expand exponential $U = 1+i\phi/F_0+\cdots$, $\partial_\mu U = i\partial_\mu\phi/F_0 + \cdots$, $\Rightarrow$

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \frac{F_0^2}{4}\text{Tr}\left[\frac{i\partial_\mu\phi}{F_0}\left(-\frac{i\partial^\mu\phi}{F_0}\right)\right]+\cdots = \frac{1}{4}\text{Tr}(\lambda_a\partial_\mu\phi_a\lambda_b\partial^\mu\phi_b)+\cdots\\ &= \frac{1}{4}\partial_\mu\phi_a\partial^\mu\phi_b\underbrace{\text{Tr}(\lambda_a\lambda_b)}_{2\delta_{ab}}+\cdots = \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a+\mathcal{L}_{\text{int}}.\end{aligned}$$

  No other terms containing only two fields $\Rightarrow$ eight fields ϕ_a describe eight independent *massless* particles.

- What about other structures?

– $\mathrm{Tr}(UU^\dagger)$ is a constant.

– Total derivatives have no dynamical significance. Thus

$$\mathrm{Tr}[(\partial_\mu\partial^\mu U)U^\dagger] = \partial_\mu[\mathrm{Tr}(\partial^\mu U U^\dagger)] - \mathrm{Tr}(\partial^\mu U\partial_\mu U^\dagger).$$

– Product of two invariant traces is excluded at lowest order, because $\mathrm{Tr}(\partial_\mu U U^\dagger) = 0$ (Exercise).

- Discussion of vector and axial-vector currents associated with global $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry.

  Parameterize infinitesimal transformations as

$$L = 1 - i\epsilon_a^L \frac{\lambda_a}{2}, \qquad (5.11)$$

$$R = 1 - i\epsilon_a^R \frac{\lambda_a}{2}. \qquad (5.12)$$

Construction of J^μ_{aL}: Set $\epsilon^R_a = 0$ and choose $\epsilon^L_a = \epsilon^L_a(x)$. $\Rightarrow$

$$
\begin{aligned}
U &\mapsto U' = RUL^\dagger = U\left(1 + i\epsilon^L_a \frac{\lambda_a}{2}\right),\\
U^\dagger &\mapsto U'^\dagger = \left(1 - i\epsilon^L_a \frac{\lambda_a}{2}\right) U^\dagger,\\
\partial_\mu U &\mapsto \partial_\mu U' = \partial_\mu U\left(1 + i\epsilon^L_a \frac{\lambda_a}{2}\right) + U i\partial_\mu \epsilon^L_a \frac{\lambda_a}{2},\\
\partial_\mu U^\dagger &\mapsto \partial_\mu U'^\dagger = \left(1 - i\epsilon^L_a \frac{\lambda_a}{2}\right) \partial_\mu U^\dagger - i\partial_\mu \epsilon^L_a \frac{\lambda_a}{2} U^\dagger.
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
\delta\mathcal{L}_{\text{eff}} &= \frac{F_0^2}{4}\text{Tr}\left[U i\partial_\mu \epsilon^L_a \frac{\lambda_a}{2}\partial^\mu U^\dagger + \partial_\mu U\left(-i\partial^\mu \epsilon^L_a \frac{\lambda_a}{2} U^\dagger\right)\right]\\
&= \frac{F_0^2}{4} i\partial_\mu \epsilon^L_a \text{Tr}\left[\frac{\lambda_a}{2}(\partial^\mu U^\dagger U - U^\dagger \partial^\mu U)\right]
\end{aligned}
$$

$$\stackrel{*}{=} \frac{F_0^2}{4} i \partial_\mu \epsilon_a^L \mathrm{Tr}\left(\lambda_a \partial^\mu U^\dagger U\right). \tag{5.13}$$

$*$: $U^\dagger U = 1 \Rightarrow \partial^\mu (U^\dagger U) = 0 \Rightarrow \partial^\mu U^\dagger U = -U^\dagger \partial^\mu U$.

Left currents

$$J^\mu_{aL} = \frac{\partial \delta \mathcal{L}_{\text{eff}}}{\partial \partial_\mu \epsilon_a^L} = i \frac{F_0^2}{4} \mathrm{Tr}\left(\lambda_a \partial^\mu U^\dagger U\right). \tag{5.14}$$

Right currents (Exercise)

$$J^\mu_{aR} = \frac{\partial \delta \mathcal{L}_{\text{eff}}}{\partial \partial_\mu \Theta_a^R} = -i \frac{F_0^2}{4} \mathrm{Tr}\left(\lambda_a U \partial^\mu U^\dagger\right). \tag{5.15}$$

Vector and axial-vector currents

$$J^\mu_{aV} = J^\mu_{aR} + J^\mu_{aL} = -i \frac{F_0^2}{4} \mathrm{Tr}\left(\lambda_a [U, \partial^\mu U^\dagger]\right), \tag{5.16}$$

$$J^\mu_{aA} = J^\mu_{aR} - J^\mu_{aL} = -i \frac{F_0^2}{4} \mathrm{Tr}\left(\lambda_a \{U, \partial^\mu U^\dagger\}\right). \tag{5.17}$$

$\mathcal{L}_{\text{eff}}$ is invariant under $\text{SU}(3)_L \times \text{SU}(3)_R \Rightarrow$ left and right currents are conserved $\Rightarrow$ vector and axial-vector currents are conserved.

Q: What happens under $\phi \to -\phi$?
A:

1.
$$J^\mu_{aV} \overset{\phi \mapsto -\phi}{\mapsto} -i\frac{F_0^2}{4}\text{Tr}[\lambda_a(U^\dagger\partial^\mu U - \partial^\mu U U^\dagger)]$$
$$= -i\frac{F_0^2}{4}\text{Tr}[\lambda_a(-\partial^\mu U^\dagger U + U\partial^\mu U^\dagger)] = J^\mu_{aV},$$

i.e. even in the number of Goldstone bosons.

2.
$$J^\mu_{aA} \overset{\phi \mapsto -\phi}{\mapsto} -i\frac{F_0^2}{4}\text{Tr}[\lambda_a(U^\dagger\partial^\mu U + \partial^\mu U U^\dagger)]$$
$$= i\frac{F_0^2}{4}\text{Tr}[\lambda_a(\partial^\mu U^\dagger U + U\partial^\mu U^\dagger)] = -J^\mu_{aA},$$

i.e. odd in the number of Goldstone bosons.

Expand J^μ_{aA} in fields,

$$J^\mu_{aA} = -i\frac{F_0^2}{4}\mathrm{Tr}\left(\lambda_a\left\{1+\cdots,-i\frac{\lambda_b\partial^\mu\phi_b}{F_0}+\cdots\right\}\right) = -F_0\partial^\mu\phi_a+\cdots$$

$\Rightarrow$ axial-vector current has non-vanishing matrix element when evaluated between vacuum and one-Goldstone-boson state [see Eq. (4.30)]:

$$\begin{aligned}\langle 0|J^\mu_{aA}(x)|\phi_p(p)\rangle &= \langle 0| - F_0\partial^\mu\phi_a(x)|\phi_b(p)\rangle \\ &= -F_0\partial^\mu\exp(-ip\cdot x)\delta_{ab} = ip^\mu F_0\exp(-ip\cdot x)\delta_{ab}.\end{aligned}$$

- So far: Perfect $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry.

  Now: Include explicit symmetry breaking due quark-mass term of

QCD,

$$\mathcal{L}_{\mathcal{M}} = -\bar{q}_R \mathcal{M} q_L - \bar{q}_L \mathcal{M}^\dagger q_R, \quad \mathcal{M} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}. \tag{5.18}$$

Argument due to Georgi: Although $\mathcal{M}$ is in reality just constant matrix and does not transform along with quark fields, $\mathcal{L}_{\mathcal{M}}$ of Eq. (5.18) *would be* invariant *if* $\mathcal{M}$ transformed as

$$\mathcal{M} \mapsto R\mathcal{M}L^\dagger. \tag{5.19}$$

Construct most general Lagrangian $\mathcal{L}(U, \mathcal{M})$ invariant under

$$U \mapsto RUL^\dagger, \quad \mathcal{M} \mapsto R\mathcal{M}L^\dagger,$$

and expand in powers of $\mathcal{M}$. At lowest order in $\mathcal{M}$

$$\mathcal{L}_{\text{s.b.}} = \frac{F_0^2 B_0}{2} \text{Tr}(\mathcal{M}U^\dagger + U\mathcal{M}^\dagger), \tag{5.20}$$

s.b. = symmetry breaking.

• Interpretation of new parameter B_0.

Consider energy density of ground state ($U = U_0 = 1$),

$$\langle \mathcal{H}_{\text{eff}} \rangle_{\text{min}} = -F_0^2 B_0 (m_u + m_d + m_s). \tag{5.21}$$

Justification of Eq. (5.21):
Construct Hamilton density corresponding to $\mathcal{L}$ of Eq. (5.10) and (5.20).
Dynamical fields ϕ_a; conjugate momenta

$$\pi_a = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a} = \frac{F_0^2}{4} \text{Tr} \left(\frac{\partial \dot{U}}{\partial \dot{\phi}_a} \dot{U}^\dagger + \dot{U} \frac{\partial \dot{U}^\dagger}{\partial \dot{\phi}_a} \right).$$

We need

$$\pi_a \dot{\phi}_a = \frac{F_0^2}{4} \text{Tr} \left(\dot{\phi}_a \frac{\partial \dot{U}}{\partial \dot{\phi}_a} \dot{U}^\dagger + \dot{U} \dot{\phi}_a \frac{\partial \dot{U}^\dagger}{\partial \dot{\phi}_a} \right).$$

Make use of

$$
\begin{aligned}
\dot{U} &= \frac{i\dot{\phi}}{F_0} - \frac{\dot{\phi}\phi + \phi\dot{\phi}}{2F_0^2} + \cdots, \\
\frac{\partial \dot{U}}{\partial \dot{\phi}_a} &= \frac{i\lambda_a}{F_0} - \frac{\lambda_a\phi + \phi\lambda_a}{2F_0^2} + \cdots, \\
\dot{\phi}_a \frac{\partial \dot{U}}{\partial \dot{\phi}_a} &= i\frac{\dot{\phi}}{F_0} - \frac{\dot{\phi}\phi + \phi\dot{\phi}}{2F_0^2} + \cdots = \dot{U}, \\
\dot{\phi}_a \frac{\partial \dot{U}^\dagger}{\partial \dot{\phi}_a} &= \dot{U}^\dagger.
\end{aligned}
$$

$\Rightarrow$ Hamilton density

$$
\begin{aligned}
\mathcal{H} &= \pi_a \dot{\phi}_a - \mathcal{L} \\
&= \frac{F_0^2}{4} \mathrm{Tr}(\dot{U}\dot{U}^\dagger + \dot{U}\dot{U}^\dagger) - \mathcal{L}
\end{aligned}
$$

$$= \underbrace{\frac{F_0^2}{4}\mathrm{Tr}(\dot{U}\dot{U}^\dagger) + \frac{F_0^2}{4}\mathrm{Tr}(\vec{\nabla}U \cdot \vec{\nabla}U^\dagger)}_{\geq 0} \underbrace{-\frac{F_0^2 B_0}{2}\mathrm{Tr}(\mathcal{M}U^\dagger + U\mathcal{M})}_{\mathcal{V}}.$$

Hamilton density is minimized by constant and uniform fields. Determine minimum of last term:

$$\frac{\partial \mathcal{V}}{\partial \phi_a} = \frac{\partial}{\partial \phi_a}\left[-\frac{F_0^2 B_0}{2}\mathrm{Tr}(\mathcal{M}U^\dagger + U\mathcal{M})\right] = 0.$$

Make use of

$$\mathrm{Tr}(\mathcal{M}U^\dagger + U\mathcal{M}) = 2\mathrm{Tr}\left[\mathcal{M}\left(1 - \frac{\phi^2}{2F_0^2} + \frac{\phi^4}{24F_0^4} + \cdots\right)\right].$$

Consider

$$\frac{\partial}{\partial \phi_a}\mathrm{Tr}\left[\mathcal{M}\left(1 - \frac{\phi^2}{2F_0^2} + \frac{\phi^4}{24F_0^4} + \cdots\right)\right]$$

$$= \mathrm{Tr}\left[\mathcal{M}\left(-\frac{\lambda_a\phi + \phi\lambda_a}{2F_0^2} + \frac{\lambda_a\phi^3 + \phi\lambda_a\phi^2 + \phi^2\lambda_a\phi + \phi^3\lambda_a}{24F_0^4} + \cdots\right)\right].$$

Parameterize

$$\mathcal{M} = m_0\lambda_0 + m_3\lambda_3 + m_8\lambda_8,$$

where

$$\begin{aligned} m_0 &= \frac{m_u + m_d + m_s}{\sqrt{6}}, \\ m_3 &= \frac{m_u - m_d}{2}, \\ m_8 &= \frac{\frac{m_u+m_d}{2} - m_s}{\sqrt{3}}. \end{aligned}$$

Ansatz for solution:

$$\phi = \phi_0 + \frac{1}{F_0^2}\phi_2 + \frac{1}{F_0^4}\phi_4 + \cdots.$$

Organize in powers of $1/F_0^2$. Write $\phi_0 = \lambda_b \phi_{0b}$. Terms proportional to $1/F_0^2$ (factor $-1/2$ omitted):

$$\mathrm{Tr}[\mathcal{M}(\lambda_a\phi_0+\phi_0\lambda_a)] = \mathrm{Tr}[\mathcal{M}(\underbrace{\lambda_a\lambda_b+\lambda_b\lambda_a}_{\frac{4}{3}\delta_{ab}+2d_{abc}\lambda_c})]\phi_{0b}$$

$$= \left(\frac{4}{3}\delta_{ab}(m_u+m_d+m_s)+4m_3d_{ab3}+4m_8d_{ab8}\right)\phi_{0b}=0.$$

8 equations ($a=1,\cdots,8$) in 8 unknowns ϕ_{0b}. Example $a=1$:

$$\frac{4}{3}(m_u+m_d+m_s)\phi_{01}+4m_3\underbrace{d_{1b3}}_{0}\phi_{03}+4m_8\underbrace{d_{1b8}}_{d_{118}\delta_{b1}=\frac{1}{\sqrt{3}}\delta_{b1}}\phi_{0b}$$

$$= \frac{4}{3}(m_u+m_d+m_s)\phi_{01}+4\frac{\frac{m_u+m_d}{2}-m_s}{\sqrt{3}}\frac{1}{\sqrt{3}}\phi_{01}$$

$$= 2(m_u+m_d)\phi_{01}=0. \quad \Rightarrow \quad \phi_{01}=0.$$

Proceed analogously for remaining cases. $\Rightarrow$ For non-vanishing quark masses

$$\phi_{0b} = 0, \quad b = 1, \cdots, 8.$$

Now consider $1/F_0^4$ terms:

$$\mathrm{Tr}\{\mathcal{M}[(\lambda_a\phi_2 + \phi_2\lambda_a) - \frac{1}{12}\underbrace{(\lambda_a\phi_0^3 + \phi_0\lambda_a\phi_0^2 + \phi_0^2\lambda_a\phi_0 + \phi_0^3\lambda_a)}_{0,\text{ because }\phi_0=0}]\} = 0.$$

Calculation for ϕ_2 as for ϕ_0 above. $\Rightarrow \phi_2 = 0$. And so on. In total we obtain

$$\phi = 0$$

as the configuration minimizing $\mathcal{H}$ and thus Eq. (5.21).
$\phi = 0$ is indeed minimum. Verified by taking second derivative of $\mathcal{V}$ and showing

$$\left.\frac{\partial^2\mathcal{V}}{\partial\phi_a\partial\phi_b}\right|_{\phi=0}\phi_a\phi_b \geq 0\,\forall\,\phi.$$

Compare derivative of Eq. (5.21) with respect to m_q with corresponding quantity in QCD (make use of Hellmann-Feynman theorem, Exercise),

$$\left.\frac{\partial\langle 0|\mathcal{H}_{\rm QCD}|0\rangle}{\partial m_q}\right|_{m_u=m_d=m_s=0}=\frac{1}{3}\langle 0|\bar{q}q|0\rangle_0=\frac{1}{3}\langle\bar{q}q\rangle_0.$$

$\langle\bar{q}q\rangle_0$: scalar singlet quark condensate in chiral limit [see Eq. (4.27)].

$\Rightarrow$

$$3F_0^2B_0=-\langle\bar{q}q\rangle_0. \tag{5.22}$$

- Remarks

1. $\mathrm{Tr}(\mathcal{M})$ is not invariant.
2. $\mathrm{Tr}(\mathcal{M}U^\dagger-U\mathcal{M}^\dagger)$ has wrong behavior under parity $\phi(t,\vec{x})\to-\phi(t,-\vec{x})$,

$$\mathrm{Tr}[\mathcal{M}U^\dagger(t,\vec{x})-U(t,\vec{x})\mathcal{M}^\dagger]\quad\stackrel{P}{\mapsto}\quad\mathrm{Tr}[\mathcal{M}U(t,-\vec{x})-U^\dagger(t,-\vec{x})\mathcal{M}^\dagger]$$

$$\overset{\mathcal{M}=\mathcal{M}^\dagger}{=} -\mathrm{Tr}[\mathcal{M}U^\dagger(t,-\vec{x}) - U(t,-\vec{x})\mathcal{M}^\dagger].$$

3. Because of $\mathcal{M} = \mathcal{M}^\dagger$, $\mathcal{L}_{\text{s.b.}}$ contains only terms even in ϕ (Exercise).

- Masses of Goldstone bosons.

Identify terms of second order in fields in $\mathcal{L}_{\text{s.b.}}$,

$$\mathcal{L}_{\text{s.b}} = -\frac{B_0}{2}\mathrm{Tr}(\phi^2\mathcal{M}) + \cdots . \tag{5.23}$$

Exercise

$$\begin{aligned}\mathrm{Tr}(\phi^2\mathcal{M}) \;=\; & 2(m_u+m_d)\pi^+\pi^- + 2(m_u+m_s)K^+K^- + 2(m_d+m_s)K^0\bar{K}^0 \\ & +(m_u+m_d)\pi^0\pi^0 + \frac{2}{\sqrt{3}}(m_u-m_d)\pi^0\eta + \frac{m_u+m_d+4m_s}{3}\eta^2.\end{aligned}$$

Isospin-symmetric limit $m_u = m_d = \hat{m} \Rightarrow \pi^0\eta$ term vanishes i.e. no π^0-η mixing.

$\Rightarrow$

$$
\begin{aligned}
M_\pi^2 &= 2B_0\hat{m}, && (5.24)\\
M_K^2 &= B_0(\hat{m}+m_s), && (5.25)\\
M_\eta^2 &= \frac{2}{3}B_0\left(\hat{m}+2m_s\right). && (5.26)
\end{aligned}
$$

In combination with $B_0 = -\langle\bar{q}q\rangle_0/(3F_0^2)$ referred to as Gell-Mann, Oakes, and Renner relations.

Also: Gell-Mann-Okubo relation (Exercise)

$$4M_K^2 = 4B_0(\hat{m}+m_s) = 2B_0(\hat{m}+2m_s)+2B_0\hat{m} = 3M_\eta^2+M_\pi^2 \quad (5.27)$$

independent of value of B_0.

Values of quark masses $\hat{m}$ and m_s cannot be extracted from Eqs. (5.24) - (5.26).

Why? Rescaling $B_0 \to \lambda B_0 \wedge m_q \to m_q/\lambda$ leaves relations invariant.

But: Ratios of quark masses. Using $M_\pi = 135$ MeV, $M_K = 496$ MeV, and $M_\eta = 547$ MeV $\Rightarrow$ (Exercise)

$$\frac{M_K^2}{M_\pi^2} = \frac{\hat{m} + m_s}{2m} \Rightarrow \frac{m_s}{\hat{m}} = 25.9,$$
$$\frac{M_\eta^2}{M_\pi^2} = \frac{2m_s + \hat{m}}{3\hat{m}} \Rightarrow \frac{m_s}{\hat{m}} = 24.3. \quad (5.28)$$

- Remark on $\langle \bar{q}q \rangle_0$

$\langle \bar{q}q \rangle_0 \neq 0$ sufficient but not necessary condition for ssb in QCD. $\mathcal{L}_{\text{s.b.}} \Rightarrow$ shift of vacuum energy but also finite Goldstone boson masses. Both effects proportional to B_0. $\text{Tr}(\mathcal{M})$ term would have decoupled vacuum energy shift from Goldstone boson masses. However, forbidden by symmetry argument.

Analogy

	Heisenberg ferromagnet	QCD
Symmetry of $\hat{H}$	O(3)	$\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$
Symmetry of $\|0\rangle$	O(2)	$\mathrm{SU}(3)_V$
VEV	$\langle \vec{M} \rangle$	$\langle \bar{q}q \rangle_0$
Explicit s. b.	external magnetic field	quark masses
interaction	$-\langle \vec{M} \rangle \cdot \vec{H}$	$\langle \mathcal{H}_{\text{eff}} \rangle$ of Eq. (5.21)

In principle: B_0 could vanish or be rather small. $\Rightarrow$ Quadratic masses of Goldstone bosons might be dominated by terms which are nonlinear in quark masses, i.e., by higher-order terms in expansion of $\mathcal{L}(U, \mathcal{M})$. $\Rightarrow$ Generalized chiral perturbation theory.
Analogue would be antiferromagnet which shows ssb with $\langle \vec{M} \rangle = 0$.
Analysis of recent data on $K^+ \to \pi^+\pi^- e^+\nu_e$ in terms of the isoscalar s-wave scattering length a_0^0 (Colangelo, Gasser, Leutwyler) $\Rightarrow \langle \bar{q}q \rangle_0$ is indeed leading order parameter of spontaneously broken chiral symmetry.

5.3 Effective Lagrangians and Weinberg's Power Counting Scheme

Reference:

- S. Weinberg, Physica A **96**, 327 (1979).

Perturbative calculations in effective field theory require **two main ingredients**:

1. Knowledge of the **most general effective Lagrangian**.
2. Consistent **expansion scheme** for observables.

Mesonic chiral perturbation theory:
$\mathcal{L}_{\text{eff}}$ organized as string of terms

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \cdots, \tag{5.29}$$

subscripts refer to order in momentum and quark-mass expansion.
Index 2: Two derivatives or one quark-mass term.

Index 4: Four derivatives, two derivatives and one quark-mass term, two quark-mass terms.
Feynman rules: Derivatives generate four-momenta.
Quark-mass term $\simeq$ two derivatives because of Eqs. (5.24) - (5.26), $M^2 \sim m_q$, and on-shell condition $p^2 = M^2$.
Mesonic sector: Chiral orders are always even $[\mathcal{O}(q^{2k}),\ k \geq 1]$ because Lorentz indices of derivatives always have to be contracted and quark-mass terms count as $\mathcal{O}(q^2)$.

Weinberg's power counting scheme

Q: How do different diagrams compare?

Analyze given diagram under

1. linear rescaling of all *external* momenta, $p_i \mapsto t p_i$,
2. quadratic rescaling of light quark masses, $m_q \mapsto t^2 m_q$ (corresponds to $M^2 \mapsto t^2 M^2$).

Chiral dimension D:

$$\mathcal{M}(tp_i, t^2 m_q) = t^D \mathcal{M}(p_i, m_q) = \mathcal{O}(q^D). \tag{5.30}$$

For small enough momenta (and masses) contributions with increasing D become less important.

$$\begin{aligned} D &= nN_L - 2N_I + \sum_{k=1}^{\infty} 2kN_{2k} && (5.31) \\ &= 2 + (n-2)N_L + \sum_{k=1}^{\infty} 2(k-1)N_{2k} && (5.32) \\ &\geq 2 \text{ in 4 dimensions.} \end{aligned}$$

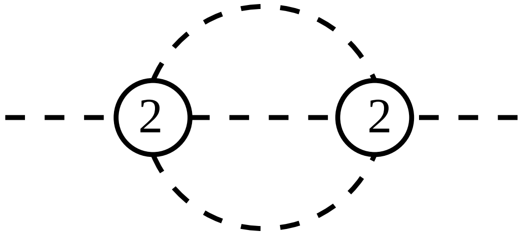
- n: Number of space-time dimensions.
- N_L: Number of independent loops.
- N_I: Number of internal Goldstone boson lines.

- N_{2k}: Number of vertices from $\mathcal{L}_{2k}$.
- Loops suppressed by $(n-2)N_L$.
- Relation between the momentum and loop expansion:

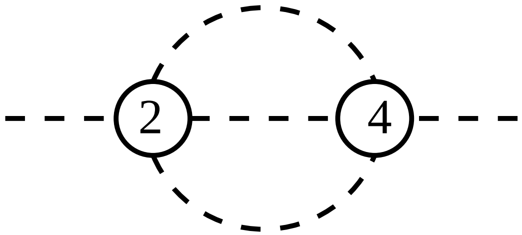
  1. $\mathcal{O}(q^2)$: No loops.
  2. $\mathcal{O}(q^4)$: No loops and 1 loop.
  3. $\mathcal{O}(q^6)$: No loops, 1 loop, and 2 loops.
  4. etc.

- Perturbative scheme in terms of **external momenta** and **quark masses** ($\rightarrow$ meson masses^2) which are small compared to some scale [here: $4\pi F_0 = \mathcal{O}$ (1 GeV)].

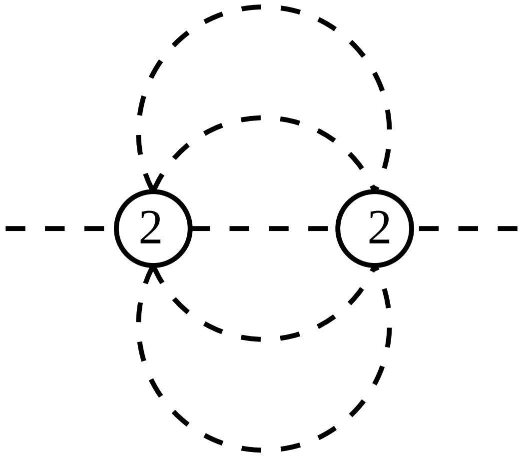
Examples ($n = 4$ dimensions):



$$
\begin{aligned}
D &= 4\cdot 2 - 2\cdot 3 + 2\cdot 2 = 6 \\
  &= 2 + 2\cdot 2 + (2-2)\cdot 2
\end{aligned}
$$



$$D = 4\cdot 2 - 2\cdot 3 + 1\cdot 2 + 1\cdot 4 = 8$$



$$D = 4 \cdot 4 - 2 \cdot 5 + 2 \cdot 2 = 10$$

Proof: Start from Feynman rules for evaluating S-matrix element.

- Internal lines:

$$\int \mathrm{d}^4 k \frac{1}{k^2 - M^2 + i\epsilon} \quad \overset{M^2 \to t^2 M^2}{\to} \quad \int \mathrm{d}^4 k \frac{1}{t^2(k^2/t^2 - M^2 + i\epsilon)}$$

$$k \underline{\underline{=}} tl \qquad t^2 \int \mathrm{d}^4 l \frac{1}{l^2 - M^2 + i\epsilon}.$$

- Vertices with $2k$ derivatives or k quark-mass terms:

$$\delta^4(q) q^{2k} \to t^{2k-4} \delta^4(q) q^{2k},$$

  – since $p \to tp$ if q is an external momentum,

  – and $k = tl$ if q is an internal momentum (see above).

- These are rules to calculate $S \sim \delta^4(P)\mathcal{M}$.
  Add 4 to compensate for overall momentum-conserving delta function.

- Scaling behavior of the contribution to $\mathcal{M}$ of a given diagram:

$$D = 4 + 2N_I + \sum_{k=1}^{\infty} N_{2k}(2k - 4).$$

- Relation between # of independent loops, # of internal lines, and total # of vertices $N_V = \sum_{k=1}^{\infty} N_{2k}$:

$$N_L = N_I - (N_V - 1).$$

Remark: Product of N_V momentum-conserving δ functions contains overall momentum conservation. $\Rightarrow$ $N_V - 1$ rather than N_V restrictions on internal momenta.

- Apply to

$$-4 \sum_{k=1}^{\infty} N_{2k} = -4N_V = 4(N_L - N_I - 1).$$

$\Rightarrow$ Eq. (5.31):

$$D = 4N_L - 2N_I + \sum_{k=1}^{\infty} 2kN_{2k}.$$

- Apply to

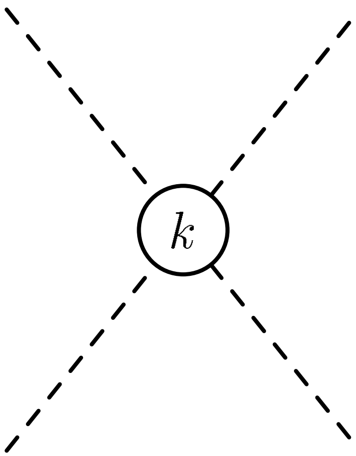
$$-4\sum_{k=1}^{\infty} N_{2k} = -2\sum_{k=1}^{\infty} N_{2k} + 2(N_L - N_I - 1).$$

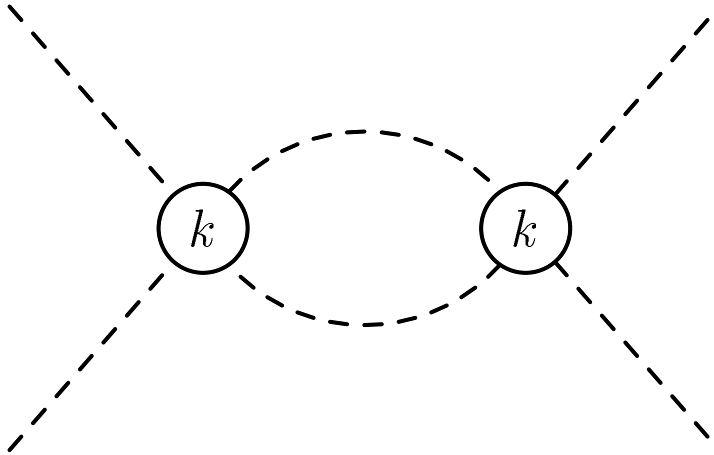
$\Rightarrow$ Eq. (5.32):

$$D = 2 + \sum_{k=1}^{\infty} 2(k-1)N_{2k} + 2N_L \geq 2.$$

In particular, diagrams containing loops are suppressed due to the term $2N_L$.

- Remark: Minimal $k > 0$ important.





$$D = 0 - 0 + k = k = \begin{cases} 0 \text{ for } k = 0, \\ 2 \text{ for } k = 2. \end{cases}$$

$$D = 4 - 2 \cdot 2 + 2 \cdot k = 2k = \begin{cases} 0 \text{ for } k = 0, \\ 4 \text{ for } k = 2. \end{cases}$$

Loop diagram is only suppressed if $k_{\text{min}} > 0$.

5.4 Construction of the Effective Lagrangian

References:

- J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465 (1985)
- H. W. Fearing and S. Scherer, Phys. Rev. D **53**, 315 (1996)
- J. Bijnens, G. Colangelo, and G. Ecker, JHEP **9902**, 020 (1999)
- T. Ebertshäuser, *Mesonic Chiral Perturbation Theory: Odd Intrinsic Parity Sector*, PhD thesis, Johannes Gutenberg-Universität, Mainz, Germany, 2001, http://archimed.uni-mainz.de/
- T. Ebertshäuser, H. W. Fearing, and S. Scherer, Phys. Rev. D **65**, 054033 (2002)
- J. Bijnens, L. Girlanda, and P. Talavera, Eur. Phys. J. C **23**, 539 (2002)

So far: Lowest-order effective Lagrangian for *global* $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry.
Sec. 3.4: Ward identities of QCD are obtained from *locally* invariant generating functional involving a coupling to external fields.
Follow Gasser and Leutwyler: Promote global symmetry of effective Lagrangian to a local one,

$$L \to V_L(x),\ R \to V_R(x),$$

and introduce coupling to the *same* external fields v, a, s, and p as in QCD [see Eq. (3.59)].
Collect Goldstone bosons in special unitary matrix

$$U(x) = \exp\left(i\frac{\phi(x)}{F_0}\right).$$

Transformation behavior under $G = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$, parity P, and charge conjugation C:

$$\begin{aligned} U &\stackrel{G}{\mapsto} V_R U V_L^\dagger, \\ U(\vec{x},t) &\stackrel{P}{\mapsto} U^\dagger(-\vec{x},t), \\ U &\stackrel{C}{\mapsto} U^T \quad (\text{Exercise}). \end{aligned}$$

Let object A transform as $V_R A V_L^\dagger$. Define covariant derivative $D_\mu A$ as (Exercise)

$$\begin{aligned} D_\mu A &\equiv \partial_\mu A - i r_\mu A + i A l_\mu \\ &\mapsto V_R(\partial_\mu A - i r_\mu A + i A l_\mu) V_L^\dagger = V_R (D_\mu A) V_L^\dagger. \end{aligned} \tag{5.33}$$

Defining property: Covariant derivative should transform as object it acts on.
In particular

$$D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu. \tag{5.34}$$

Introduce field strength tensors

$$f^R_{\mu\nu} \equiv \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu] \overset{G}{\mapsto} V_R f^R_{\mu\nu} V_R^\dagger, \quad (5.35)$$

$$f^L_{\mu\nu} \equiv \partial_\mu l_\nu - \partial_\nu l_\mu - i[l_\mu, l_\nu] \overset{G}{\mapsto} V_L f^L_{\mu\nu} V_L^\dagger. \quad (5.36)$$

Field strength tensors are traceless,

$$\mathrm{Tr}(f^L_{\mu\nu}) = \mathrm{Tr}(f^R_{\mu\nu}) = 0,$$

because $\mathrm{Tr}(l_\mu) = \mathrm{Tr}(r_\mu) = 0$ and trace of any commutator vanishes.
Introduce linear combination $\chi = 2B_0(s + ip)$.
E.g., pure QCD: $\chi = 2B_0 \mathrm{diag}(m_u, m_d, m_s)$.
Construct effective Lagrangian in terms of U, $U^\dagger$, χ, $\chi^\dagger$, $f^R_{\mu\nu}$, $f^L_{\mu\nu}$ and covariant derivatives of these objects.

Construction of invariants

Suppose we have matrices $A, B, C, \ldots$, all of which transform as

$$A \overset{G}{\mapsto} V_R A V_L^\dagger,$$

$$B \stackrel{G}{\mapsto} V_R B V_L^\dagger,$$

$$\ldots$$

Form invariants by "multiplying" in the following way:

$$\mathrm{Tr}(AB^\dagger) \stackrel{G}{\mapsto} \mathrm{Tr}(V_R A \underbrace{V_L^\dagger V_L}_{1} B^\dagger V_R^\dagger) = \mathrm{Tr}(V_R^\dagger V_R A B^\dagger) = \mathrm{Tr}(AB^\dagger).$$

- Generalization to more terms is obvious.
- Product of invariant traces is invariant,

$$\mathrm{Tr}(AB^\dagger CD^\dagger), \quad \mathrm{Tr}(AB^\dagger)\mathrm{Tr}(CD^\dagger), \quad \cdots. \tag{5.37}$$

List of (selected) ingredients

- Assign (chiral) orders:

$$\begin{aligned} U &= \mathcal{O}(q^0), \\ D_\mu U &= \mathcal{O}(q), \end{aligned}$$

$$\begin{aligned} r_\mu, l_\mu &= \mathcal{O}(q), \\ f_{\mu\nu}^{L/R} &= \mathcal{O}(q^2), \\ \chi &= \mathcal{O}(q^2). \end{aligned}$$

- Each covariant derivative produces power of q.
- Identify terms which can be related by total derivatives.
- List of objects A up to and including order q^2 which transform as $A' = V_R A V_L^\dagger$:

$$U,\ D_\mu U,\ D_\mu D_\nu U,\ \chi,\ U f_{\mu\nu}^L,\ f_{\mu\nu}^R U.$$

- Construction of chirally invariant expressions (to order q^2):

$$\begin{aligned} \mathcal{O}(q^0) &: \ \mathrm{Tr}\left(UU^\dagger\right) = \mathrm{Tr}(\mathbb{1}) = const. \\ \mathcal{O}(q) &: \ \mathrm{Tr}\left(D_\mu U U^\dagger\right) = 0, \quad (\text{Exercise}) \end{aligned}$$

important: excludes terms of the

$$
\begin{aligned}
& \text{type } \mathrm{Tr}[\mathcal{O}(q)] \times \mathrm{Tr}(\ldots)\ , \\
\mathcal{O}(q^2)\ :\ & \mathrm{Tr}\left(D_\mu D_\nu U U^\dagger\right) = -\mathrm{Tr}\left[D_\nu U (D_\mu U)^\dagger\right] \text{ (Exercise)}, \\
& \mathrm{Tr}\left[D_\mu U (D_\nu U)^\dagger\right]\ , \\
& \mathrm{Tr}\left[U (D_\mu D_\nu U)^\dagger\right] = -\mathrm{Tr}\left[D_\mu U (D_\nu U)^\dagger\right] \text{ (Exercise)}, \\
& \mathrm{Tr}\left(\chi U^\dagger\right)\ , \\
& \mathrm{Tr}\left(U \chi^\dagger\right)\ , \\
& \mathrm{Tr}\left[(U f^L_{\mu\nu}) U^\dagger\right] = \mathrm{Tr}\left(f^L_{\mu\nu}\right) = 0, \\
& \mathrm{Tr}\left(f^R_{\mu\nu}\right) = 0.
\end{aligned}
$$

- Lorentz invariance: Indices have to be contracted.

- Candidates:

$$
\begin{aligned}
& \mathrm{Tr}\left[D_\mu U (D^\mu U)^\dagger\right]\ , \\
& \mathrm{Tr}\left(\chi U^\dagger \pm U \chi^\dagger\right)\ .
\end{aligned}
$$

- Parity:

$$\mathcal{L}(\vec{x},t) \stackrel{P}{\mapsto} \mathcal{L}(-\vec{x},t).$$

  $\mathrm{Tr}(\chi U^\dagger - U\chi^\dagger)$ has wrong parity.

- Charge conjugation [no additional constraint at $\mathcal{O}(q^2)$] (Exercise).

Lowest–order Lagrangian $\mathcal{L}_2$

$$\mathcal{L}_2 = \frac{F_0^2}{4}\mathrm{Tr}\left[D_\mu U (D^\mu U)^\dagger\right] + \frac{F_0^2}{4}\mathrm{Tr}\left(\chi U^\dagger + U\chi^\dagger\right). \tag{5.38}$$

At $\mathcal{O}(q^2)$ two parameters:

$$F_0 \approx 93\,\mathrm{MeV}, \quad 3F_0^2 B_0 = -\langle 0|\bar{q}q|0\rangle_0$$

- $\mathcal{L}_2$ has predictive power! $\pi\pi$ scattering etc. (Exercise).

Lowest-order equation of motion

Essentially as in Assignment 1, 3.

Consider small variations of SU(3) matrix:

$$U'(x) = U(x) + \delta U(x) = \left(1 + i\sum_{a=1}^{8} \Delta_a(x)\lambda_a\right) U(x), \qquad (5.39)$$

$\Delta_a(x)$: Real functions. Matrix U' satisfies both conditions

$$U'U'^\dagger = 1, \quad \det(U') = 1, \qquad (5.40)$$

up to and including terms linear in Δ_a.

Apply principle of stationary action. Variation of action

$$\begin{aligned}
\delta S &= \frac{F_0^2}{4}\int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}^3x\, \mathrm{Tr}\left[D_\mu \delta U (D^\mu U)^\dagger + D_\mu U (D^\mu \delta U)^\dagger + \chi \delta U^\dagger + \delta U \chi^\dagger\right] \\
&\overset{*}{=} \frac{F_0^2}{4}\int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}^3x\, \mathrm{Tr}\left[-\delta U (D_\mu D^\mu U)^\dagger - D_\mu D^\mu U \delta U^\dagger + \chi \delta U^\dagger + \delta U \chi^\dagger\right] \\
&\overset{**}{=} i\frac{F_0^2}{4}\int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}^3x \sum_{a=1}^{8} \Delta_a(x)
\end{aligned}$$

$$\times\mathrm{Tr}\left\{\lambda_a[D_\mu D^\mu U U^\dagger - U(D_\mu D^\mu U)^\dagger - \chi U^\dagger + U\chi^\dagger]\right\}. \qquad (5.41)$$

$*$: partial integration + standard boundary conditions $\Delta_a(t_1,\vec{x}) = \Delta_a(t_2,\vec{x}) = 0$ + divergence theorem + definition of covariant derivative of Eq. (5.33).

Example:

$$\begin{aligned}
\mathrm{Tr}[D_\mu \delta U (D^\mu U)^\dagger] &= \mathrm{Tr}[(\partial_\mu \delta U - i r_\mu \delta U + i\delta U l_\mu)(D^\mu U)^\dagger] \\
&= \partial_\mu \mathrm{Tr}[\delta U (D^\mu U)^\dagger] \\
&\quad -\mathrm{Tr}\{\delta U[\partial_\mu (D^\mu U)^\dagger + i(D^\mu U)^\dagger r_\mu - i l_\mu (D^\mu U)^\dagger]\} \\
&= \text{tot. der.} - \mathrm{Tr}[\delta U (D_\mu D^\mu U)^\dagger].
\end{aligned}$$

$**$: $\delta U^\dagger = -U^\dagger \delta U U^\dagger$ + invariance of trace with respect to cyclic permutations.

Functions $\Delta_a(x)$ may be chosen arbitrarily $\Rightarrow$ eight Euler-Lagrange equations

$$\mathrm{Tr}\left\{\lambda_a[D^2UU^\dagger - U(D^2U)^\dagger - \chi U^\dagger + U\chi^\dagger]\right\} = 0, \quad a = 1, \cdots, 8. \tag{5.42}$$

Any 3×3 matrix A can be written as

$$A = a_0 1_{3\times 3} + \sum_{i=1}^{8} a_i\lambda_i, \quad a_0 = \frac{1}{3}\mathrm{Tr}(A), \quad a_i = \frac{1}{2}\mathrm{Tr}(\lambda_i A). \tag{5.43}$$

$\Rightarrow$ Compact matrix form

$$\mathcal{O}^{(2)}_{\mathrm{EOM}}(U) \equiv D^2UU^\dagger - U(D^2U)^\dagger - \chi U^\dagger + U\chi^\dagger + \frac{1}{3}\mathrm{Tr}(\chi U^\dagger - U\chi^\dagger) = 0. \tag{5.44}$$

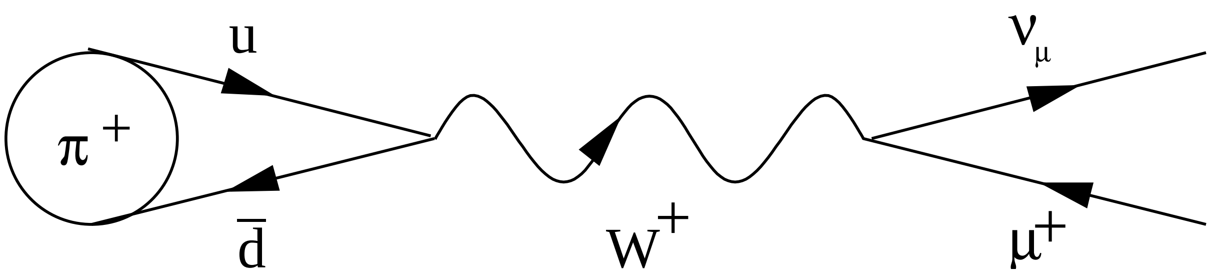
Remark: $\mathrm{Tr}[D^2UU^\dagger - U(D^2U)^\dagger] = 0$ (Exercise).

5.5 Applications at Lowest Order

Eq. (5.32): $D = 2 + 2N_L + \sum_{k=1}^{\infty} 2(k-1)N_{2k}$. Lowest order: $D = 2$, i.e. tree-level diagrams with vertices from $\mathcal{L}_2$.

5.5.1 Pion Decay $\pi^+ \to \mu^+\nu_\mu$

Diagrammatic representation in terms of d.o.f. of Standard Model:



- Interaction of massive charged weak bosons $\mathcal{W}_\rho^\pm = (\mathcal{W}_{1\rho} \mp i\mathcal{W}_{2\rho})/\sqrt{2}$ with leptons

$$\mathcal{L}_{\rm CC}^{(l)} = -\frac{g}{2\sqrt{2}} \left[\mathcal{W}_\rho^+ \bar{\nu}_\mu \gamma^\rho (1-\gamma_5)\mu + \mathcal{W}_\rho^- \bar{\mu}\gamma^\rho(1-\gamma_5)\nu_\mu \right].$$

CC: Charged current.
Fermi constant is related to gauge coupling g and W mass as

$$G_F = \sqrt{2}\frac{g^2}{8M_W^2} = 1.166\,37(1) \times 10^{-5}\,\mathrm{GeV}^{-2}.$$

- Coupling of W bosons to quarks

$$\mathcal{L}_{\mathrm{CC}}^{(q)} = -\frac{g}{2\sqrt{2}}\left\{\mathcal{W}_\rho^+[V_{ud}\bar{u}\gamma^\rho(1-\gamma_5)d + V_{us}\bar{u}\gamma^\rho(1-\gamma_5)s] + h.c.\right\},$$

$$|V_{ud}| = 0.97377 \pm 0.00027, \quad |V_{us}| = 0.2257 \pm 0.0021.$$

- Express in terms of QCD Lagrangian with coupling to external fields.
Set

$$r_\mu = 0, \quad l_\mu = -\frac{g}{\sqrt{2}}(\mathcal{W}_\mu^+ T_+ + h.c.)$$

in $\mathcal{L}_{\text{ext}}$ [see Eq. (3.59)], where

$$T_+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- Coupling of W bosons to Goldstone bosons.
  Insert covariant derivative

$$D_\mu U = \partial_\mu U + iU l_\mu$$

into $\mathcal{L}_2$ and identify terms $\sim W\phi$:

$$\frac{F_0^2}{4}\text{Tr}[D_\mu U (D^\mu U)^\dagger] \overset{U^\dagger\partial^\mu U=-\partial^\mu U^\dagger U}{=} i\frac{F_0^2}{2}\text{Tr}(l_\mu \partial^\mu U^\dagger U) + \cdots$$

$$\overset{\partial^\mu U^\dagger = -i\partial^\mu\phi/F_0+\cdots}{=} \frac{F_0}{2}\text{Tr}(l_\mu \partial^\mu \phi) + \cdots.$$

$\Rightarrow$

$$\mathcal{L}_{W\phi} = -\frac{g}{\sqrt{2}}\frac{F_0}{2}\text{Tr}[(\mathcal{W}_\mu^+ T_+ + \mathcal{W}_\mu^- T_-)\partial^\mu\phi].$$

Evaluate

$$
\begin{aligned}
&\mathrm{Tr}(T_+\partial^\mu\phi) \\
&= \mathrm{Tr}\left[\begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \partial^\mu \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}\right] \\
&= V_{ud}\sqrt{2}\partial^\mu\pi^- + V_{us}\sqrt{2}\partial^\mu K^-, \\
&\mathrm{Tr}(T_-\partial^\mu\phi) \\
&= \mathrm{Tr}\left[\begin{pmatrix} 0 & 0 & 0 \\ V_{ud} & 0 & 0 \\ V_{us} & 0 & 0 \end{pmatrix} \partial^\mu \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}\right] \\
&= V_{ud}\sqrt{2}\partial^\mu\pi^+ + V_{us}\sqrt{2}\partial^\mu K^+.
\end{aligned}
$$

(V_{ud} and V_{us} real.)

$\Rightarrow$ Interaction Lagrangian

$$\mathcal{L}_{W\phi} = -g\frac{F_0}{2}[\mathcal{W}_\mu^+(V_{ud}\partial^\mu\pi^- + V_{us}\partial^\mu K^-) + \mathcal{W}_\mu^-(V_{ud}\partial^\mu\pi^+ + V_{us}\partial^\mu K^+)]. \tag{5.45}$$

- Expand Feynman propagator for W bosons

$$\frac{-g_{\mu\nu} + \frac{k_\mu k_\nu}{M_W^2}}{k^2 - M_W^2} = \frac{g_{\mu\nu}}{M_W^2} + O(\frac{kk}{M_W^4}).$$

- Neglect terms of higher order in (momentum/M_W)^2.
- Feynman rule for invariant amplitude for weak pion decay

$$\begin{aligned}\mathcal{M} &= \underbrace{i\left[-\frac{g}{2\sqrt{2}}\bar{u}_{\nu_\mu}\gamma^\rho(1-\gamma_5)v_{\mu^+}\right]}_{\text{leptonic vertex}} \quad \underbrace{\frac{ig_{\rho\sigma}}{M_W^2}}_{W\text{ propagator}} \quad \underbrace{i\left[-g\frac{F_0}{2}V_{ud}(-ip^\sigma)\right]}_{\text{hadronic vertex}} \\ &= -G_F V_{ud} F_0 \bar{u}_{\nu_\mu}\not{p}(1-\gamma_5)v_{\mu^+},\end{aligned}$$

p: four-momentum of pion.

- Decay rate (Exercise)

$$\frac{1}{\tau} = \frac{G_F^2 |V_{ud}|^2}{4\pi} F_0^2 M_\pi m_\mu^2 \left(1 - \frac{m_\mu^2}{M_\pi^2}\right)^2 .$$

- F_0: pion-decay constant in the chiral limit. Measures strength of matrix element of axial-vector current between one-Goldstone boson state and vacuum.

- Degeneracy of a single coupling constant F_0 removed at $\mathcal{O}(q^4)$.

- Empirical numbers:

$$\begin{aligned} F_\pi &= 92.4\,\text{MeV}, \\ F_K &= 113\,\text{MeV}. \end{aligned}$$

5.5.2 Pion-Pion Scattering

References:

- S. Weinberg, Phys. Rev. Lett. **17**, 616 (1966)
- J. Gasser and H. Leutwyler, Annals Phys. **158**, 142 (1984)
- M. A. Preston, *Physics of the Nucleus* (Addison-Wesley, Reading, MA, 1962)
- J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, and M. E. Sainio, Phys. Lett. B **374**, 210 (1996)
- G. Colangelo, J. Gasser, and H. Leutwyler, Phys. Rev. Lett. **86**, 5008 (2001)
- G. Colangelo, J. Gasser, and H. Leutwyler, Nucl. Phys. **B603**, 125 (2001)

Consider $\mathcal{L}_2$ with $r_\mu = l_\mu = 0$,

$$\mathcal{L}_2 = \frac{F^2}{4}\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \frac{F^2}{4}\mathrm{Tr}\left(\chi U^\dagger + U\chi^\dagger\right)$$

in the SU(2) sector with

$$\chi = 2B \underbrace{\begin{pmatrix} \hat{m} & 0 \\ 0 & \hat{m} \end{pmatrix}}_{\mathcal{M}}$$

and

$$U = \exp\left(i\frac{\phi}{F}\right), \quad \phi = \sum_{i=1}^{3} \tau_i \phi_i =: \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}.$$

Remark on chiral limit:

- In the SU(2) sector it is common to express quantities in the chiral limit without index 0, e. g., F and B. By this one means the SU(2) chiral limit, i. e. $m_u = m_d = 0$ but m_s at its physical value.

- In the SU(3) sector the quantities F_0 and B_0 denote the chiral limit for all three quarks: $m_u = m_d = m_s = 0$.

Substitution $U \leftrightarrow U^\dagger$. $\Rightarrow$ $\mathcal{L}_2$ contains even powers of ϕ only:

$$\mathcal{L}_2 = \mathcal{L}_2^{2\phi} + \mathcal{L}_2^{4\phi} + \cdots .$$

- $\mathcal{L}_2$ does not produce a vertex with 3 Goldstone bosons. $\Rightarrow$ At $D = 2$, no s-, u-, and t-channel pole diagrams.
- At $D = 2$, $\pi\pi$ scattering is generated by a 4 Goldstone boson interaction term.

Expand

$$U = 1 + i\frac{\phi}{F} - \frac{1}{2}\frac{\phi^2}{F^2} - \frac{i}{6}\frac{\phi^3}{F^3} + \frac{1}{24}\frac{\phi^4}{F^4} + \cdots$$

and identify $\mathcal{L}_2^{4\phi}$ as (Exercise)

$$\mathcal{L}_2^{4\phi} = \frac{1}{48F^2}\left[\mathrm{Tr}([\phi, \partial_\mu\phi][\phi, \partial^\mu\phi]) + 2B\mathrm{Tr}(\mathcal{M}\phi^4)\right].$$

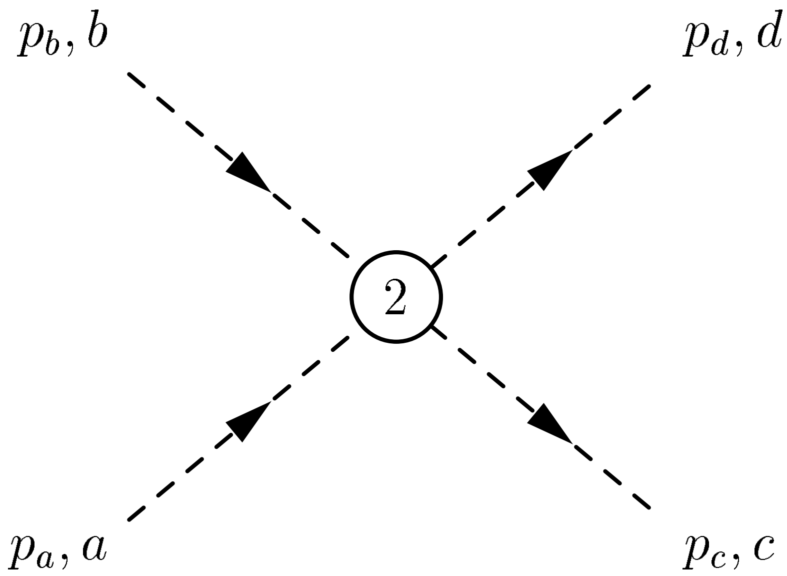
Remark: Substituting $F \to F_0$, $B \to B_0$ and the relevant expressions for ϕ and the quark mass matrix $\mathcal{M}$ the corresponding formula for SU(3) looks identical.
Insert $\phi = \tau_i \phi_i$. Make use of

$$
\begin{aligned}
[\phi, \partial_\mu \phi] &= 2i\epsilon_{ijk}\phi_i \partial_\mu \phi_j \tau_k, \\
\mathrm{Tr}(\tau_k \tau_n) &= 2\delta_{kn}, \\
\epsilon_{ijk}\epsilon_{lmk} &= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}, \\
\phi^2 &= \phi_i \phi_i.
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
\mathcal{L}_2^{4\phi} &= -\frac{1}{6F^2}\epsilon_{ijk}\phi_i \partial_\mu \phi_j \epsilon_{lmk}\phi_l \partial^\mu \phi_m + \frac{M^2}{24F^2}\phi_i\phi_i\phi_j\phi_j \\
&= \frac{1}{6F^2}(\phi_i \partial^\mu \phi_i \partial_\mu \phi_j \phi_j - \phi_i \phi_i \partial_\mu \phi_j \partial^\mu \phi_j) + \frac{M^2}{24F^2}\phi_i\phi_i\phi_j\phi_j,
\end{aligned}
$$

where $M^2 = 2B\hat{m}$.



Feynman rule for Cartesian isospin indices a, b, c, and d from "$i\mathcal{L}$".
Example

$$\langle p_c, c; p_d, d | \phi_i \partial^\mu \phi_i \partial_\mu \phi_j \phi_j | p_a, a; p_b, b \rangle$$

24 combinations of combining 4 fields with 4 quanta. E.g.

$$\langle p_c, c; p_d, d | \phi_i \, \partial^\mu \phi_i \, \partial_\mu \phi_j \, \phi_j | p_a, a; p_b, b \rangle$$

$$\Rightarrow \; \delta_{ic} i p_d^\mu \delta_{id}\delta_{ja}(-ip_{b\mu})\delta_{jb} = p_d \cdot p_b \, \delta_{cd}\delta_{ab}.$$

Complete result

$$\begin{aligned}
\mathcal{M} \;=\; & i\Bigg[\frac{1}{6F^2}\,(2\,[\delta_{ab}\delta_{cd}(-ip_a - ip_b)\cdot(ip_c + ip_d) \\
& \qquad +\delta_{ac}\delta_{bd}(-ip_a + ip_c)\cdot(-ip_b + ip_d) \\
& \qquad +\delta_{ad}\delta_{bc}(-ip_a + ip_d)\cdot(-ip_b + ip_c)] \\
& \quad -4\,\{\delta_{ab}\delta_{cd}\,[(-ip_a)\cdot(-ip_b) + (ip_c)\cdot(ip_d)] \\
& \qquad +\delta_{ac}\delta_{bd}[(-ip_a)\cdot(ip_c) + (-ip_b)\cdot(ip_d)] \\
& \qquad +\delta_{ad}\delta_{bc}[(-ip_a)\cdot(ip_d) + (-ip_b)\cdot(ip_c)]\}) \\
& +\frac{M^2}{24F^2} 8(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\Bigg] \\
=\; & \frac{i}{3F^2}\,\{\delta_{ab}\delta_{cd}[(p_a + p_b)^2 + 2p_a\cdot p_b + 2p_c\cdot p_d + M^2] \\
& \quad +\delta_{ac}\delta_{bd}[(p_a - p_c)^2 - 2p_a\cdot p_c - 2p_b\cdot p_d + M^2]
\end{aligned}$$

$$
\begin{aligned}
&\quad +\delta_{ad}\delta_{bc}[(p_a - p_d)^2 - 2p_a \cdot p_d - 2p_b \cdot p_c + M^2]\Big\} \\
&= \frac{i}{3F^2}\left[\delta_{ab}\delta_{cd}(3s - p_a^2 - p_b^2 - p_c^2 - p_d^2 + M^2)\right. \\
&\quad +\delta_{ac}\delta_{bd}(3t - p_a^2 - p_c^2 - p_b^2 - p_d^2 + M^2) \\
&\quad \left. +\delta_{ad}\delta_{bc}(3u - p_a^2 - p_d^2 - p_b^2 - p_c^2 + M^2)\right] \\
&= i\left[\delta_{ab}\delta_{cd}\frac{s - M^2}{F^2} + \delta_{ac}\delta_{bd}\frac{t - M^2}{F^2} + \delta_{ad}\delta_{bc}\frac{u - M^2}{F^2}\right] \\
&\quad -\frac{i}{3F^2}\left(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right)\left(\Lambda_a + \Lambda_b + \Lambda_c + \Lambda_d\right), \qquad (5.46)
\end{aligned}
$$

where $\Lambda_k = p_k^2 - M^2$.

Mandelstam variables

$$
\begin{aligned}
s &= (p_a + p_b)^2 = (p_c + p_d)^2, \\
t &= (p_a - p_c)^2 = (p_d - p_b)^2, \\
u &= (p_a - p_d)^2 = (p_c - p_b)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
2p_a \cdot p_b &= s - p_a^2 - p_b^2, \quad 2p_c \cdot p_d = s - p_c^2 - p_d^2, \\
-2p_a \cdot p_c &= t - p_a^2 - p_c^2, \quad -2p_b \cdot p_d = t - p_b^2 - p_d^2, \\
-2p_a \cdot p_d &= u - p_a^2 - p_d^2, \quad -2p_b \cdot p_c = u - p_b^2 - p_c^2.
\end{aligned}
$$

The last line of the Feynman rule disappears, if the external lines satisfy on-mass-shell conditions.
Scattering process $\pi_a(p_a) + \pi_b(p_b) \to \pi_c(p_c) + \pi_d(p_d)$ at $\mathcal{O}(q^2)$:

$$
T = \delta_{ab}\delta_{cd}\frac{s - M_\pi^2}{F_\pi^2} + \delta_{ac}\delta_{bd}\frac{t - M_\pi^2}{F_\pi^2} + \delta_{ad}\delta_{bc}\frac{u - M_\pi^2}{F_\pi^2}.
$$

$(iT = \mathcal{M})$. We replaced

$$
\begin{aligned}
F \to F_\pi, &\quad F_\pi = F[1 + \mathcal{O}(q^2)], \\
M^2 \to M_\pi^2, &\quad M_\pi^2 = M^2[1 + \mathcal{O}(q^2)],
\end{aligned}
$$

because difference is of $\mathcal{O}(q^4)$ in T.

Consider (theoretical) limit $M_\pi^2, s, t, u \to 0$:

$$T \to 0.$$

- Goldstone bosons interact "weakly" at low energies.

Isospin symmetry. $\Rightarrow$ Most general parametrization

$$T = \delta_{ab}\delta_{cd}A(s,t,u) + \delta_{ac}\delta_{bd}A(t,s,u) + \delta_{ad}\delta_{bc}A(u,t,s)$$

with $A(s,t,u) = A(s,u,t)$.
Isospin channels:

$$\begin{aligned} T^{I=0} &= 3A(s,t,u) + A(t,u,s) + A(u,s,t), \\ T^{I=1} &= A(t,u,s) - A(u,s,t), \\ T^{I=2} &= A(t,u,s) + A(u,s,t). \end{aligned}$$

s-wave $\pi\pi$ scattering lengths (Convention in ChPT differs by a factor $(-M_\pi)$ from the usual definition of a scattering length in the effective range expansion.)

$$T^{I=0}|_{\rm thr} = 32\pi a_0^0,$$
$$T^{I=2}|_{\rm thr} = 32\pi a_0^2.$$

Lower index 0: s wave; upper index: Isospin. ($T^{I=1}|_{\rm thr}$ vanishes because of Bose symmetry.)

- $\pi^+\pi^+$ scattering described by $T^{I=2}$.
- Other physical reactions may be determined using the appropriate Clebsch-Gordan coefficients.

Prediction at $\mathcal{O}(q^2)$:

$$A(s,t,u) = \frac{s - M_\pi^2}{F_\pi^2}.$$

At threshold

$$s_{\rm thr} = (2M_\pi)^2$$

and thus

$$A(s_{\rm thr}, t_{\rm thr}, u_{\rm thr}) = \frac{3M_\pi^2}{F_\pi^2}.$$

- $I = 0$: Consider linear combination

$$\begin{aligned}
&[3A(s,t,u) + A(t,u,s) + A(u,s,t)]_{\rm thr} \\
&= [2A(s,t,u) + A(s,t,u) + A(t,u,s) + A(u,s,t)]_{\rm thr} \\
&= \frac{6M_\pi^2}{F_\pi^2} + \frac{[s+t+u-3M_\pi^2]_{\rm thr}}{F_\pi^2} \\
&= \frac{7M_\pi^2}{F_\pi^2}
\end{aligned}$$

- $I = 2$: Consider linear combination

$$\begin{aligned}
&[A(t,u,s) + A(u,s,t)]_{\rm thr} \\
&= [A(t,u,s) + A(u,s,t) + A(s,t,u) - A(s,t,u)]_{\rm thr}
\end{aligned}$$

$$= \frac{M_\pi^2}{F_\pi^2} - \frac{3M_\pi^2}{F_\pi^2}$$
$$= -\frac{2M_\pi^2}{F_\pi^2}.$$

- $\Rightarrow$ Famous results of current algebra for the scattering lengths (S. Weinberg, Phys. Rev. Lett. **17**, 616 (1966)):

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.156, \quad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} = -0.045.$$

(with $F_\pi = 93.2$ MeV and $M_\pi = 139.57$ MeV)

- Absolute prediction of chiral symmetry! Once we know F_π (from pion decay) we can predict the scattering lengths.
- Different from Wigner-Eckart theorem which predicts relations among processes of the same type.

Experimental data

$\pi^\pm p \to \pi^\pm\pi^+ n$:[1] $a_0^0 = 0.204 \pm 0.014\,(\text{stat}) \pm 0.008\,(\text{syst})$,

$K^+ \to \pi^+\pi^- e^+\nu_e$:[2] $a_0^0 = 0.216 \pm 0.013\,(\text{stat}) \pm 0.002\,(\text{syst}) \pm 0.002\,(\text{theor})$,

$\pi^+\pi^-$ atom lifetime:[3] $|a_0^0 - a_0^2| = 0.264^{+0.033}_{-0.020}$,

$K^\pm \to \pi^\pm\pi^0\pi^0$:[4] $a_0^0 - a_0^2 = 0.268 \pm 0.010\,(\text{stat}) \pm 0.004\,(\text{syst}) \pm 0.013\,(\text{ext})$,

$a_0^2 = -0.041 \pm 0.022\,(\text{stat}) \pm 0.014\,(\text{syst})$.

Predictions for the s-wave scattering lengths at $\mathcal{O}(q^6)$[5]

$$a_0^0 = \overbrace{0.156}^{\mathcal{O}(q^2)} + \overbrace{\underbrace{0.039}_{\text{L}} + \underbrace{0.005}_{\text{anal.}}}^{\mathcal{O}(q^4):\ +28\%} + \overbrace{\underbrace{0.013}_{k_i} + \underbrace{0.003}_{\text{L}} + \underbrace{0.001}_{\text{anal.}}}^{\mathcal{O}(q^6):\ +8.5\%} = \overbrace{0.217}^{\text{total}},$$

---

[1] M. Kermani et al. [CHAOS Collaboration], Phys. Rev. C 58, 3431 (1998)

[2] S. Pislak et al., Phys. Rev. D 67, 072004 (2003)

[3] B. Adeva et al. [DIRAC Collaboration], Phys. Lett. B 619, 50 (2005)

[4] J. R. Batley et al. [NA48/2 Collaboration], Phys. Lett. B 633, 173 (2006)

[5] J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, and M. E. Sainio, Phys. Lett. B 374, 210 (1996)

$$a_0^0 - a_0^2 = \overbrace{0.201}^{\mathcal{O}(q^2)} + \overbrace{\underbrace{0.036}_{\text{L}} + \underbrace{0.006}_{\text{anal.}}}^{\mathcal{O}(q^4):\ +21\%} + \overbrace{\underbrace{0.012}_{k_i} + \underbrace{0.003}_{\text{L}} + \underbrace{0.001}_{\text{anal.}}}^{\mathcal{O}(q^6):\ +6.6\%} = \overbrace{0.258}^{\text{total}} .$$

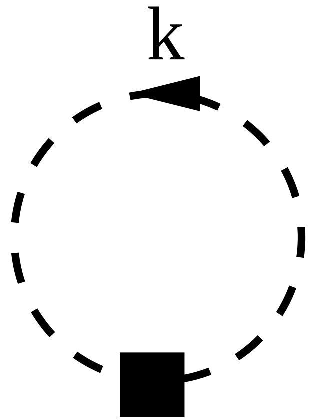
- $\mathcal{O}(q^4)$
  - chiral logarithms of one-loop diagrams (L)
  - analytic contributions from one-loop diagrams + tree graphs from $\mathcal{L}_4$ (anal.)
- $\mathcal{O}(q^6)$
  - loop corrections involving double chiral logarithms (k_i)
  - loop corrections with chiral logarithms (L)
  - analytic contributions (anal.)

5.6 Dimensional Regularization: Basics

References:

- G. 't Hooft and M. J. Veltman, Nucl. Phys. **B44**, 189 (1972)
- G. Leibbrandt, Rev. Mod. Phys. **47**, 849 (1975)
- G. 't Hooft and M. J. Veltman, Nucl. Phys. **B153**, 365 (1979)
- T. P. Cheng and L. F. Li, *Gauge Theory of Elementary Particle Physics* (Clarendon, Oxford, 1984), chapter 2
- J. C. Collins, *Renormalization* (Cambridge University Press, Cambridge, 1984), chapter 4
- M. J. Veltman, *Diagrammatica. The Path to Feynman Rules* (Cambridge University Press, Cambridge, 1994)
- Any modern book on quantum field theory

$D \geq 4$: We need to discuss loops!
Simple example



Consider integral

$$I = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - M^2 + i0^+}. \tag{5.47}$$

Introduce

$$a \equiv \sqrt{\vec{k}^2 + M^2} > 0$$

so that

$$
\begin{aligned}
k^2 - M^2 + i0^+ &= k_0^2 - \vec{k}^2 - M^2 + i0^+ \\
&= k_0^2 - a^2 + i0^+ \\
&= k_0^2 - (a - i0^+)^2 \\
&= [k_0 + (a - i0^+)][k_0 - (a - i0^+)].
\end{aligned}
$$

Define

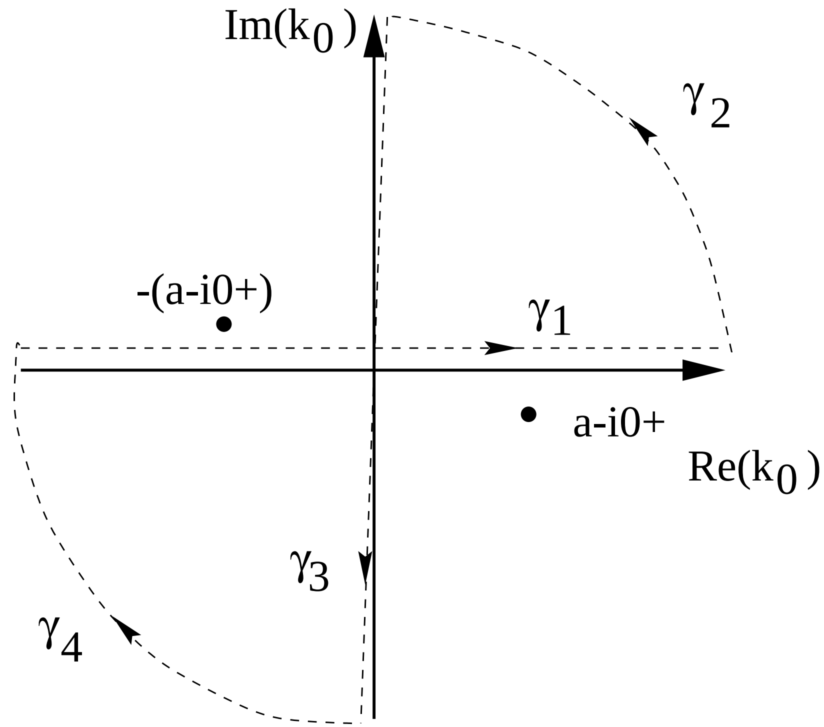
$$
f(k_0) = \frac{1}{[k_0 + (a - i0^+)][k_0 - (a - i0^+)]}.
$$

Determine $\int_{-\infty}^{\infty} \mathrm{d}k_0 f(k_0)$ as part of the calculation of I.
Consider f in the complex k_0 plane and make use of Cauchy's theorem

$$
\oint_C \mathrm{d}z f(z) = 0 \tag{5.48}
$$

for functions which are differentiable in every point inside the closed contour C.

Choose contour as



$$0 = \sum_{i=1}^{4} \int_{\gamma_i} \mathrm{d}z f(z).$$

Make use of

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f[\gamma(t)]\gamma'(t)\mathrm{d}t$$

to obtain for individual integrals:

$$\gamma_1(t) = t,\ \gamma_1'(t) = 1,\ a = -\infty,\ b = \infty : \quad \int_{\gamma_1} f(z)\mathrm{d}z = \int_{-\infty}^{\infty} f(t)\mathrm{d}t,$$

$$\gamma_2(t) = Re^{it},\ \gamma_2'(t) = iRe^{it},\ a = 0,\ b = \frac{\pi}{2} :$$

$$\int_{\gamma_2} f(z)\mathrm{d}z = \lim_{R\to\infty} \int_0^{\frac{\pi}{2}} f(Re^{it}) iRe^{it}\mathrm{d}t = 0,\ \text{because}\ \lim_{R\to\infty} \underbrace{Rf(Re^{it})}_{\sim \frac{1}{R}} = 0,$$

$$\gamma_3(t) = it,\ \gamma_3'(t) = i,\ a = +\infty,\ b = -\infty : \quad \int_{\gamma_3} f(z)\mathrm{d}z = \int_{\infty}^{-\infty} f(it)i\mathrm{d}t,$$

$$\gamma_4(t) = Re^{it},\ \gamma_4'(t) = iRe^{it},\ a = \frac{3}{2}\pi,\ b = \pi :$$

$$\int_{\gamma_4} f(z)\mathrm{d}z = 0 \text{ analogous to } \gamma_2.$$

Combine with Eq. (5.48) $\Rightarrow$ so-called Wick rotation

$$\int_{-\infty}^{\infty} f(t)\mathrm{d}t = -i\int_{\infty}^{-\infty} \mathrm{d}t f(it) = i\int_{-\infty}^{\infty} \mathrm{d}t f(it). \tag{5.49}$$

Intermediate result

$$\begin{aligned} I &= \frac{1}{(2\pi)^4} i \int_{-\infty}^{\infty} \mathrm{d}k_0 \int \mathrm{d}^3k \frac{i}{(ik_0)^2 - \vec{k}^2 - M^2 + i0^+} \\ &= \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{1}{l^2 + M^2 - i0^+}. \end{aligned}$$

$l^2 = l_1^2 + l_2^2 + l_3^2 + l_4^2$ denotes Euclidian scalar product. In this *special* case: Integrand does not have a pole $\Rightarrow$ we can omit the $-i0^+$.

- I diverges for large values of l [ultraviolet (UV) divergence].

UV degree of divergence can be estimated by simply counting powers of momenta.
If the integral behaves asymptotically as

$$\begin{aligned}
\int \mathrm{d}^4 l / l^2 &: \quad \text{diverges quadratically} \\
\int \mathrm{d}^4 l / l^3 &: \quad \text{diverges linearly} \\
\int \mathrm{d}^4 l / l^4 &: \quad \text{diverges logarithmically}
\end{aligned}$$

I diverges quadratically.

Various methods to regularize divergent integrals.
Here: *Dimensional* regularization, because it preserves algebraic relations among Green functions (Ward identities) if underlying symmetries do not depend on # of space-time dimensions.

Dimensional regularization: Generalize integral from 4 to n dimensions.
Introduce polar coordinates

$$
\begin{aligned}
l_1 &= l\cos(\theta_1), \\
l_2 &= l\sin(\theta_1)\cos(\theta_2), \\
l_3 &= l\sin(\theta_1)\sin(\theta_2)\cos(\theta_3), \\
&\vdots \\
l_{n-1} &= l\sin(\theta_1)\sin(\theta_2)\cdots\cos(\theta_{n-1}), \\
l_n &= l\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{n-1}),
\end{aligned} \tag{5.50}
$$

where $0 \leq l$, $\theta_i \in [0,\pi], i = 1,\cdots,n-2$, $\theta_{n-1} \in [0,2\pi]$.
General integral is symbolically of the form

$$
\int \mathrm{d}^n l \cdots = \int_0^\infty l^{n-1}\mathrm{d}l \int_0^{2\pi} \mathrm{d}\theta_{n-1} \int_0^\pi \mathrm{d}\theta_{n-2}\sin(\theta_{n-2})\cdots\int_0^\pi \mathrm{d}\theta_1 \sin^{n-2}(\theta_1)\cdots. \tag{5.51}
$$

If integrand does not depend on angles, angular integration can explicitly be carried out. Make use of (Exercise)

$$\int_0^{\pi} \sin^m(\theta)\mathrm{d}\theta = \frac{\sqrt{\pi}\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}$$

(shown by induction).

Result of angular integration

$$\begin{aligned}\int_0^{2\pi} \mathrm{d}\theta_{n-1} \cdots \int_0^{\pi} \mathrm{d}\theta_1 \sin^{n-2}(\theta_1) \;&=\; 2\pi \underbrace{\frac{\sqrt{\pi}\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)} \frac{\sqrt{\pi}\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)} \cdots \frac{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}}_{(n-2)\text{ factors}} \\ &=\; 2\pi \frac{\pi^{\frac{n-2}{2}}}{\Gamma\left(\frac{n}{2}\right)} = 2\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}. \qquad (5.52)\end{aligned}$$

Check for $n = 3$:

$$4\pi = 2\frac{\pi}{\frac{1}{2}} = 2\frac{\pi^{\frac{3}{2}}}{\frac{\sqrt{\pi}}{2}} = 2\frac{\pi^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)}.$$

Define integral for n dimensions (n integer) as

$$I_n(M^2, \mu^2) = \mu^{4-n} \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{i}{k^2 - M^2 + i0^+}. \tag{5.53}$$

Scale μ: Unit of mass, 't Hooft parameter, renormalization scale (integral has the same dimension for arbitrary n). [Integral of Eq. (5.53) is convergent only for $n = 1$.]

Wick rotation + angular integration $\Rightarrow$ integral formally reads

$$I_n(M^2, \mu^2) = \mu^{4-n} 2\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \frac{1}{(2\pi)^n} \int_0^\infty \mathrm{d}l \frac{l^{n-1}}{l^2 + M^2}.$$

For later use, investigate (more general) integral

$$\int_0^\infty \frac{l^{n-1}\mathrm{d}l}{(l^2+M^2)^\alpha} = \frac{1}{(M^2)^\alpha}\int_0^\infty \frac{l^{n-1}\mathrm{d}l}{(\frac{l^2}{M^2}+1)^\alpha} = \frac{1}{2}(M^2)^{\frac{n}{2}-\alpha}\int_0^\infty \frac{t^{\frac{n}{2}-1}\mathrm{d}t}{(t+1)^\alpha}, \tag{5.54}$$

with substitution $t \equiv l^2/M^2$.
Beta function

$$B(x,y) = \int_0^\infty \frac{t^{x-1}\mathrm{d}t}{(1+t)^{x+y}} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{5.55}$$

integral converges for $x>0$, $y>0$ and diverges if $x \le 0$ or $y \le 0$.

Non-positive values of x or y: Use analytic continuation in terms of Gamma function to *define* Beta function and thus integral of Eq. (5.54).

Recall: $\Gamma(z)$ is single valued and analytic over entire complex plane, save for the points $z=-n$, $n=0,1,2,\cdots$, where it possesses simple poles with residue $(-1)^n/n!$.

$x = n/2$, $x + y = \alpha$ and $y = \alpha - n/2 \Rightarrow$

$$\int_0^\infty \frac{l^{n-1}\mathrm{d}l}{(l^2 + M^2)^\alpha} = \frac{1}{2}(M^2)^{\frac{n}{2}-\alpha}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\alpha - \frac{n}{2}\right)}{\Gamma(\alpha)} \tag{5.56}$$

$\alpha = 1 \Rightarrow$

$$\begin{aligned} I_n(M^2, \mu^2) &= \mu^{4-n} \underbrace{2\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}}_{\text{angular integration}} \frac{1}{(2\pi)^n}\frac{1}{2}(M^2)^{\frac{n}{2}-1}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(1 - \frac{n}{2}\right)}{\underbrace{\Gamma(1)}_{1}} \\ &= \frac{\mu^{4-n}}{(4\pi)^{\frac{n}{2}}}(M^2)^{\frac{n}{2}-1}\Gamma\left(1 - \frac{n}{2}\right). \end{aligned} \tag{5.57}$$

$a^z = \exp[\ln(a)z]$, $a \in \mathbb{R}^+$ is an analytic function in $\mathbb{C}$.
Making use of

$$\mu^{4-n} = (\mu^2)^{2-\frac{n}{2}}, \quad (M^2)^{\frac{n}{2}-1} = M^2(M^2)^{\frac{n}{2}-2}, \quad (4\pi)^{\frac{n}{2}} = (4\pi)^2(4\pi)^{\frac{n}{2}-2},$$

we define (as a function of a *complex* variable n)

$$I(M^2, \mu^2, n) = \frac{M^2}{(4\pi)^2}\left(\frac{4\pi\mu^2}{M^2}\right)^{2-\frac{n}{2}} \Gamma\left(1 - \frac{n}{2}\right).$$

$n \to 4$: Gamma function has a pole.
How is this pole is approached?
Important property: $\Gamma(z+1) = z\Gamma(z)$

$$\Gamma\left(1 - \frac{n}{2}\right) = \frac{\Gamma\left(1 - \frac{n}{2} + 1\right)}{1 - \frac{n}{2}} = \frac{\Gamma\left(2 - \frac{n}{2} + 1\right)}{\left(1 - \frac{n}{2}\right)\left(2 - \frac{n}{2}\right)} = \frac{\Gamma\left(1 + \frac{\epsilon}{2}\right)}{(-1)\left(1 - \frac{\epsilon}{2}\right)\frac{\epsilon}{2}}$$

where $\epsilon \equiv 4 - n$.

Make use of $a^x = \exp[\ln(a)x] = 1 + \ln(a)x + O(x^2)$.
Expand integral for small ϵ

$$I(M^2, \mu^2, n) = \frac{M^2}{16\pi^2}\left[1 + \frac{\epsilon}{2}\ln\left(\frac{4\pi\mu^2}{M^2}\right) + O(\epsilon^2)\right]$$

$$\times \left(-\frac{2}{\epsilon}\right) \left[1 + \frac{\epsilon}{2} + O(\epsilon^2)\right] \left[\underbrace{\Gamma(1)}_{1} + \frac{\epsilon}{2}\Gamma'(1) + O(\epsilon^2)\right]$$

$$= \frac{M^2}{16\pi^2} \left[-\frac{2}{\epsilon} \underbrace{-\Gamma'(1)}_{\gamma_E = 0.5772\cdots} -1 - \ln(4\pi) + \ln\left(\frac{M^2}{\mu^2}\right) + O(\epsilon)\right],$$

γ_E: Euler's constant.

Final result

$$I(M^2, \mu^2, n) = \frac{M^2}{16\pi^2} \left[R + \ln\left(\frac{M^2}{\mu^2}\right)\right] + O(n-4), \tag{5.58}$$

where

$$\underbrace{R}_{\widetilde{\mathrm{MS}}} = \underbrace{\underbrace{\frac{2}{n-4}}_{\mathrm{MS}} - [\ln(4\pi) + \Gamma'(1)]}_{\overline{\mathrm{MS}}} - 1. \tag{5.59}$$

Using the same techniques ⇒ very useful expression for the more general integral (Exercise)

$$\int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{(k^2)^p}{(k^2 - M^2 + i0^+)^q} =$$
$$i(-)^{p-q} \frac{1}{(4\pi)^{\frac{n}{2}}} (M^2)^{p+\frac{n}{2}-q} \frac{\Gamma\left(p+\frac{n}{2}\right)\Gamma\left(q-p-\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma(q)}. \qquad (5.60)$$

Remarks:

1. In the context of combining propagators by using Feynman's trick ⇒ integrals with M^2 replaced by $A - i0^+$, A real number.
   It is important to consistently deal with the boundary condition $-i0^+$.

   Example: Consider $\ln(A - i0^+)$.
   Express z in its polar form $z = |z| \exp(i\varphi)$, demand $-\pi \leq \varphi < \pi$.

   - $A > 0$: $\ln(A - i0^+) = \ln(A)$.

- $A < 0$: Infinitesimal imaginary part indicates that $-|A|$ is reached in the third quadrant from below the real axis, $\Rightarrow$ we have to use the $-\pi$.
  Make use of $\ln(ab) = \ln(a) + \ln(b) \Rightarrow$

$$\ln(A - i0^+) = \ln(|A|) + \ln(e^{-i\pi}) = \ln(|A|) - i\pi, \quad A < 0.$$

- Summarized in a single expression

$$\ln(A - i0^+) = \ln(|A|) - i\pi\Theta(-A) \quad \text{for } A \in \mathbb{R}. \tag{5.61}$$

  Discussion is of importance for consistently determining imaginary parts of loop integrals.

2. In dim. reg. power-law divergences are set to 0.
3. Logarithmic UV divergences of one-loop integrals in dim. reg. show up as single poles in $\epsilon = 4 - n$.

5.7 Chiral Lagrangian at Order $\mathcal{O}(q^4)$

Reference:

- J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465 (1985)

Most general Lagrangian at $\mathcal{O}(q^4)$:

$$
\begin{aligned}
\mathcal{L}_4 = \; & L_1 \left\{ \mathrm{Tr}[D_\mu U (D^\mu U)^\dagger] \right\}^2 \\
& + L_2 \mathrm{Tr}\left[D_\mu U (D_\nu U)^\dagger \right] \mathrm{Tr}\left[D^\mu U (D^\nu U)^\dagger \right] \\
& + L_3 \mathrm{Tr}\left[D_\mu U (D^\mu U)^\dagger D_\nu U (D^\nu U)^\dagger \right] \\
& + L_4 \mathrm{Tr}\left[D_\mu U (D^\mu U)^\dagger \right] \mathrm{Tr}\left(\chi U^\dagger + U \chi^\dagger \right) \\
& + L_5 \mathrm{Tr}\left[D_\mu U (D^\mu U)^\dagger (\chi U^\dagger + U \chi^\dagger) \right] \\
& + L_6 \left[\mathrm{Tr}\left(\chi U^\dagger + U \chi^\dagger \right) \right]^2 \\
& + L_7 \left[\mathrm{Tr}\left(\chi U^\dagger - U \chi^\dagger \right) \right]^2 \\
& + L_8 \mathrm{Tr}\left(U \chi^\dagger U \chi^\dagger + \chi U^\dagger \chi U^\dagger \right)
\end{aligned}
$$

$$
\begin{aligned}
&-iL_9 \mathrm{Tr}\left[f^R_{\mu\nu} D^\mu U (D^\nu U)^\dagger + f^L_{\mu\nu} (D^\mu U)^\dagger D^\nu U\right] \\
&+L_{10} \mathrm{Tr}\left(U f^L_{\mu\nu} U^\dagger f_R^{\mu\nu}\right) \\
&+H_1 \mathrm{Tr}\left(f^R_{\mu\nu} f_R^{\mu\nu} + f^L_{\mu\nu} f_L^{\mu\nu}\right) \\
&+H_2 \mathrm{Tr}\left(\chi\chi^\dagger\right) . \qquad (5.62)
\end{aligned}
$$

- Numerical values of low-energy coupling constants L_i (LECs) are not determined by chiral symmetry.
- In principle, determined in terms of (remaining) parameters of QCD, namely, heavy-quark masses and QCD scale Λ_{QCD} (g_3).
- Sources of information on LECs:
  - Fix by using empirical input.
  - Estimate from QCD-inspired models.
  - Calculate from lattice QCD.

S. Weinberg, *The Quantum Theory of Fields*, Vol. I, Chap. 12:
"... the cancellation of ultraviolet divergences does not really depend on renormalizability; as long as we include every one of the infinite number of interactions allowed by symmetries, the so-called non-renormalizable theories are actually just as renormalizable as renormalizable theories."

Weinberg's power counting $\Rightarrow$ (renormalized) one-loop graphs with vertices from $\mathcal{L}_2$ are of $\mathcal{O}(q^4)$. By construction Eq. (5.62) represents the most general Lagrangian at $\mathcal{O}(q^4)$.

$\Rightarrow$ Adjust (renormalize) parameters of $\mathcal{L}_4$ to cancel one-loop divergences:

$$L_i = L_i^r + \frac{\Gamma_i}{32\pi^2}R, \quad i = 1, \cdots, 10, \tag{5.63}$$

$$H_i = H_i^r + \frac{\Delta_i}{32\pi^2}R, \quad i = 1, 2, \tag{5.64}$$

where

$$R = \frac{2}{n-4} - [\ln(4\pi) - \gamma_E + 1]. \tag{5.65}$$

L_i bare parameters vs. L_i^r renormalized parameters.

Coefficient	Empirical Value	Γ_i
L_1^r	0.4 ± 0.3	$\frac{3}{32}$
L_2^r	1.35 ± 0.3	$\frac{3}{16}$
L_3^r	-3.5 ± 1.1	0
L_4^r	-0.3 ± 0.5	$\frac{1}{8}$
L_5^r	1.4 ± 0.5	$\frac{3}{8}$
L_6^r	-0.2 ± 0.3	$\frac{11}{144}$
L_7^r	-0.4 ± 0.2	0
L_8^r	0.9 ± 0.3	$\frac{5}{48}$
L_9^r	6.9 ± 0.7	$\frac{1}{4}$
L_{10}^r	-5.5 ± 0.7	$-\frac{1}{4}$

Renormalized low-energy coupling constants L_i^r in units of 10^{-3} at the scale $\mu = M_\rho$, see J. Bijnens, G. Ecker, and J. Gasser, *The Second DAΦNE Physics Handbook*, Vol. I, Chap. 3. $\Delta_1 = -1/8$, $\Delta_2 = 5/24$.

- Except for L_3 and L_7, L_i and H_i are required in the renormalization of one-loop graphs.
- H_1 and H_2 contain only external fields, are of no physical relevance.
- Renormalized coefficients L_i^r depend on scale μ introduced by dimensional regularization [see Eq. (5.58)].
  Values at two different scales μ_1 and μ_2 are related by

$$L_i^r(\mu_2) = L_i^r(\mu_1) + \frac{\Gamma_i}{16\pi^2} \ln\left(\frac{\mu_1}{\mu_2}\right). \tag{5.66}$$

Scale dependence of coefficients and of finite part of loop-diagrams compensate each other. Physical observables are scale independent.

5.8 Application at Order $\mathcal{O}(q^4)$: Masses of the Goldstone Bosons

Reference:

- J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465 (1985)

Masses at $\mathcal{O}(q^4)$ will allow us to illustrate various properties typical of ChPT:

1. Relation between bare LECs L_i and renormalized LECs L_i^r: Divergences of one-loop diagrams are canceled.
2. Scale dependence of $L_i^r(\mu)$ and of finite contributions of one-loop diagrams combine to scale-independent predictions for physical observables.
3. A perturbation expansion in the explicit symmetry breaking with respect to a symmetry that is realized in the Nambu-Goldstone mode generates corrections which are non-analytic in the symmetry break-

ing parameter, here the quark masses [see L. F. Li and H. Pagels, Phys. Rev. Lett. **26**, 1204 (1971)].

- Recall [see, e.g., T. P. Cheng and L. F. Li, *Gauge Theory of Elementary Particle Physics* (Clarendon, Oxford, 1984), chapter 2]

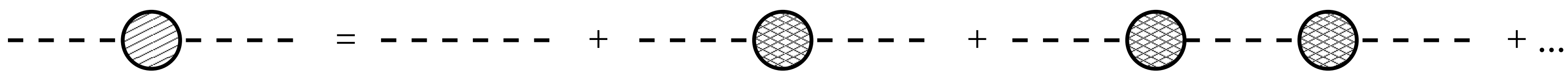
Propagator of a (pseudo-) scalar field is defined as the Fourier transform of the two-point Green function:

$$i\Delta(p) = \int \mathrm{d}^4 x e^{-ip\cdot x} \langle 0|T\left[\Phi_0(x)\Phi_0(0)\right]|0\rangle. \tag{5.67}$$

Index 0: Bare unrenormalized field (do not confuse with free field). Recall propagator of free field with mass m (Assignment 3)

$$i\Delta_F(p) = \frac{i}{p^2 - m^2 + i0^+}.$$

Full propagator in terms of the so-called proper self-energy insertions $-i\Sigma(p^2)$:



$-i\Sigma(p^2)$ consists of one-particle-irreducible diagrams: Diagrams which do not fall apart into two separate pieces when cutting an arbitrary internal line.

Summation via a geometric series

$$
\begin{aligned}
i\Delta(p) &= \frac{i}{p^2 - M_0^2 + i0^+} \\
&\quad + \frac{i}{p^2 - M_0^2 + i0^+} \underbrace{[-i\Sigma(p^2)] \frac{i}{p^2 - M_0^2 + i0^+}}_{x} \\
&\quad + \cdots \\
&= \frac{i}{p^2 - M_0^2 + i0^+} \underbrace{[1 + x + x^2 + \cdots]}_{1/(1-x)}
\end{aligned}
$$

$$= \frac{i}{p^2 - M_0^2 - \Sigma(p^2) + i0^+}. \tag{5.68}$$

Physical (or pole) mass (including interaction) is defined as the position of the pole of Eq. (5.68),

$$M^2 - M_0^2 - \Sigma(M^2) \stackrel{!}{=} 0. \tag{5.69}$$

Assume that $\Sigma(p^2)$ can be expanded in a series around $p^2 = \lambda^2$:

$$\Sigma(p^2) = \Sigma(\lambda^2) + (p^2 - \lambda^2)\Sigma'(\lambda^2) + \tilde{\Sigma}(p^2). \tag{5.70}$$

Remainder $\tilde{\Sigma}(p^2)$ depends on choice of λ^2, satisfies $\tilde{\Sigma}(\lambda^2) = \tilde{\Sigma}'(\lambda^2) = 0$.

$\Rightarrow$

$$i\Delta(p) = \frac{i}{p^2 - M_0^2 - \Sigma(\lambda^2) - (p^2 - \lambda^2)\Sigma'(\lambda^2) - \tilde{\Sigma}(p^2) + i0^+}. \tag{5.71}$$

Take $\lambda^2 = M^2$ in Eq. (5.71) + condition of Eq. (5.69) $\Rightarrow$

$$i\Delta(p) = \frac{i}{(p^2 - M^2)[1 - \Sigma'(M^2)] - \tilde{\Sigma}(p^2) + i0^+} = \frac{iZ_\Phi}{p^2 - M^2 - Z_\Phi\tilde{\Sigma}(p^2) + i0^+}.$$

Wave function renormalization constant

$$Z_\Phi = \frac{1}{1 - \Sigma'(M^2)}.$$

Introduce renormalized field as $\Phi_R = \Phi_0/\sqrt{Z_\Phi} \Rightarrow$ renormalized propagator

$$\begin{aligned} i\Delta_R(p) &= \int \mathrm{d}^4x e^{-ip\cdot x}\langle 0|T[\Phi_R(x)\Phi_R(0)]|0\rangle \\ &= \frac{i}{p^2 - M^2 - Z_\Phi\tilde{\Sigma}(p^2) + i0^+}. \end{aligned}$$

$\tilde{\Sigma}(M^2) = \tilde{\Sigma}'(M^2) = 0 \Rightarrow$ residue $= 1$.
In the vicinity of the pole, renormalized propagator behaves as a free propagator with physical squared mass M^2.

At lowest order ($D = 2$), the propagator simply reads

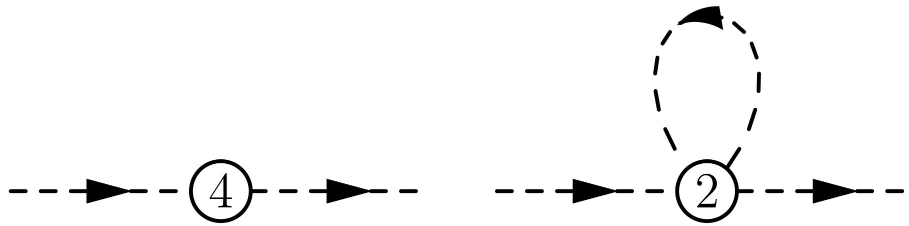
$$i\Delta(p) = \frac{i}{p^2 - M_0^2 + i0^+}, \tag{5.72}$$

with lowest-order squared masses M_0^2

$$\begin{aligned} M_{\pi,2}^2 &= 2B_0\hat{m}, \\ M_{K,2}^2 &= B_0(\hat{m} + m_s), \\ M_{\eta,2}^2 &= \frac{2}{3}B_0\left(\hat{m} + 2m_s\right). \end{aligned}$$

(Subscript 2 refers to chiral order 2. We assume isospin symmetry.)

$\mathcal{L}_2$ and $\mathcal{L}_4$ (without external fields) generate vertices with an even number of Goldstone bosons only $\Rightarrow$ self-energy contributions at $D = 4$:



We need

$$\mathcal{L}_{\rm int} = \mathcal{L}_2^{4\phi} + \mathcal{L}_4^{2\phi}, \tag{5.73}$$

where

$$\mathcal{L}_2^{4\phi} = \frac{1}{24F_0^2}\left\{\mathrm{Tr}([\phi, \partial_\mu\phi]\phi\partial^\mu\phi) + B_0\mathrm{Tr}(\mathcal{M}\phi^4)\right\}. \tag{5.74}$$

Q: Which terms of $\mathcal{L}_4$ contribute?
A:

- Terms $\sim L_9,\ L_{10},\ H_1,\ H_2$ do not, because they either contain field-strength tensors or external fields only.
- $\partial_\mu U = O(\phi) \Rightarrow$ terms $\sim L_1,\ L_2,\ L_3$ are $O(\phi^4) \Rightarrow$ do not contribute.

- Candidates: L_4 - L_8
- Example:

$$L_4 \mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger)\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger) =$$
$$L_4 \frac{2}{F_0^2}[\partial_\mu \eta \partial^\mu \eta + \partial_\mu \pi^0 \partial^\mu \pi^0 + 2\partial_\mu \pi^+ \partial^\mu \pi^- + 2\partial_\mu K^+ \partial^\mu K^-$$
$$+2\partial_\mu K^0 \partial^\mu \bar{K}^0 + O(\phi^4)][4B_0(2\hat{m} + m_s) + O(\phi^2)].$$

- Remaining terms as Exercise.

We obtain for $\mathcal{L}_4^{2\phi}$

$$\begin{aligned} \mathcal{L}_4^{2\phi} &= -\frac{1}{2}\left(a_\eta \eta^2 + b_\eta \partial_\mu \eta \partial^\mu \eta\right) \\ &\quad -\frac{1}{2}\left(a_\pi \pi^0 \pi^0 + b_\pi \partial_\mu \pi^0 \partial^\mu \pi^0\right) \\ &\quad -a_\pi \pi^+ \pi^- - b_\pi \partial_\mu \pi^+ \partial^\mu \pi^- \\ &\quad -a_K K^+ K^- - b_K \partial_\mu K^+ \partial^\mu K^- \end{aligned}$$

$$-a_K K^0 \bar{K}^0 - b_K \partial_\mu K^0 \partial^\mu \bar{K}^0, \tag{5.75}$$

with constants a_ϕ and b_ϕ

$$\begin{aligned}
a_\eta &= \frac{64 B_0^2}{3 F_0^2} \left[(2\hat{m} + m_s)(\hat{m} + 2m_s) L_6 + 2(\hat{m} - m_s)^2 L_7 + (\hat{m}^2 + 2m_s^2) L_8 \right], \\
b_\eta &= -\frac{16 B_0}{F_0^2} \left[(2\hat{m} + m_s) L_4 + \frac{1}{3}(\hat{m} + 2m_s) L_5 \right], \\
a_\pi &= \frac{64 B_0^2}{F_0^2} \left[(2\hat{m} + m_s)\hat{m} L_6 + \hat{m}^2 L_8 \right], \\
b_\pi &= -\frac{16 B_0}{F_0^2} \left[(2\hat{m} + m_s) L_4 + \hat{m} L_5 \right], \\
a_K &= \frac{32 B_0^2}{F_0^2} \left[(2\hat{m} + m_s)(\hat{m} + m_s) L_6 + \frac{1}{2}(\hat{m} + m_s)^2 L_8 \right], \\
b_K &= -\frac{16 B_0}{F_0^2} \left[(2\hat{m} + m_s) L_4 + \frac{1}{2}(\hat{m} + m_s) L_5 \right].
\end{aligned} \tag{5.76}$$

Self energies at $\mathcal{O}(q^4)$ are of the form

$$\Sigma_\phi(p^2) = A_\phi + B_\phi p^2. \tag{5.77}$$

Constants A_ϕ and B_ϕ receive a tree-level contribution from $\mathcal{L}_4$ and a one-loop contribution with a vertex from $\mathcal{L}_2$.

- Tree-level contribution of $\mathcal{L}_4$: Lagrangians of Eq. (5.75) contain either exactly two derivatives of the fields or no derivatives at all.

  Example η. Feynman rule from $i\mathcal{L}$. $\partial_\mu\phi$ generates $-ip_\mu$ (ip_μ) for incoming (outgoing) line:

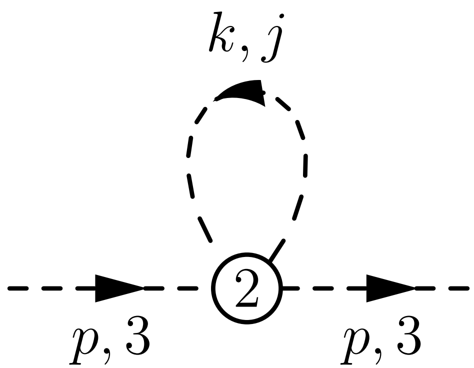
  $$-i\Sigma_\eta^{\text{tree}}(p^2) = i2\left[-\frac{1}{2}a_\eta - \frac{1}{2}b_\eta(ip_\mu)(-ip^\mu)\right] = -i(a_\eta + b_\eta p^2).$$

  Factor of 2 takes account of two combinations of contracting the fields with external lines.

- Argument for one-loop contribution: Lagrangian $\mathcal{L}_2^{4\phi}$ contains either two derivatives or no derivatives, symbolically $\phi\phi\partial\phi\partial\phi$ or ϕ^4, re-

spectively.
First term $\Rightarrow M^2$ if both ϕs are contracted with external lines, and p^2 if both $\partial\phi$s are contracted with external lines. "Mixed" situation vanishes upon integration.
Second term $\Rightarrow$ no momentum dependence.

Example: Pion-loop contribution to π^0 self energy



Apply Feynman rule of Eq. (5.46) for $a = c = 3$, $p_a = p_c = p$, $b = d = j$, and $p_b = p_d = k$:

$$
\begin{aligned}
&\frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{3F_0^2}\sum_{j=1}^{3}\Bigg\{\underbrace{\delta^{3j}\delta^{3j}}_{\to 1}[(p+k)^2 + 2p\cdot k + 2p\cdot k + M_{\pi,2}^2] \\
&+ \underbrace{\delta^{33}\delta^{jj}}_{\to 3}[(p-p)^2 - 2p^2 - 2k^2 + M_{\pi,2}^2] \\
&\underbrace{\delta^{3j}\delta^{3j}}_{\to 1}[(p-k)^2 - 2p\cdot k - 2k\cdot p + M_{\pi,2}^2]\Bigg\}\frac{i}{k^2 - M_{\pi,2}^2 + i0^+} \\
= \ &\frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{3F_0^2}\left(-4p^2 - 4k^2 + 5M_{\pi,2}^2\right)\frac{i}{k^2 - M_{\pi,2}^2 + i0^+}. \qquad (5.78)
\end{aligned}
$$

Explanation of symmetry factor 1/2:

Feynman rule of Eq. (5.46) results from $4! = 24$ distinct combinations of contracting four field operators with four external lines. Two lines have to be selected as internal lines: $\Rightarrow$ 6 possibilities to choose one pair out of 4 field operators to form internal lines. For the two remaining operators one has two possibilities of contracting them with external lines. $\Rightarrow 6 \times 2 = 24/2$ combinations.

Integral of Eq. (5.78) diverges $\Rightarrow$ dimensional regularization. Besides I we need

$$\mu^{4-n} i \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{k_\mu k_\nu}{k^2 - M^2 + i0^+} = g_{\mu\nu} C. \tag{5.79}$$

Integral contains no external momenta, $g_{\mu\nu}$ is the only symmetric second-rank tensor.
Determination of C (simplest example of Veltman-Passarino procedure).
Contract Eq. (5.79) with $g^{\mu\nu}$ in n dimensions and add $0 = -M^2 + M^2$ in numerator:

$$\mu^{4-n} i \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{k^2 - M^2 + M^2}{k^2 - M^2 + i0^+} = \underbrace{g^{\mu\nu} g_{\mu\nu}}_{n} C.$$

Make use of

$$\mu^{4-n} i \int \frac{\mathrm{d}^n k}{(2\pi)^n} = 0$$

in dimensional regularization which is "shown" as follows. Consider (more general) integral

$$\int \mathrm{d}^n k (k^2)^p,$$

substitute $k = \lambda k'$ ($\lambda > 0$), relabel $k' = k$

$$\cdots = \lambda^{n+2p} \int \mathrm{d}^n k (k^2)^p. \tag{5.80}$$

$\lambda > 0$ arbitrary $\wedge$ for fixed p, the result is to hold for arbitrary $n \Rightarrow 0$ in dimensional regularization. (Note: This has the character of a prescription. Integral does not depend on any scale; its analytic continuation is ill-defined in the sense that there is no dimension n where it is meaningful. It is ultraviolet divergent for $n + 2p \geq 0$ and infrared divergent for $n + 2p \leq 0$.)

$\Rightarrow$

$$\mu^{4-n} i \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{k_\mu k_\nu}{k^2 - M^2 + i0^+} = \frac{M^2}{n} g_{\mu\nu} \mu^{4-n} \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{i}{k^2 - M^2 + i0^+}$$
$$= \frac{M^2}{n} g_{\mu\nu} I(M^2, \mu^2, n),$$

i.e.

$$C = \frac{M^2}{n} I(M^2, \mu^2, n).$$

Pion-loop contribution to the π^0 self energy

$$\frac{i}{6F_0^2}(-4p^2 + M_{\pi,2}^2) I(M_{\pi,2}^2, \mu^2, n).$$

- Contribution indeed of the type $A + Bp^2$.
- Diverges as $n \to 4$.

Systematic analysis of *all* loop contributions + tree-level contributions
$\Rightarrow$

$$
\begin{aligned}
A_\pi &= \frac{M_{\pi,2}^2}{F_0^2}\Big\{\underbrace{-\frac{1}{6}I(M_{\pi,2}^2)-\frac{1}{6}I(M_{\eta,2}^2)-\frac{1}{3}I(M_{K,2}^2)}_{\text{loop contribution}}\\
&\quad\underbrace{+32[(2\hat{m}+m_s)B_0L_6+\hat{m}B_0L_8]}_{\text{tree-level contribution}}\Big\},\\
B_\pi &= \frac{2}{3}\frac{I(M_{\pi,2}^2)}{F_0^2}+\frac{1}{3}\frac{I(M_{K,2}^2)}{F_0^2}-\frac{16B_0}{F_0^2}\left[(2\hat{m}+m_s)L_4+\hat{m}L_5\right],\\
A_K &= \frac{M_{K,2}^2}{F_0^2}\Big\{\frac{1}{12}I(M_{\eta,2}^2)-\frac{1}{4}I(M_{\pi,2}^2)-\frac{1}{2}I(M_{K,2}^2)\\
&\quad+32\left[(2\hat{m}+m_s)B_0L_6+\frac{1}{2}(\hat{m}+m_s)B_0L_8\right]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
B_K &= \frac{1}{4}\frac{I(M_{\eta,2}^2)}{F_0^2} + \frac{1}{4}\frac{I(M_{\pi,2}^2)}{F_0^2} + \frac{1}{2}\frac{I(M_{K,2}^2)}{F_0^2} \\
&\quad -16\frac{B_0}{F_0^2}\left[(2\hat{m}+m_s)L_4 + \frac{1}{2}(\hat{m}+m_s)L_5\right], \\
A_\eta &= \frac{M_{\eta,2}^2}{F_0^2}\left[-\frac{2}{3}I(M_{\eta,2}^2)\right] + \frac{M_{\pi,2}^2}{F_0^2}\left[\frac{1}{6}I(M_{\eta,2}^2) - \frac{1}{2}I(M_{\pi,2}^2) + \frac{1}{3}I(M_{K,2}^2)\right] \\
&\quad +\frac{M_{\eta,2}^2}{F_0^2}[16M_{\eta,2}^2L_8 + 32(2\hat{m}+m_s)B_0L_6] \\
&\quad +\frac{128}{9}\frac{B_0^2(\hat{m}-m_s)^2}{F_0^2}(3L_7+L_8), \\
B_\eta &= \frac{I(M_{K,2}^2)}{F_0^2} - \frac{16}{F_0^2}(2\hat{m}+m_s)B_0L_4 - 8\frac{M_{\eta,2}^2}{F_0^2}L_5. \qquad (5.81)
\end{aligned}
$$

(Dependence on μ^2 and n in integrals $I(M^2, \mu^2, n)$ suppressed.)

- Integrals I and bare coefficients L_i (with the exception of L_7) have $1/(n-4)$ poles and finite pieces.
- Coefficients A_ϕ and B_ϕ are *not* finite as $n \to 4$ (no observables!).

Determine masses at $\mathcal{O}(q^4)$

$$M^2 = M_2^2 + \Sigma(M^2) \tag{5.82}$$

using predictions of Eq. (5.77) for self energies,

$$M^2 = M_2^2 + A + BM^2 \Rightarrow M^2 = \frac{M_2^2 + A}{1 - B} = M_2^2(1 + B) + A + \mathcal{O}(q^6),$$

because $A = \mathcal{O}(q^4)$ und $B = \mathcal{O}(q^2)$. Express bare coefficients L_i in terms of renormalized coefficients by using Eq. (5.63) (Exercise) $\Rightarrow$

$$M_{\pi,4}^2 \;=\; M_{\pi,2}^2\Big\{1 + \frac{M_{\pi,2}^2}{32\pi^2 F_0^2}\ln\left(\frac{M_{\pi,2}^2}{\mu^2}\right) - \frac{M_{\eta,2}^2}{96\pi^2 F_0^2}\ln\left(\frac{M_{\eta,2}^2}{\mu^2}\right)$$

$$+\frac{16}{F_0^2}\left[(2\hat{m}+m_s)B_0(2L_6^r-L_4^r)+\hat{m}B_0(2L_8^r-L_5^r)\right]\Big\}, \tag{5.83}$$

$$\begin{aligned} M_{K,4}^2 &= M_{K,2}^2\Big\{1+\frac{M_{\eta,2}^2}{48\pi^2F_0^2}\ln\left(\frac{M_{\eta,2}^2}{\mu^2}\right) \\ &\quad +\frac{16}{F_0^2}\left[(2\hat{m}+m_s)B_0(2L_6^r-L_4^r)+\frac{1}{2}(\hat{m}+m_s)B_0(2L_8^r-L_5^r)\right]\Big\}, \end{aligned} \tag{5.84}$$

$$\begin{aligned} M_{\eta,4}^2 &= M_{\eta,2}^2\left[1+\frac{M_{K,2}^2}{16\pi^2F_0^2}\ln\left(\frac{M_{K,2}^2}{\mu^2}\right)-\frac{M_{\eta,2}^2}{24\pi^2F_0^2}\ln\left(\frac{M_{\eta,2}^2}{\mu^2}\right)\right. \\ &\quad \left.+\frac{16}{F_0^2}(2\hat{m}+m_s)B_0(2L_6^r-L_4^r)+8\frac{M_{\eta,2}^2}{F_0^2}(2L_8^r-L_5^r)\right] \\ &\quad +M_{\pi,2}^2\left[\frac{M_{\eta,2}^2}{96\pi^2F_0^2}\ln\left(\frac{M_{\eta,2}^2}{\mu^2}\right)-\frac{M_{\pi,2}^2}{32\pi^2F_0^2}\ln\left(\frac{M_{\pi,2}^2}{\mu^2}\right)\right. \end{aligned}$$

$$
+\frac{M_{K,2}^2}{48\pi^2 F_0^2}\ln\left(\frac{M_{K,2}^2}{\mu^2}\right)\Bigg]
$$
$$
+\frac{128}{9}\frac{B_0^2(\hat{m}-m_s)^2}{F_0^2}(3L_7^r+L_8^r). \tag{5.85}
$$

Remarks

- Physical masses are finite $\Rightarrow$ bare coefficients L_i must be infinite in order to cancel infinities resulting from divergent loop integrals.
- Masses vanish if quark masses are sent to zero. [Self interaction in $\mathcal{L}_2$ (in the absence of quark masses) does not generate Goldstone boson masses at higher order.]
- Two types of quark-mass dependence

  1. Analytic terms $\sim m_q^2 \times L_i^r$.

2. Non-analytic terms $\sim m_q^2 \ln(m_q)$—so-called chiral logarithms—which do not involve new parameters.

Li and Pagels: Perturbation theory around a symmetry which is realized in Nambu-Goldstone mode results in both analytic as well as non-analytic expressions in the perturbation.

- Scale dependence of L_i^r is by construction such that it cancels the scale dependence of chiral logarithms. Physical observables do not depend on the scale μ!

From Eq. (5.66),

$$L_i^r(\mu) = L_i^r(\mu') + \frac{\Gamma_i}{16\pi^2} \ln\left(\frac{\mu'}{\mu}\right),$$

we obtain

$$\frac{\mathrm{d}L_i^r(\mu)}{\mathrm{d}\mu} = -\frac{\Gamma_i}{16\pi^2\mu}.$$

Also

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\ln\left(\frac{M^2}{\mu^2}\right) = 2\frac{\mathrm{d}}{\mathrm{d}\mu}\left[\ln(M) - \ln(\mu)\right] = -\frac{2}{\mu}.$$

Example pion mass:

$$\begin{aligned}
\frac{\mathrm{d}M^2_{\pi,4}}{\mathrm{d}\mu} &= \frac{M^2_{\pi,2}}{16\pi^2\mu F_0^2}\Big\{\frac{M^2_{\pi,2}}{2}(-2) - \frac{M^2_{\eta,2}}{6}(-2) \\
&\quad +16[(2\hat{m}+m_s)B_0(-2\Gamma_6+\Gamma_4) + \hat{m}B_0(-2\Gamma_8+\Gamma_5)]\Big\} \\
&= \frac{M^2_{\pi,2}}{16\pi^2\mu F_0^2}\Big\{-2B_0\hat{m} + \frac{2}{9}(\hat{m}+2m_s)B_0 \\
&\quad +16\Big[(2\hat{m}+m_s)B_0\underbrace{\left(-2\frac{11}{144}+\frac{1}{8}\right)}_{-\frac{1}{36}} + \hat{m}B_0\underbrace{\left(-2\frac{5}{48}+\frac{3}{8}\right)}_{\frac{1}{6}}\Big]\Big\} \\
&= \frac{M^2_{\pi,2}}{16\pi^2\mu F_0^2}\Big\{B_0\hat{m}\left(-2+\frac{2}{9}-\frac{8}{9}+\frac{8}{3}\right) + B_0 m_s\left(\frac{4}{9}-\frac{16}{36}\right)\Big\} = 0.
\end{aligned}$$

Chapter 6

Chiral Perturbation Theory for Baryons

- So far: Purely mesonic sector involving interaction of Goldstone bosons with each other and with external fields.
- Now: Matrix elements with a single baryon in the initial and final states.

6.1 Transformation Properties of the Fields

References:

- S. Weinberg, Phys. Rev. **166**, 1568 (1968)
- S. R. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1969)
- C. G. Callan, S. R. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2247 (1969)
- H. Georgi, *Weak Interactions and Modern Particle Theory* (Benjamin/Cummings, Menlo Park, 1984)
- J. Gasser, M. E. Sainio, and A. Švarc, Nucl. Phys. **B307**, 779 (1988)

Aim: Most general description of interaction of baryons with Goldstone bosons and external fields at low energies.

Discussion for nucleons [SU(2) ChPT] and baryon octet [SU(3) ChPT]. Consider nucleon doublet and octet of $\frac{1}{2}^+$ baryons (see Fig. 4.2)

$$\Psi = \begin{pmatrix} p \\ n \end{pmatrix}, \tag{6.1}$$

$$B = \sum_{a=1}^{8} \frac{\lambda_a B_a}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix}. \quad (6.2)$$

- Each entry is a Dirac field.
- $B \neq B^\dagger$.

Representation of groups $H = \text{SU}(2)_V/\text{SU}(3)_V$ on $\{\Psi\}/\{B\}$ [see also Eq. (5.6)]:

$$\Psi \mapsto V\Psi, \quad V \in \text{SU}(2)_V, \quad (6.3)$$

$$B \mapsto VBV^\dagger, \quad V \in \text{SU}(3)_V. \quad (6.4)$$

- Ψ transforms under the fundamental representation of SU(2).
- B transforms under the adjoint representation of SU(3).

Realization of $\text{SU(N)}_L \times \text{SU(N)}_R$ (see textbook by Georgi for more details). Start with $G = \text{SU}(2)_L \times \text{SU}(2)_R$.

Recall

$$U \mapsto RUL^\dagger$$

defines nonlinear realization of G. Introduce $u^2(x) = U(x)$ and define SU(2)-valued function $K(L, R, U)$ by

$$u \mapsto u' = \sqrt{RUL^\dagger} \equiv RuK^{-1}(L, R, U), \tag{6.5}$$

i.e.

$$K(L, R, U) = u'^{-1}Ru = \sqrt{RUL^\dagger}^{-1} R\sqrt{U}.$$

Claim:

$$\varphi(g) : \begin{pmatrix} U \\ \Psi \end{pmatrix} \mapsto \begin{pmatrix} U' \\ \Psi' \end{pmatrix} = \begin{pmatrix} RUL^\dagger \\ K(L, R, U)\Psi \end{pmatrix} \tag{6.6}$$

defines operation of G on the set $\{(U, \Psi)\}$.
Verification:

- Identity leaves (U, Ψ) invariant.

- Homomorphism property

$$\begin{aligned}
\varphi(g_1)\varphi(g_2)\begin{pmatrix} U \\ \Psi \end{pmatrix} &= \varphi(g_1)\begin{pmatrix} R_2 U L_2^\dagger \\ K(L_2, R_2, U)\Psi \end{pmatrix} \\
&= \begin{pmatrix} R_1 R_2 U L_2^\dagger L_1^\dagger \\ K(L_1, R_1, R_2 U L_2^\dagger) K(L_2, R_2, U)\Psi \end{pmatrix} \\
&= \begin{pmatrix} R_1 R_2 U (L_1 L_2)^\dagger \\ K(L_1 L_2, R_1 R_2, U)\Psi \end{pmatrix} \\
&= \varphi(g_1 g_2)\begin{pmatrix} U \\ \Psi \end{pmatrix}.
\end{aligned}$$

We made use of (Exercise)

$$K(L_1, R_1, R_2 U L_2^\dagger) K(L_2, R_2, U) = K((L_1 L_2), (R_1 R_2), U).$$

Remarks:

- For general group element $g = (L, R)$ the transformation behavior of Ψ depends on U!

- Special case of isospin transformation: $R = L = V$. $\Rightarrow u' = VuV^\dagger$, because
$$U' = u'^2 = VuV^\dagger VuV^\dagger = Vu^2V^\dagger = VUV^\dagger.$$
Compare with Eq. (6.5) $\Rightarrow K^{-1}(V,V,U) = V^\dagger$ or $K(V,V,U) = V$ $\Rightarrow \Psi$ transforms linearly as isospin doublet under the isospin subgroup $\mathrm{SU}(2)_V$ of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$.

  General feature: Transformation behavior under the subgroup which leaves the ground state invariant is independent of U.

- Various realizations may be connected to each other using field redefinitions.

Analogously: For $G = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ one uses nonlinear realization
$$\varphi(g) : \begin{pmatrix} U \\ B \end{pmatrix} \mapsto \begin{pmatrix} U' \\ B' \end{pmatrix} = \begin{pmatrix} RUL^\dagger \\ K(L,R,U)BK^\dagger(L,R,U) \end{pmatrix}, \qquad (6.7)$$

where K is defined completely analogously to Eq. (6.5) after inserting corresponding SU(3) matrices.

Generalization to other multipletts: Define transformation behavior under subgroup H in terms of V (and $V^\dagger$). Replace $V \rightarrow K$ (and $V^\dagger \rightarrow K^\dagger$).

6.2 Baryonic Effective Lagrangian at Lowest Order

References:

- J. Gasser, M. E. Sainio, and A. Švarc, Nucl. Phys. **B307**, 779 (1988)
- A. Krause, Helv. Phys. Acta **63**, 3 (1990)
- E. Jenkins and A. V. Manohar, Phys. Lett. B **255**, 558 (1991)
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- H. Georgi, *Weak Interactions and Modern Particle Theory* (Benjamin/Cummings, Menlo Park, 1984)

- B. Borasoy, Phys. Rev. D **59**, 054021 (1999)

Aim: Construction of effective πN Lagrangian $\mathcal{L}_{\pi N}^{(1)}$ assuming *local* $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_V$ symmetry.

Transformation behavior of external fields

$$
\begin{aligned}
r_\mu &\mapsto V_R r_\mu V_R^\dagger + i V_R \partial_\mu V_R^\dagger, \\
l_\mu &\mapsto V_L l_\mu V_L^\dagger + i V_L \partial_\mu V_L^\dagger, \\
v_\mu^{(s)} &\mapsto v_\mu^{(s)} - \partial_\mu \Theta, \\
s + ip &\mapsto V_R (s + ip) V_L^\dagger, \\
s - ip &\mapsto V_L (s - ip) V_R^\dagger.
\end{aligned}
$$

Transformation behavior of nucleon doublet and of U

$$\begin{pmatrix} U(x) \\ \Psi(x) \end{pmatrix} \mapsto \begin{pmatrix} V_R(x)U(x)V_L^\dagger(x) \\ \exp[-i\Theta(x)]K[V_L(x),V_R(x),U(x)]\Psi(x) \end{pmatrix} \tag{6.8}$$

Introduce covariant derivative $D_\mu\Psi$ transforming as Ψ:

$$D_\mu\Psi(x) \mapsto [D_\mu\Psi(x)]' \stackrel{!}{=} \exp[-i\Theta(x)]K[V_L(x),V_R(x),U(x)]D_\mu\Psi(x). \tag{6.9}$$

Since K depends on V_L, V_R, and U, we expect that covariant derivative contains u, $u^\dagger$, and their derivatives.

Introduce so-called chiral connection (note $\partial_\mu u u^\dagger = -u\partial_\mu u^\dagger$),

$$\Gamma_\mu = \frac{1}{2}\left[u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - il_\mu)u^\dagger\right], \tag{6.10}$$

and define

$$D_\mu\Psi = (\partial_\mu + \Gamma_\mu - iv_\mu^{(s)})\Psi. \tag{6.11}$$

Need to show

$$D'_\mu\Psi' \;=\; [\partial_\mu + \Gamma'_\mu - i(v_\mu^{(s)} - \partial_\mu\Theta)]\exp(-i\Theta)K\Psi$$

$$= \exp(-i\Theta)K(\partial_\mu + \Gamma_\mu - iv_\mu^{(s)})\Psi. \qquad (6.12)$$

Make use of product rule

$$\partial_\mu[\exp(-i\Theta)K\Psi] = -i\partial_\mu\Theta\exp(-i\Theta)K\Psi + \exp(-i\Theta)\partial_\mu K\Psi + \exp(-i\Theta)K\partial_\mu\Psi,$$

multiply by $\exp(i\Theta) \Rightarrow$ condition

$$\partial_\mu K = K\Gamma_\mu - \Gamma'_\mu K.$$

Eq. (6.5),

$$K = u'^\dagger V_R u = \underbrace{u'u'^\dagger}_{1} u'^\dagger V_R u = u'U'^\dagger V_R u = u'V_L \underbrace{U^\dagger}_{u^\dagger u^\dagger} \underbrace{V_R^\dagger V_R}_{1} u = u'V_L u^\dagger,$$

$\Rightarrow$

$$\begin{aligned} 2(K\Gamma_\mu - \Gamma'_\mu K) &= K\left[u^\dagger(\partial_\mu - ir_\mu)u\right] - \left[u'^\dagger(\partial_\mu - iV_R r_\mu V_R^\dagger + V_R\partial_\mu V_R^\dagger)u'\right]K \\ &\quad +(R \to L, r_\mu \to l_\mu, u \leftrightarrow u^\dagger, u' \leftrightarrow u'^\dagger) \\ &= u'^\dagger V_R(\partial_\mu u - ir_\mu u) - u'^\dagger\partial_\mu u' \underbrace{K}_{u'^\dagger V_R u} \end{aligned}$$

$$
\begin{aligned}
&+iu'^\dagger V_R r_\mu \underbrace{V_R^\dagger u' K}_{u} - u'^\dagger V_R \partial_\mu V_R^\dagger \underbrace{u' K}_{V_R u} \\
&+(R \to L, r_\mu \to l_\mu, u \leftrightarrow u^\dagger, u' \leftrightarrow u'^\dagger) \\
= \; & u'^\dagger V_R \partial_\mu u - iu'^\dagger V_R r_\mu u - \underbrace{u'^\dagger \partial_\mu u' u'^\dagger}_{-\partial_\mu u'^\dagger} V_R u \\
&+iu'^\dagger V_R r_\mu u - u'^\dagger \underbrace{V_R \partial_\mu V_R^\dagger V_R}_{-\partial_\mu V_R} u \\
&+(R \to L, r_\mu \to l_\mu, u \leftrightarrow u^\dagger, u' \leftrightarrow u'^\dagger) \\
= \; & u'^\dagger V_R \partial_\mu u + \partial_\mu u'^\dagger V_R u + u'^\dagger \partial_\mu V_R u \\
&+(R \to L, u \leftrightarrow u^\dagger, u' \leftrightarrow u'^\dagger) \\
= \; & \partial_\mu (u'^\dagger V_R u + u' V_L u^\dagger) = 2\partial_\mu K. \quad \surd
\end{aligned}
$$

At $\mathcal{O}(q)$: Another Hermitian building block, so-called chiral vielbein,

$$
u_\mu := i\left[u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger\right]. \tag{6.13}
$$

- Under parity transforms as axial vector

$$u_\mu \overset{P}{\mapsto} i\left[u(\partial^\mu - il^\mu)u^\dagger - u^\dagger(\partial^\mu - ir^\mu)u\right] = -u^\mu.$$

- Under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_V$ transforms as (Exercise)

$$u_\mu \mapsto K u_\mu K^\dagger.$$

Structure of the most general effective πN Lagrangian describing processes with a single nucleon in initial and final states:

$$\bar{\Psi}\widehat{O}\Psi.$$

$\widehat{O}$ is operator acting in Dirac- and flavor space transforming under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_V$ as $K\widehat{O}K^\dagger$. Lagrangian must be a Hermitian Lorentz scalar which is even under the discrete symmetries C, P, and T.

Lagrangian with smallest number of derivatives

$$\mathcal{L}_{\pi N}^{(1)} = \bar{\Psi}\left(i\not{D} - m + \frac{\mathrm{g}_A}{2}\gamma^\mu\gamma_5 u_\mu\right)\Psi. \tag{6.14}$$

Two parameters

- Chiral limit m of nucleon mass [physical nucleon mass: $m_N = 939$ MeV. Theoretical analysis: $m \approx 883$ MeV (for fixed $m_s \neq 0$)].
- Chiral limit $\mathtt{g}_A$ of axial-vector coupling constant g_A. Physical value determined from neutron beta decay: $g_A = 1.2695 \pm 0.0029$.

Overall normalization: External field and pion fields $\to 0 \Rightarrow$ free Lagrangian with mass m.

Discussion of power counting

Consider chiral limit: There is no reason for nucleon mass to become small $\Rightarrow \partial^0$ acting on nucleon field does not produce "small" quantity $\Rightarrow$ new features of chiral power counting in baryonic sector.

Counting of bilinears $\bar{\Psi}\Gamma\Psi$

Consider matrix elements of positive-energy plane-wave solutions to the free Dirac equation in the Dirac representation:

$$\psi^{(+)}(\vec{x},t) = \exp(-ip_N \cdot x)\sqrt{E_N + m_N}\begin{pmatrix} \chi \\ \frac{\vec{\sigma}\cdot\vec{p}_N}{E_N+m_N}\chi \end{pmatrix}. \tag{6.15}$$

- χ two-component Pauli spinor;
- $p_N^\mu = (E_N, \vec{p}_N)$ with $E_N = \sqrt{\vec{p}_N^{\,2} + m_N^2}$;
- Nonrelativistic limit: Lower (small) component suppressed with $|\vec{p}_N|/m_N$ relative to upper (large) component.

Divide 16 4×4 matrices into

$$\begin{aligned}\mathcal{E} &= \{\mathbb{1}, \gamma^0, \gamma^5\gamma^i, \sigma^{ij}\},\\ \mathcal{O} &= \{\gamma^5, \gamma^5\gamma^0, \gamma^i, \sigma^{0i}\},\end{aligned}$$

where

$$
\begin{aligned}
\mathbb{1} &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \\
\gamma^0 &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \\
\gamma^5\gamma^i &= \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \\
\sigma^{ij} &= \epsilon_{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \\
\gamma^5 &= \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \\
\gamma^5\gamma^0 &= \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \\
\gamma^i &= \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},
\end{aligned}
$$

$$\sigma^{0i} = i \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}.$$

Odd matrices couple large with small components; do not couple large with large components.
Even matrices couple large with large components; do not couple large with small components.
$i\partial^\mu$ produces $p_N^\mu = (m_N, \vec{0}) + (E_N - m_N, \vec{p}_N)$. First term $\mathcal{O}(q^0)$ and second term $\mathcal{O}(q)$.

Summary of chiral power counting (of new elements):

$$\Psi, \bar{\Psi} = \mathcal{O}(q^0),\ D_\mu \Psi = \mathcal{O}(q^0),\ (i\not{D} - m)\Psi = \mathcal{O}(q),$$
$$1, \gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu} = \mathcal{O}(q^0),\ \gamma_5 = \mathcal{O}(q). \tag{6.16}$$

The given order is always the *minimal* one. For example, γ_μ has $\mathcal{O}(q^0)$ piece (γ_0) and $\mathcal{O}(q)$ piece (γ_i).

Construction of $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ Lagrangian similar. Consider building blocks transforming as $K \cdots K^\dagger$. Take products, sandwich between $\bar{B}$ and B and take trace.

Lowest-order Lagrangian

$$\mathcal{L}_{MB}^{(1)} = \mathrm{Tr}\left[\bar{B}\,(i\not{D} - M_0)\,B\right] - \frac{D}{2}\mathrm{Tr}\left(\bar{B}\gamma^\mu\gamma_5\{u_\mu, B\}\right) - \frac{F}{2}\mathrm{Tr}\left(\bar{B}\gamma^\mu\gamma_5[u_\mu, B]\right). \tag{6.17}$$

- M_0 mass of baryon octet in chiral limit.
- Covariant derivative defined as

$$D_\mu B = \partial_\mu B + [\Gamma_\mu, B], \tag{6.18}$$

  with Γ_μ from Eq. (6.10) [for $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$].
- Constants D and F from fit to semileptonic decays $B \to B' + e^- + \bar{\nu}_e$ at tree level:

$$D = 0.80, \quad F = 0.50. \tag{6.19}$$

Further values used in the literature: ($D = 0.75$, $F = 0.50$), ($D = 0.804$, $F = 0.463$).

6.3 Pion Nucleon Scattering at Lowest Order

References:

- S. Weinberg, Phys. Rev. Lett. **17**, 616 (1966)
- Y. Tomozawa, Nuovo Cim. **46 A**, 707 (1966)
- M. Mojžiš, Eur. Phys. J. C **2**, 181 (1998)
- N. Fettes and U.-G. Meißner, Nucl. Phys. **A676**, 311 (2000)
- T. Becher and H. Leutwyler, JHEP **0106**, 017 (2001)
- R. Koch, Nucl. Phys. **A448**, 707 (1986)
- E. Matsinos, Phys. Rev. C **56**, 3014 (1997)

- H. C. Schröder *et al.*, Eur. Phys. J. C **21**, 473 (2001)

Effective Lagrangian of Eq. (6.14) reproduces famous Weinberg-Tomozawa predictions for s-wave scattering lengths.

General parameterization of invariant amplitude $\mathcal{M} = iT$ for the process $\pi^a(q) + N(p) \to \pi^b(q') + N(p')$

$$
\begin{aligned}
T^{ab}(p,q;p',q') \;=\; & \bar{u}(p')\Big\{ \underbrace{\frac{1}{2}\{\tau^b,\tau^a\}}_{\delta^{ab}} A^+(\nu,\nu_B) + \underbrace{\frac{1}{2}[\tau^b,\tau^a]}_{-i\epsilon_{abc}\tau^c} A^-(\nu,\nu_B) \\
& + \frac{1}{2}(\not{q} + \not{q}')\left[\delta^{ab}B^+(\nu,\nu_B) - i\epsilon_{abc}\tau^c B^-(\nu,\nu_B)\right]\Big\} u(p).
\end{aligned}
\tag{6.20}
$$

Two independent scalar kinematical variables

$$
\nu \;=\; \frac{s-u}{4m_N} = \frac{(p+p')\cdot q}{2m_N} = \frac{(p+p')\cdot q'}{2m_N}, \tag{6.21}
$$

$$\nu_B \; = \; -\frac{q \cdot q'}{2m_N} = \frac{t - 2M_\pi^2}{4m_N}, \tag{6.22}$$

with Mandelstam variables

$$\begin{aligned} s &= (p+q)^2, \\ t &= (p'-p)^2, \\ u &= (p'-q)^2, \\ s+t+u &= 2m_N^2 + 2M_\pi^2. \end{aligned}$$

Pion-crossing symmetry

$$T^{ab}(p,q;p',q') = T^{ba}(p,-q';p',-q).$$

(Exercise) $\Rightarrow$

$$\begin{aligned} &A^+(-\nu,\nu_B) = A^+(\nu,\nu_B), \quad A^-(-\nu,\nu_B) = -A^-(\nu,\nu_B), \\ &B^+(-\nu,\nu_B) = -B^+(\nu,\nu_B), \quad B^-(-\nu,\nu_B) = B^-(\nu,\nu_B). \end{aligned}$$

Isospin decomposition

$$\langle I', I'_3|T|I, I_3\rangle = T^I \delta_{II'}\delta_{I_3 I'_3}.$$

Relation between the two sets

$$\begin{aligned} T^{\frac{1}{2}} &= T^+ + 2T^-, \\ T^{\frac{3}{2}} &= T^+ - T^-. \end{aligned} \tag{6.23}$$

- **Evaluation of tree-level approximation to πN scattering obtained from $\mathcal{L}^{(1)}_{\pi N}$**

Consider chiral vielbein without external fields (Exercise)

$$u_\mu = -\frac{\vec{\tau}\cdot\partial_\mu\vec{\phi}}{F} + \mathcal{O}(\phi^3) \tag{6.24}$$

and chiral connection (Exercise)

$$\Gamma_\mu = \frac{i}{4F^2}\vec{\tau}\cdot\vec{\phi}\times\partial_\mu\vec{\phi} + \mathcal{O}(\phi^4). \tag{6.25}$$

$\Rightarrow$

$$\mathcal{L}_{\rm int} = -\frac{1}{2}\frac{\mathrm{g}_A}{F}\bar{\Psi}\gamma^\mu\gamma_5\tau^b\partial_\mu\phi^b\Psi - \frac{1}{4F^2}\bar{\Psi}\gamma^\mu \underbrace{\vec{\tau}\cdot\vec{\phi}\times\partial_\mu\vec{\phi}}_{\epsilon_{cde}\tau^c\phi^d\partial_\mu\phi^e} \Psi. \qquad (6.26)$$

- First term: Pseudovector pion-nucleon coupling.
- Second term: Weinberg-Tomozawa contact interaction.

Feynman rules

- for an incoming pion with four-momentum q and Cartesian isospin index a:

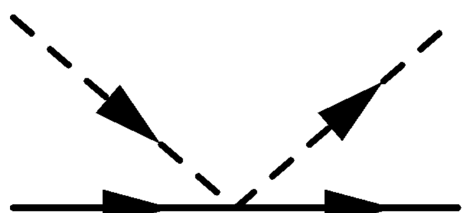
$$-\frac{1}{2}\frac{\mathrm{g}_A}{F}\not{q}\gamma_5\tau^a, \qquad (6.27)$$

- for an incoming pion with q, a and an outgoing pion with q', b:

$$i\left(-\frac{1}{4F^2}\right)\gamma^\mu\epsilon_{cde}\tau^c\left(\delta^{da}\delta^{eb}iq'_\mu + \delta^{db}\delta^{ea}(-iq)_\mu\right) = \frac{\not{q}+\not{q}'}{4F^2}\epsilon_{abc}\tau^c. \qquad (6.28)$$

Calculation at lowest order: $m \to m_N$, $F \to F_\pi$, and $\mathtt{g}_A \to g_A$, because difference is $\mathcal{O}(q^2)$.

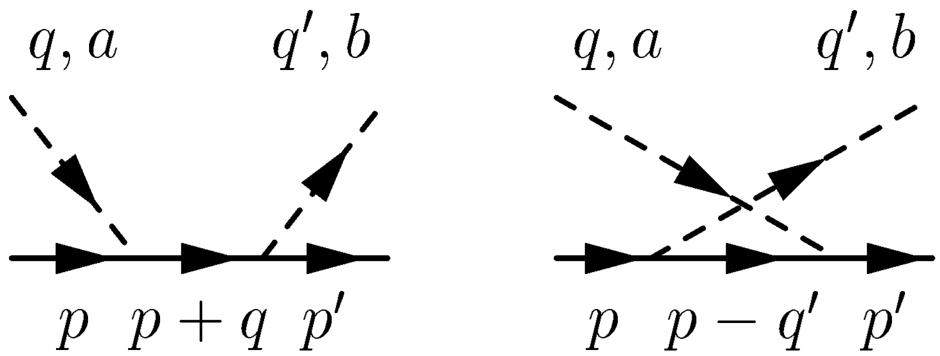
Contact contribution



$$\mathcal{M}_{\rm cont} = \bar{u}(p')\frac{\not q + \not q'}{4F_\pi^2} \underbrace{\epsilon_{abc}\tau^c}_{i\frac{1}{2}[\tau^b,\tau^a]} u(p) = i\frac{1}{2F_\pi^2}\bar{u}(p')\frac{1}{2}[\tau^b,\tau^a]\frac{1}{2}(\not q + \not q')u(p). \tag{6.29}$$

Remark: "Conventional" calculation (PS or PV couplings) does not generate such a term.

s- and u-channel nucleon-pole diagrams



$$
\begin{aligned}
\mathcal{M}_{s+u} &= \frac{g_A^2}{4F_\pi^2}\bar{u}(p')\tau^b\tau^a(-)(-\not{q}')\gamma_5\frac{i}{\not{p}'+\not{q}'-m_N}(-\not{q})\gamma_5 u(p) \\
&\quad +\frac{g_A^2}{4F_\pi^2}\bar{u}(p')\tau^a\tau^b(-\not{q})\gamma_5\frac{i}{\not{p}'-\not{q}-m_N}(-)(-\not{q}')\gamma_5 u(p)
\end{aligned}
$$

$$
\begin{aligned}
&= i\frac{g_A^2}{4F_\pi^2}\bar{u}(p')\tau^b\tau^a(-\not{q}')\gamma_5\frac{1}{\not{p}' + \not{q}' - m_N}\not{q}\gamma_5 u(p) \\
&\quad +i\frac{g_A^2}{4F_\pi^2}\bar{u}(p')\tau^a\tau^b\not{q}\gamma_5\frac{1}{\not{p}' - \not{q} - m_N}(-\not{q}')\gamma_5 u(p). \qquad (6.30)
\end{aligned}
$$

s- and u-channel contributions related via pion crossing $a \leftrightarrow b$ and $q \leftrightarrow -q'$. In the following: Explicitly calculate s-channel contribution, obtain u-channel contribution through crossing.

Make use of Dirac equation and rewrite

$$
\begin{aligned}
\not{q}\gamma_5 u(p) &= (\not{p}' + \not{q}' - m_N + m_N - \not{p})\gamma_5 u(p) \\
&= (\not{p}' + \not{q}' - m_N)\gamma_5 u(p) + 2m_N\gamma_5 u(p).
\end{aligned}
$$

$\Rightarrow$

$$
\mathcal{M}_s = i\frac{g_A^2}{4F_\pi^2}\bar{u}(p')\tau^b\tau^a(-\not{q}')\gamma_5\frac{1}{\not{p}' + \not{q}' - m_N}
$$

$$\times \left[(\not{p}' + \not{q}' - m_N) + 2m_N\right]\gamma_5 u(p)$$

$$\stackrel{\gamma_5^2=1}{=} \; i\frac{g_A^2}{4F_\pi^2}\bar{u}(p')\tau^b\tau^a\left[(-\not{q}') + (-\not{q}')\gamma_5\frac{1}{\not{p}' + \not{q}' - m_N}2m_N\gamma_5\right]u(p).$$

Repeat procedure

$$\bar{u}(p')\not{q}'\gamma_5 = \bar{u}(p')[-2m_N\gamma_5 - \gamma_5(\not{p} + \not{q} - m_N)].$$

$\Rightarrow$

$$\mathcal{M}_s = i\frac{g_A^2}{4F_\pi^2}\bar{u}(p')\tau^b\tau^a[(-\not{q}') + \underbrace{4m_N^2\gamma_5\frac{1}{\not{p}' + \not{q}' - m_N}\gamma_5}_{\text{PS coupling}} + 2m_N]u(p). \tag{6.31}$$

PS result using Goldberger-Treiman relation

$$\frac{\mathrm{g}_A}{F} = \frac{\mathrm{g}_{\pi N}}{m}.$$

$\mathrm{g}_{\pi N}$: Pion nucleon coupling constant in chiral limit.

$$s - m_N^2 = 2m_N(\nu - \nu_B),$$

$\Rightarrow$

$$\begin{aligned}
&\bar{u}(p')\gamma_5 \frac{1}{\not p' + \not q' - m_N} \gamma_5 u(p) \\
&= \bar{u}(p')\gamma_5 \frac{\not p' + \not q' + m_N}{(p' + q')^2 - m_N^2} \gamma_5 u(p) \\
&\overset{\text{Dirac eq.}}{=} \bar{u}(p')\gamma_5 \frac{\not q'}{(p' + q')^2 - m_N^2} \gamma_5 u(p) \\
&= -\bar{u}(p') \frac{\not q'}{(p' + q')^2 - m_N^2} u(p) \\
&\overset{\text{momentum cons.}}{=} -\bar{u}(p') \frac{1}{2} \frac{\not q' + \not p + \not q - \not p'}{(p' + q')^2 - m_N^2} u(p) \\
&\overset{\text{Dirac eq.}}{=} \frac{1}{2m_N(\nu - \nu_B)} \left[-\frac{1}{2} \bar{u}(p')(\not q + \not q')u(p) \right].
\end{aligned}$$

$\Rightarrow$

$$\mathcal{M}_s = i\frac{g_A^2}{4F_\pi^2}\bar{u}(p')\tau^b\tau^a\left[2m_N + \frac{1}{2}(\not{q} + \not{q}')\left(-1 - \frac{2m_N}{\nu - \nu_B}\right)\right]u(p). \tag{6.32}$$

u-channel result through substitution $a \leftrightarrow b$ and $q \leftrightarrow -q'$

$$\mathcal{M}_u = i\frac{g_A^2}{4F_\pi^2}\bar{u}(p')\tau^a\tau^b\left[2m_N + \frac{1}{2}(\not{q} + \not{q}')\left(1 - \frac{2m_N}{\nu + \nu_B}\right)\right]u(p). \tag{6.33}$$

No t-channel contribution, since 3π vertex does not exist.

Combine s- and u-channel contributions using

$$\tau^b\tau^a = \frac{1}{2}\{\tau^b, \tau^a\} + \frac{1}{2}[\tau^b, \tau^a], \quad \tau^a\tau^b = \frac{1}{2}\{\tau^b, \tau^a\} - \frac{1}{2}[\tau^b, \tau^a],$$

and

$$\frac{1}{\nu - \nu_B} \pm \frac{1}{\nu + \nu_B} = \frac{\left\{\begin{array}{c} 2\nu \\ 2\nu_B \end{array}\right\}}{\nu^2 - \nu_B^2}.$$

Summary of contributions to $A^{\pm}$ and $B^{\pm}$:

amplitude\origin	PS	ΔPV	contact	sum
A^+	0	$\frac{g_A^2 m_N}{F_\pi^2}$	0	$\frac{g_A^2 m_N}{F_\pi^2}$
A^-	0	0	0	0
B^+	$-\frac{g_A^2}{F_\pi^2}\frac{m_N \nu}{\nu^2-\nu_B^2}$	0	0	$-\frac{g_A^2}{F_\pi^2}\frac{m_N \nu}{\nu^2-\nu_B^2}$
B^-	$-\frac{g_A^2}{F_\pi^2}\frac{m_N \nu_B}{\nu^2-\nu_B^2}$	$-\frac{g_A^2}{2F_\pi^2}$	$\frac{1}{2F_\pi^2}$	$\frac{1-g_A^2}{2F_\pi^2} - \frac{g_A^2}{F_\pi^2}\frac{m_N \nu_B}{\nu^2-\nu_B^2}$

- Scattering lengths

Consider threshold kinematics

$$p^\mu = p'^\mu = (m_N, 0), \quad q^\mu = q'^\mu = (M_\pi, 0), \quad \nu|_{\rm thr} = M_\pi, \quad \nu_B|_{\rm thr} = -\frac{M_\pi^2}{2m_N}. \tag{6.34}$$

Using

$$u(p) \to \sqrt{2m_N}\begin{pmatrix}\chi \\ 0\end{pmatrix}, \quad \bar{u}(p') \to \sqrt{2m_N}\begin{pmatrix}\chi'^\dagger & 0\end{pmatrix}$$

$\Rightarrow$ matrix elements at threshold

$$T|_{\rm thr} = 2m_N\chi'^\dagger\left[\delta^{ab}\left(A^+ + M_\pi B^+\right) - i\epsilon_{abc}\tau^c\left(A^- + M_\pi B^-\right)\right]_{\rm thr}\chi. \tag{6.35}$$

Using

$$\left[\nu^2 - \nu_B^2\right]_{\rm thr} = M_\pi^2\left(1 - \frac{\mu^2}{4}\right), \quad \mu = \frac{M_\pi}{m_N} \approx \frac{1}{7}$$

$\Rightarrow$

$$T\big|_{\rm thr} \;=\; 2m_N\chi'^\dagger\Big[\delta^{ab}\,\underbrace{\Big(\frac{g_A^2 m_N}{F_\pi^2}+\underbrace{M_\pi\left(-\frac{g_A^2}{F_\pi^2}\right)\frac{m_N}{M_\pi}\frac{1}{1-\frac{\mu^2}{4}}}_{\rm PS}\Big)}_{\rm ChPT\,=\,PV}$$

$$-i\epsilon_{abc}\tau^c M_\pi\,\underbrace{\Big(\frac{1}{2F_\pi^2}\underbrace{-\frac{g_A^2}{2F_\pi^2}\underbrace{-\frac{g_A^2}{F_\pi^2}\left(-\frac{1}{2}\right)\frac{1}{1-\frac{\mu^2}{4}}}_{\rm PS}}_{\rm PV}\Big)}_{\rm ChPT}\Big]\chi. \quad (6.36)$$

Discussion of s-wave scattering lengths.
Consider differential cross section in center of mass system

$$\frac{d\sigma}{d\Omega}=\frac{|\vec{q}\,'|}{|\vec{q}\,|}\left(\frac{1}{8\pi\sqrt{s}}\right)^2|T|^2. \quad (6.37)$$

At threshold

$$\left.\frac{d\sigma}{d\Omega}\right|_{\rm thr} = \left(\frac{1}{8\pi(m_N+M_\pi)}\right)^2 |T|^2_{\rm thr} \stackrel{!}{=} |a|^2. \tag{6.38}$$

Definition (via multipole decomposition)

$$a^\pm_{0+} = \frac{1}{8\pi(m_N+M_\pi)} T^\pm|_{\rm thr} = \frac{1}{4\pi(1+\mu)}\left[A^\pm + M_\pi B^\pm\right]_{\rm thr}. \tag{6.39}$$

Index 0+: πN system is in s wave ($l=0$) with total angular momentum $1/2 = 0 + 1/2$.

Results of table $\Rightarrow$

$$a^-_{0+} = \frac{M_\pi}{8\pi(1+\mu)F_\pi^2}\left(1+\frac{g_A^2\mu^2}{4}\frac{1}{1-\frac{\mu^2}{4}}\right) = \frac{M_\pi}{8\pi(1+\mu)F_\pi^2}[1+\mathcal{O}(q^2)], \tag{6.40}$$

$$a^+_{0+} = -\frac{g_A^2 M_\pi}{16\pi(1+\mu)F_\pi^2}\frac{\mu}{1-\frac{\mu^2}{4}} = \mathcal{O}(q^2). \tag{6.41}$$

Linear combinations $a^{\frac{1}{2}} = a^+_{0+} + 2a^-_{0+}$ and $a^{\frac{3}{2}} = a^+_{0+} - a^-_{0+}$:

$$\begin{aligned} a^{\frac{1}{2}} &= \frac{M_\pi}{4\pi(1+\mu)F_\pi^2} + \mathcal{O}(M_\pi^2), \\ a^{\frac{3}{2}} &= -\frac{M_\pi}{8\pi(1+\mu)F_\pi^2} + \mathcal{O}(M_\pi^2), \end{aligned}$$

satisfy Weinberg-Tomozawa relation

$$a^I = -\frac{M_\pi}{8\pi(1+\mu)F_\pi^2}[I(I+1) - \frac{3}{4} - 2], \tag{6.42}$$

because

$$\begin{aligned} I = \frac{1}{2}: &\quad \frac{1}{2}\cdot\frac{3}{2} - \frac{3}{4} - 2 = -2, \\ I = \frac{3}{2}: &\quad \frac{3}{2}\cdot\frac{5}{2} - \frac{3}{4} - 2 = 1. \end{aligned}$$

[The result, in principle, holds for a general target of isospin T (except for the pion) after replacing $3/4$ by $T(T+1)$ and μ by M_π/M_T.]

Remarks:

- As in $\pi\pi$ scattering, scattering lengths vanish in the chiral limit. Interaction of Goldstone bosons with other hadrons vanishes in zero-energy limit.

- PS coupling

  - $a_{0+}^{+} \sim$ proportional m_N instead of $\sim \mu M_\pi$. Contradiction to requirements of chiral symmetry.
  - a_{0+}^{-} is too large by a factor of g_A^2 in comparison with the two-pion contact term of Eq. (6.29) (Weinberg-Tomozawa term) induced by nonlinear realization of chiral symmetry.

- PV coupling

  - Totally wrong result for a_{0+}^{-}, because it misses the two-pion contact term of Eq. (6.29).

scattering length	a_{0+}^{+} [MeV^{-1}]	a_{0+}^{-} [MeV^{-1}]
tree level result	-6.80×10^{-5}	$+5.71 \times 10^{-4}$
ChPT $\mathcal{O}(q)$	0	$+5.66 \times 10^{-4}$
HBChPT $\mathcal{O}(q^2)$ [3]	-1.3×10^{-4}	$+5.5 \times 10^{-4}$
HBChPT $\mathcal{O}(q^3)$ [3]	$(-7 \pm 9) \times 10^{-5}$	$(+6.7 \pm 1.0) \times 10^{-4}$
HBChPT $\mathcal{O}(q^4)$ [I] [4]	-6.9×10^{-5}	$+6.47 \times 10^{-4}$
HBChPT $\mathcal{O}(q^4)$ [II] [4]	$+3.2 \times 10^{-5}$	$+5.52 \times 10^{-4}$
HBChPT $\mathcal{O}(q^4)$ [III] [4]	$+1.9 \times 10^{-5}$	$+6.21 \times 10^{-4}$
RChPT $\mathcal{O}(q^4)$ (a) [5]	-6.0×10^{-5}	$+6.55 \times 10^{-4}$
RChPT $\mathcal{O}(q^4)$ (b) [5]	-9.4×10^{-5}	$+6.55 \times 10^{-4}$
PS	-1.23×10^{-2}	$+9.14 \times 10^{-4}$
PV	-6.80×10^{-5}	$+5.06 \times 10^{-6}$
empirical values [6]	$(-7 \pm 1) \times 10^{-5}$	$(6.6 \pm 0.1) \times 10^{-4}$
empirical values [7]	$(2.04 \pm 1.17) \times 10^{-5}$	$(5.71 \pm 0.12) \times 10^{-4}$ $(5.92 \pm 0.11) \times 10^{-4}$
experiment [8]	$(-2.7 \pm 3.6) \times 10^{-5}$	$(+6.59 \pm 0.30) \times 10^{-4}$

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Numerical results using

$$g_A = 1.267, \quad F_\pi = 92.4\,\text{MeV}, \quad m_N = m_p = 938.3\,\text{MeV}, \quad , M_\pi = M_{\pi^+} = 139.6\,\text{MeV}.$$

6.4 Example for a Loop Diagram

References:

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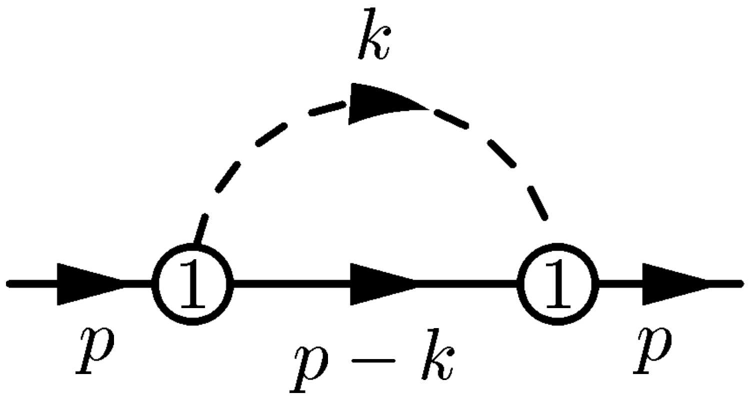
6.4.1 Power Counting

- **Power counting:** Associate chiral order D with a diagram
  - Square of the lowest-order pion mass: $M^2 = B(m_u + m_d) \sim \mathcal{O}(q^2)$
  - Nucleon mass in the chiral limit $m \sim \mathcal{O}(q^0)$
  - Loop integration in n dimensions $\sim \mathcal{O}(q^n)$
  - Vertex from $\mathcal{L}_\pi^{(2k)} \sim \mathcal{O}(q^{2k})$
  - Vertex from $\mathcal{L}_{\pi N}^{(k)} \sim \mathcal{O}(q^k)$
  - Nucleon propagator $\sim \mathcal{O}(q^{-1})$
  - Pion propagator $\sim \mathcal{O}(q^{-2})$

6.4.2 One-Loop Correction to the Nucleon Mass

- Example: Contribution to nucleon mass



Goal: $D = n \cdot 1 - 2 \cdot 1 - 1 \cdot 1 + 2 \cdot 1 = n - 1$

How this is achieved, see Assignment 13.

6.4.3 The Generation of Counterterms *

The renormalization of the effective field theory (of pions and nucleons) is performed by expressing all the bare parameters and bare fields of the effective Lagrangian in terms of renormalized quantities [see J. C. Collins, *Renormalization* (Cambridge University Press, Cambridge, 1984) for details]. In this process, one generates counterterms which are responsible for the absorption of all the divergences occurring in the calculation of loop diagrams. In order to illustrate the procedure let us discuss $\mathcal{L}_{\pi N}^{(1)}$ and consider the free part in combination with the πN interaction term with the smallest number of pion fields,

$$\mathcal{L}_{\pi N}^{(1)} = \bar{\Psi}_0 \left(i\gamma_\mu \partial^\mu - m_0 - \frac{1}{2} \frac{\mathrm{g}_{A0}}{F_0} \gamma_\mu \gamma_5 \tau^a \partial^\mu \pi_0^a \right) \Psi_0 + \cdots, \tag{6.43}$$

given in terms of bare fields and parameters denoted by subscripts 0. Introducing renormalized fields (we work in the isospin-symmetric limit) through

$$\Psi = \frac{\Psi_0}{\sqrt{Z_\Psi}}, \quad \pi^a = \frac{\pi_0^a}{\sqrt{Z_\pi}}, \tag{6.44}$$

we express the field redefinition constants $\sqrt{Z_\Psi}$ and $\sqrt{Z_\pi}$ and the bare quantities in terms of renormalized parameters:

$$\begin{aligned} Z_\Psi &= 1 + \delta Z_\Psi \left(m, \mathrm{g}_A, g_i, \nu \right), \\ Z_\pi &= 1 + \delta Z_\pi \left(m, \mathrm{g}_A, g_i, \nu \right), \\ m_0 &= m(\nu) + \delta m \left(m, \mathrm{g}_A, g_i, \nu \right), \\ \mathrm{g}_{A0} &= \mathrm{g}_A(\nu) + \delta g_A \left(m, \mathrm{g}_A, g_i, \nu \right), \end{aligned} \tag{6.45}$$

where g_i, $i = 1, \cdots \infty$, collectively denote all the renormalized parameters which correspond to bare parameters g_{i0} of the full effective Lagrangian. The parameter ν indicates the dependence on the choice of the renormalization prescription.[1] Substituting Eqs. (6.44) and (6.45) into Eq. (6.43), we obtain

$$\mathcal{L}_{\pi N}^{(1)} = \mathcal{L}_{\text{basic}} + \mathcal{L}_{\text{ct}} + \cdots \tag{6.46}$$

with the so-called basic and counterterm Lagrangians, respectively,[2]

$$\mathcal{L}_{\text{basic}} = \bar{\Psi}\left(i\gamma_\mu\partial^\mu - m - \frac{1}{2}\frac{\mathrm{g}_A}{F}\gamma_\mu\gamma_5\tau^a\partial^\mu\pi^a\right)\Psi, \tag{6.47}$$

$$\mathcal{L}_{\text{ct}} = \delta Z_\Psi\bar{\Psi}i\gamma_\mu\partial^\mu\Psi - \delta\{m\}\bar{\Psi}\Psi - \frac{1}{2}\delta\left\{\frac{\mathrm{g}_A}{F}\right\}\bar{\Psi}\gamma_\mu\gamma_5\tau^a\partial^\mu\pi^a\Psi, \tag{6.48}$$

where we introduced the abbreviations

$$\begin{aligned}\delta\{m\} &\equiv \delta Z_\Psi m + Z_\Psi\delta m,\\ \delta\left\{\frac{\mathrm{g}_A}{F}\right\} &\equiv \delta Z_\Psi\frac{\mathrm{g}_A}{F}\sqrt{Z_\pi} + Z_\Psi\left(\frac{\mathrm{g}_{A0}}{F_0} - \frac{\mathrm{g}_A}{F}\right)\sqrt{Z_\pi} + \frac{\mathrm{g}_A}{F}(\sqrt{Z_\pi} - 1).\end{aligned}$$

In Eq. (6.47), m, g_A, and F denote the chiral limit of the physical nucleon mass, the axial-vector coupling constant, and the pion-decay constant, respectively. Expanding the counterterm Lagrangian of Eq. (6.48) in powers of the renormalized coupling constants generates an infinite series, the individual terms of which are responsible for the subtraction of loop diagrams.

[1]Note that our choice $m(\nu) = m$, where m is the nucleon pole mass in the chiral limit, is only one among an infinite number of possibilities.

[2]Collins uses a slightly different convention which is obtained through the replacement $(\delta Z_\Psi m + Z_\Psi\delta m) \to \delta m$.