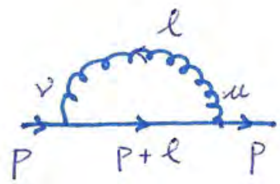
RENORMALIZATION IN QCD

⇒ LOOP DIAGRAMS

↳ LOOP DIAGRAMS CORRESPOND WITH QUANTUM CORRECTIONS

- **QUARK SELF-ENERGY**

FREE QUARK PROPAGATOR $\frac{i\,(\not p + m)}{p^2 - m^2}$



GLUON 'DRESSING' OF THE QUARK
(SELF ENERGY) MODIFIES ITS PROPERTIES
$\Sigma(p)$

$$-i\,\Sigma(p) \equiv \int \frac{d^4\ell}{(2\pi)^4}\, \frac{i}{\ell^2}\left[-g_{\mu\nu} + (1-\xi)\,\ell_\mu \ell_\nu/\ell^2\right]$$

$$\cdot\, \frac{1}{3}\,\underbrace{\mathrm{Tr}\left\{\frac{\lambda_a}{2}\frac{\lambda_a}{2}\right\}}_{\frac{1}{2}\cdot\underbrace{\delta_{aa}}_{=8}} \cdot \left[-ig\gamma^\mu\right] \frac{i(\not p + \not \ell + m)}{\left[(p+\ell)^2 - m^2\right]}\left[-ig\gamma^\nu\right]$$

AVERAGE OVER 3 COLORS

COLOR FACTOR : $\frac{4}{3}$

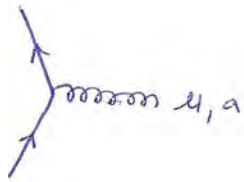
FOR $\ell \gg$ PART OF LOOP

$$-i\,\Sigma(p) \longrightarrow \int d^4\ell\, \frac{1}{\ell^2}\, \frac{\not\ell}{\ell^2} \Rightarrow \text{LINEARLY DIVERGENT}$$

• QUARK – GLUON VERTEX

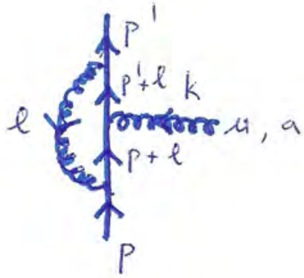
↳ LOWEST ORDER

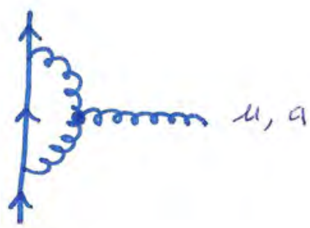
$O(g)$



$-ig \frac{\lambda_a}{2} \gamma^\mu$

↳ $O(g^3)$





$-ig \Lambda_a^{(1)\mu}$ $\qquad$ $-ig \Lambda_a^{(2)\mu}$

$$-ig \Lambda_a^{(1)\mu} = \int \frac{d^4 l}{(2\pi)^4} \left[-ig \frac{\lambda_b}{2} \gamma^\beta\right] \frac{i(\not{p}' + \not{l} + m)}{(p'+l)^2 - m^2} \left[-ig \frac{\lambda_a}{2} \gamma^\mu\right]$$

$$\frac{i(\not{p} + \not{l} + m)}{(p+l)^2 - m^2} \left[-ig \frac{\lambda_b}{2} \gamma^\alpha\right]$$

$$\cdot \frac{i}{l^2} \left[-g_{\alpha\beta} + (1-\xi) l_\alpha l_\beta / l^2\right]$$

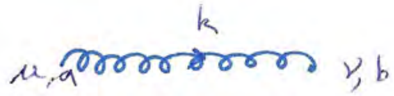
$$= -ig \left(ig^2\right) \frac{\lambda_b}{2} \frac{\lambda_a}{2} \frac{\lambda_b}{2}$$

$$\cdot \int \frac{d^4 l}{(2\pi)^4} \frac{\gamma^\beta (\not{p}' + \not{l} + m) \gamma^\mu (\not{p} + \not{l} + m) \gamma^\alpha}{\left[(p'+l)^2 - m^2\right]\left[(p+l)^2 - m^2\right] l^2} \left\{-g_{\alpha\beta} + (1-\xi) \frac{l_\alpha l_\beta}{l^2}\right\}$$

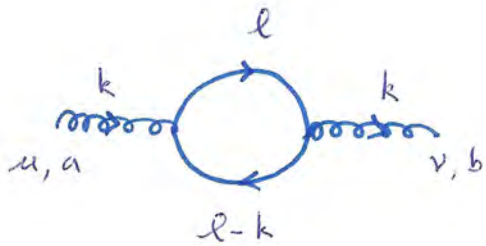
$$\xrightarrow[l \gg]{} \int d^4 l \, \frac{l^2}{l^6} \sim \int d^4 l \, \frac{1}{l^4}$$

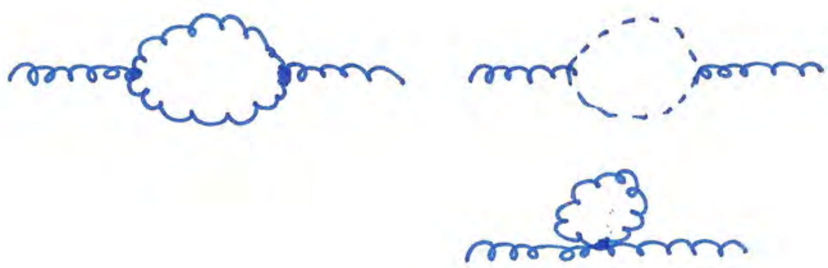
LOGARITHMIC DIVERGENCE

• GLUON SELF-ENERGY



$$\delta_{ab}\frac{i}{k^2}\left[-g_{\mu\nu}+(1-\xi)\frac{k_\mu k_\nu}{k^2}\right]$$





$$-i\Pi^{\mu\nu}_{ab}(k)_{\text{QUARK LOOP}} = \int\frac{d^4\ell}{(2\pi)^4}\underset{\substack{\uparrow\\ \text{FERMION LOOP}}}{(-1)}\,\mathrm{Tr}\left\{\left[-ig\frac{\lambda_b}{2}\gamma^\nu\right]\frac{i(\not{\ell}+m)}{\ell^2-m^2}\left[-ig\frac{\lambda_a}{2}\gamma^\mu\right]\cdot\frac{i(\not{\ell}-\not{k}+m)}{(\ell-k)^2-m^2}\right\}$$

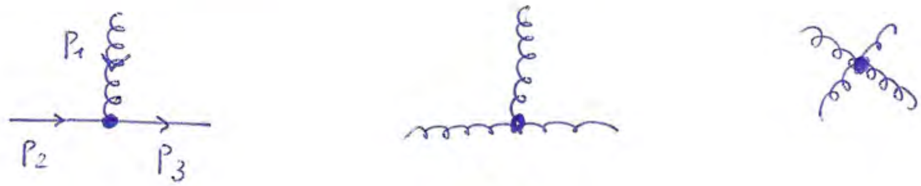
$$= -\underbrace{\mathrm{Tr}\left\{\frac{\lambda_b}{2}\frac{\lambda_a}{2}\right\}}_{\frac{1}{2}\delta_{ab}}\cdot\int\frac{d^4\ell}{(2\pi)^4}\frac{\mathrm{Tr}\left\{\gamma^\nu(\not{\ell}+m)\gamma^\mu(\not{\ell}-\not{k}+m)\right\}}{\left[\ell^2-m^2\right]\left[(\ell-k)^2-m^2\right]}$$

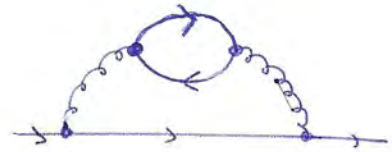
$$\xrightarrow[\ell\gg]{}\int d^4\ell\,\frac{\ell^2}{\ell^4}\sim\int d^4\ell\,\frac{1}{\ell^2}$$

QUADRATIC DIVERGENCE

⇒ DEGREE OF DIVERGENCE OF ARBITRARY DIAGRAM

FOR EVERY VERTEX OF TYPE i



e.g. ENTERS 

- EVERY MOMENTUM $\sim \int \frac{d^4 p_i}{(2\pi)^4}$

- EVERY FERMION PROPAGATOR CONNECTING 2 VERTICES LEADS TO A "MOMENTUM" POWER AT EACH VERTEX

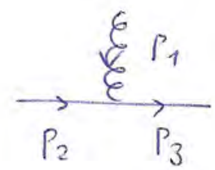
$$\frac{1}{2} \cdot \int d^4 p_i \quad \frac{\not{p}_i + m}{p_i^2 - m^2} \longrightarrow \frac{3}{2}$$

BECAUSE PROPAGATOR CONNECTS 2 VERTICES

- EVERY BOSON PROPAGATOR LEADS TO A "MOMENTUM" POWER AT EACH VERTEX

$$\frac{1}{2} \cdot \int d^4 p_i \cdot \frac{1}{p_i^2 - m^2} \left(-g_{\mu\nu} + \frac{p_{i\mu} p_{i\nu}}{p_i^2} (1-\xi) \right) \longrightarrow 1$$

- FOR EVERY VERTEX : THERE IS 1 ENERGY - MOMENTUM CONSERVATION



$\rightarrow \; p_1 + p_2 = p_3$

$\delta^4 \left(\sum_i p_i \right) \rightarrow$ REDUCING MOMENTUM POWER COUNTING BY 4 AT EACH VERTEX.

$\Downarrow$

-4

- IF WE HAVE DERIVATIVE INTERACTIONS

e.g.

$d_i = 1$

d_i = # DERIVATIVES IN INTERACTION $\mathcal{L}$

$\Downarrow$

LEADS TO MOMENTUM POWER d_i AT EACH VERTEX

$\therefore$ DEGREE OF DIVERGENCE (**DD**) WHEN SUMMING OVER ALL VERTICES

$$DD = \frac{\text{\# MOMENTUM POWERS IN NUMERATOR}}{\text{\# MOMENTUM POWERS IN DENOMINATOR}}$$

$$\mathbf{DD} = \sum_i n_i \cdot \left(d_i + \frac{3}{2} f_i + b_i - 4 \right)$$

ENERGY - MOMENTUM CONSERV.

VERTICES OF TYPE i

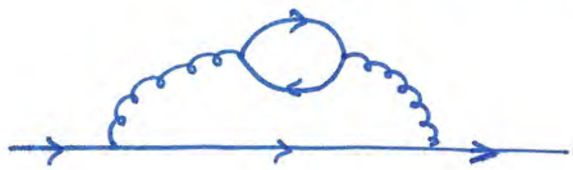
SUM OVER ALL VERTICES OF TYPE i

FERMION LINES ENTERING VERTEX i

BOSON LINES ENTERING VERTEX i

DERIVATIVES IN $\mathcal{L}$ AT VERTEX i

- CORRECT FOR EXTERNAL LINES



→ BECAUSE FOR EXTERNAL LINES : NO PROPAGATOR

→ WE SUBTRACTED ONE ENERGY-MOMENTUM CONSERVATION TOO MUCH, BECAUSE ONE CORRESPONDS WITH OVERALL ENERGY - MOMENTUM CONSERVATION

↓

LEADS TO NO REDUCTION IN MOMENTUM POWER.

$$\therefore \boxed{DD = \sum_i n_i \cdot \delta_i + \left(4 - \frac{3}{2} N_f - N_b\right)}$$

N_f : # EXTERNAL FERMION LINES

N_b : # " BOSON "

δ_i : INDEX OF DIVERGENCE OF $\mathcal{L}_i$

$$\delta_i \equiv d_i + \frac{3}{2} f_i + b_i - 4$$

- IF $\delta_i > 0$

  ↳ MORE & MORE DIVERGENCES APPEAR WHEN CALCULATING HIGHER LOOPS

  $\Downarrow$

  NON - RENORMALIZABLE THEORY

- IF $\delta_i < 0$

  ↳ THEORY BECOMES MORE CONVERGENT WHEN CALCULATING HIGHER LOOPS

  SUCH THEORIES ARE TRIVIAL

  $\Downarrow$

  SUPER - RENORMALIZABLE THEORY

- IF $\delta_i = 0$

  ↳ NO NEW DIVERGENCES APPEAR WHEN CALCULATING HIGHER LOOP DIAGRAMS COMPARED WITH 1 LOOP DIAGRAMS

  $\Downarrow$

  RENORMALIZABLE THEORY

  'PHYSICAL' THEORIES

  QED, QCD, STANDARD ELECTROWEAK GAUGE THEORY

- e.g. FOR QCD

$\mathcal{L}_{qqg}$

$f_i = 2 \quad , \quad b_i = 1 \quad , \quad d_i = 0$

$\delta_i = \frac{3}{2} f_i + b_i + d_i - 4$

$\longrightarrow \quad \delta_i = \frac{3}{2} \cdot 2 + 1 + 0 - 4 \stackrel{!}{=} 0$

$\mathcal{L}_{3g}$

$f_i = 0 \quad , \quad b_i = 3 \quad , \quad d_i = 1$

$\longrightarrow \quad \delta_i = 0 + 3 + 1 - 4 \stackrel{!}{=} 0$

$\mathcal{L}_{4g}$

$f_i = 0 \quad , \quad b_i = 4 \quad , \quad d_i = 0$

$\longrightarrow \quad \delta_i = 0 + 4 + 0 - 4 \stackrel{!}{=} 0$

$\hookrightarrow$ QCD IS RENORMALIZABLE

⇒ RENORMALIZATION IN QCD

- $$\boxed{DD = 4 - \frac{3}{2} N_q - N_g}$$

DEGREE OF DIVERGENCE OF ANY DIAGRAM

N_q : # EXTERNAL QUARK LINES

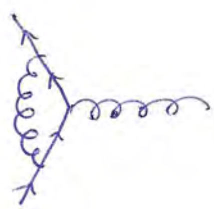
N_g : # EXTERNAL GLUON LINES

e.g.

$N_q = 2$, $N_g = 0$

$DD = 4 - \frac{3}{2} \cdot 2 - 0$

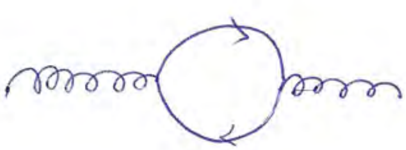
$\boxed{DD = 1}$ LINEARLY DIVERGENT



$N_q = 2$, $N_g = 1$

$DD = 4 - \frac{3}{2} \cdot 2 - 1$

$\boxed{DD = 0}$ LOGARITHMICALLY DIVERGENT



$N_q = 0$, $N_g = 2$

$DD = 4 - 0 - 2$

$\boxed{DD = 2}$ QUADRATICALLY DIVERGENT

- FOR A RENORMALIZABLE THEORY

  ↓

  ONLY A FINITE # OF DIVERGENCE TYPES APPEAR

IDEA ABSORB DIVERGENCE

BY REDEFINING A FINITE NUMBER OF PARAMETERS IN THEORY

↳ PARAMETERS (m, g) AND FIELDS (q, A^μ_a) IN $\mathcal{L}$

ARE TO BE CONSIDERED AS BARE (UNPHYSICAL) PARAMETERS

WHICH GET MODIFIED BY INTERACTION !

WRITE ORIGINAL $\mathcal{L}$ ⤳ CALL $\mathcal{L}_B$ (BARE)

$$\boxed{\mathcal{L}_B = \mathcal{L}_R + \mathcal{L}_{CT}}$$

$\mathcal{L}_R$: RENORMALIZED $\mathcal{L}$ PHYSICAL PARAMETERS / FIELDS

$\mathcal{L}_{CT}$: COUNTER TERM : CANCELS DIVERGENCE

$$\boxed{\begin{aligned}
q_B &= Z_2^{1/2}\, q \\
A^\mu_{a,B} &= Z_3^{1/2}\, A^\mu_a \\
\chi_{a,B} &= \tilde{Z}_3^{1/2}\, \chi_a \\
g_B &= Z_g\, g \\
m_B &= Z_m\, m \\
\xi_B &= Z_3\, \xi
\end{aligned}}$$

(GAUGE PARAMETER)

q_B : BARE FIELD

q : RENORMALIZED FIELD

Z_2 : FIELD RENORM. CONSTANT 'ABSORBS' DIVERGENCE

↳ e.g. $\mathcal{L}_B^{qqg} = -g_B \bar{q}_B \, \gamma^\mu \frac{\lambda_a}{2} \, q_B \, A^a_{\mu,B}$

$$= -Z_g Z_2 Z_3^{1/2} \, g \bar{q} \gamma^\mu \frac{\lambda_a}{2} q A^a_\mu$$

$$= \underbrace{-g \bar{q} \gamma^\mu \frac{\lambda_a}{2} q A^a_\mu}_{\mathcal{L}_R}$$

$$+ \underbrace{\left(Z_g Z_2 Z_3^{1/2} - 1\right) \cdot (-g) \bar{q} \gamma^\mu \frac{\lambda_a}{2} q A^a_\mu}_{\mathcal{L}_{CT}}$$

∴ INSTEAD OF CALCULATING WITH $\mathcal{L}_B$ WITH UNPHYSICAL PARAMETERS,

WE CALCULATE WITH $\mathcal{L}_R$ WITH PHYSICAL PARAMETERS

& SUPPLEMENT THIS WITH $\mathcal{L}_{CT}$ ⇒ CHOSEN IN SUCH A WAY THAT IT ABSORBS ALL DIVERGENCES

⇓

ADDITIONAL FEYNMAN RULES FOR $\mathcal{L}_{CT}$

$$\boxed{-ig\left(Z_g Z_2 Z_3^{1/2} - 1\right) \gamma^\mu \frac{\lambda_a}{2}}$$

$\hookrightarrow$ e.g

$$\mathcal{L}_{q,B} = \bar{q}_B (i\gamma^\mu \partial_\mu - m_B) q_B$$

$$= Z_2 \, \bar{q} \, (i\gamma^\mu \partial_\mu - Z_m m) \, q$$

$$= \underbrace{\bar{q} \, (i\gamma^\mu \partial_\mu - m) \, q}_{\mathcal{L}_R}$$

$$+ \underbrace{(Z_2 - 1) \, \bar{q} \, i\gamma^\mu \partial_\mu q - (Z_2 Z_m - 1) \, \bar{q} \, m \, q}_{\mathcal{L}_{CT}}$$

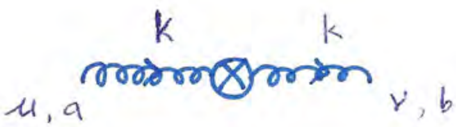
FEYNMAN RULE FOR $\mathcal{L}_{CT}$

$$\boxed{i \left[(Z_2 - 1) \, \not{p} - (Z_2 Z_m - 1) \, m \right]}$$

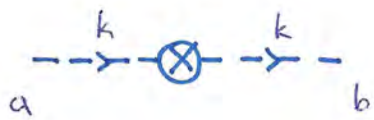
- FEYNMAN RULES FOR COUNTERTERMS

FEYNMAN RULE

$$i\left[(Z_2-1)\,\not{p} - (Z_2 Z_m - 1)\,m\right]$$

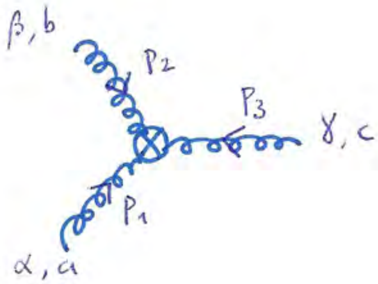


$$-i\,(Z_3-1)\,\delta_{ab}\,\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)$$



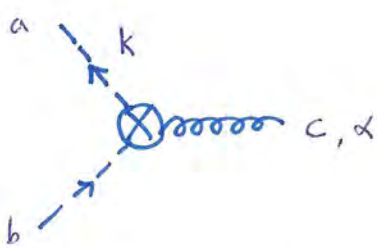
$$-i\,(\tilde{Z}_3-1)\,\delta_{ab}\,k^2$$

$$-ig\,(Z_{1F}-1)\,\gamma^\mu \frac{\lambda_a}{2} \qquad \text{WITH } Z_{1F} \equiv Z_g Z_2 Z_3^{1/2}$$

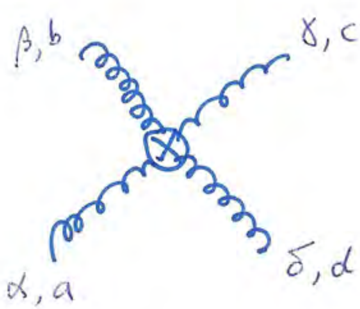


$$-g\,(Z_1-1)\,f_{abc}\,V^{\alpha\beta\gamma}(p_1,p_2,p_3)$$

$$\text{WITH } Z_1 \equiv Z_g Z_3^{3/2}$$



$$-g\,(\tilde{Z}_1-1)\,f_{abc}\,k^\alpha \qquad \text{WITH } \tilde{Z}_1 = Z_g \tilde{Z}_3 Z_3^{1/2}$$



$$-ig^2\,(Z_4-1)\,W^{\alpha\beta\gamma\delta}_{abcd} \qquad \text{WITH } Z_4 = Z_g^2 Z_3^2$$

• REGULARIZATION

CALCULATE LOOPS WITH $\mathcal{L}_R$

$\downarrow$

IN INTERMEDIATE STEP : REGULARIZE THEORY

↳ CUT-OFF METHODS $\int^{\Lambda} d^4 \ell$

$\Downarrow$

VIOLATES LORENTZ-INVARIANCE

↳ PAULI-VILLARS REGULARIZATION

e.g. $$\frac{1}{\ell^2 - m^2 + i\varepsilon} \longrightarrow \frac{1}{\ell^2 - m^2 + i\varepsilon} \cdot \Bigg(\underbrace{\frac{m^2 - \Lambda^2}{\ell^2 - \Lambda^2 + i\varepsilon}}_{\xrightarrow{\Lambda \to \infty \text{ IN END}} \, 1} \Bigg)$$

IMPROVES CONVERGENCE BY 2 POWERS

↳ DIMENSIONAL REGULARIZATION (T'HOOFT, VELTMAN)

CONSIDER $\int d^4 \ell \longrightarrow \int d^D \ell$

CALCULATE LOOPS IN D-DIMENSIONS

$D < 4 \implies$ IMPROVES CONVERGENCE

DIVERGENCE WILL APPEAR AS POLE IN " $\frac{1}{\varepsilon}$ " $\left(\varepsilon \equiv 2 - \frac{D}{2} \right)$

$D < 4 \implies \varepsilon > 0$

AT THE END : WHEN DIVERGENCE IS ABSORBED IN COUNTERTERM

$\downarrow$

ONE CAN LET $\varepsilon \to 0$ (ANALYTICAL CONTINUATION)

DIRAC ALGEBRA IN N - DIM.

$\Rightarrow$

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} I_N$$

$\Rightarrow$

$$\gamma_\mu \gamma^\mu = N I_N$$

$$\gamma_\mu \gamma_\alpha \gamma^\mu = (2-N) \gamma_\alpha$$

$$\gamma_\mu \gamma_\alpha \gamma_\beta \gamma^\mu = 4 g_{\alpha\beta} I_N + (N-4) \gamma_\alpha \gamma_\beta$$

$$\gamma_\mu \gamma_\alpha \gamma_\beta \gamma_\lambda \gamma^\mu = -2 \gamma_\lambda \gamma_\beta \gamma_\alpha - (N-4) \gamma_\alpha \gamma_\beta \gamma_\lambda$$

$\Rightarrow$ $\quad \mathrm{Tr}(I_N) = 4 + f(N)$

↳ DOES NOT AFFECT LIMIT $N \to 4$

UV DIVERGENCES

⇒ FEYNMAN PARAMETRIZATION

$$\frac{1}{A_0 A_1 \cdots A_n} = \Gamma(n+1)\int_0^1 dz_1 \int_0^{z_1} dz_2 \cdots \int_0^{z_{n-1}} dz_n \frac{1}{\left[A_0 + (A_1 - A_0) z_1 + \cdots + (A_n - A_{n-1}) z_n\right]^{n+1}}$$

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\left[B + (A-B)x\right]^{\alpha+\beta}}$$

⇒ INTEGRALS

- $$\int \frac{d^D k}{(2\pi)^D} \frac{1}{\left[k^2 - \Delta + i\varepsilon\right]^n} = \frac{i}{(4\pi)^{D/2}} \cdot \frac{\Gamma(n - D/2)}{\Gamma(n)} \frac{(-1)^n}{(\Delta - i\varepsilon)^{n - D/2}}$$

- $$\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu}{\left[k^2 - \Delta + i\varepsilon\right]^n} = 0$$

- $$\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{\left[k^2 - \Delta + i\varepsilon\right]^n} = \frac{i g^{\mu\nu}}{(4\pi)^{D/2}} \cdot \frac{\Gamma(n - \frac{D}{2} - 1)}{2\Gamma(n)} \cdot \frac{(-1)^{n-1}}{(\Delta - i\varepsilon)^{n - \frac{D}{2} - 1}}$$

⇒ Γ- FUNCTION

$$\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!}\left\{\frac{1}{\varepsilon} + \psi(n+1) + O(\varepsilon)\right\}$$

$$\psi(n+1) = 1 + \cdots + \frac{1}{n} - \gamma_E$$

$$\psi(1) = -\gamma_E \qquad (\gamma_E \simeq 0.577)$$

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E$$

$$\Gamma(1+\varepsilon) = 1 - \varepsilon\gamma_E + \frac{1}{2}\left(\gamma_E^2 + \frac{\pi^2}{6}\right)\varepsilon^2 + O(\varepsilon^3)$$

* MEASURE IN D DIM EUCLIDEAN SPACE

$\theta_1, \dots \theta_{D-2}$: $0 \to \pi$

ϕ : $0 \to 2\pi$

$$d^D \bar{p} = |\bar{p}|^{D-1} \sin^{D-2}\theta_1 \dots \sin\theta_{D-2} \; d|\bar{p}| \, d\theta_1 \dots d\theta_{D-2} \, d\phi$$

$$\int_0^\pi d\theta \; \sin^n\theta = \sqrt{\pi} \; \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$$

$$\delta^D(\bar{p}-\bar{p}_0) = \frac{1}{|\bar{p}|^{D-1}} \, \delta(|\bar{p}|-|\bar{p}_0|) \, \delta^{D-1}(\Omega_{\hat{p}} - \Omega_{\hat{p}_0})$$

'HYPERSURFACE' $\displaystyle\int_{(D-1)} d\Omega = \frac{2\pi^{D/2}}{\Gamma(D/2)}$

* BETA - FUNCTION

$$B(n,m) = \int_0^1 dx \; x^{n-1}(1-x)^{m-1} = \frac{\Gamma(n)\,\Gamma(m)}{\Gamma(n+m)}$$

* α - PARAMETRIZATION

$$\frac{1}{a^n} = \frac{1}{\Gamma(n)} \int_0^\infty d\alpha \; e^{-a\alpha} \, \alpha^{n-1}$$