Theoretical Elementary Particle Physics
Exercise 1

5 November 2018

1 $\mathbf{SU(3)}_C$ group (60 points)

1.1 SU(n) group (5 points)

Special unitary group of order n is a group of $n \times n$ unitary matrices with determinant 1. Show that the dimension of SU(n) group is equal to $n^2 - 1$.

1.2 Properties of Gell-Mann matrices (5 points)

The λ matrices are normalized such that $\mathrm{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}$ (for more properties see **Appendix A**).

(a) Check this for three combinations of matrices λ_a and λ_b.

(b) Calculate the structure constants $f_{156}, f_{345}, f_{458}$.

(c) Use Tr(AB)=Tr(BA) to show that $\mathrm{Tr}(\lambda_c[\lambda_a, \lambda_b]) = 4if_{abc}$ and, by changing the order of the λ, that the structure constants f_{abc} are antisymmetric in the exchange of two indices.

1.3 Color singlet hadrons (10 points)

Show that for quarks q_a ($a = 1, 2, 3 = r, g, b$) in the fundamental representation of color $\mathrm{SU}(3)_C$, the combinations $\bar{q}_a q_a$ (mesons) and $\varepsilon_{abc} q_a q_b q_c$ (baryons) are color singlets, i.e. invariant under $\mathrm{SU}(3)_C$ transformations: $q_a \to U_{ab} q_b$, $\bar{q}_b \to \bar{q}_b U^\dagger_{ba}$.

1.4 1-gluon exchange potential between different quark states (40 points)

We consider the interaction of 2 quarks due to exchange of one gluon. In QCD, the 8 gluons form a color octet:

$$
\begin{aligned}
&g_1 = r\bar{g}, \quad g_2 = r\bar{b}, \quad g_3 = g\bar{r}, \\
&g_4 = g\bar{b}, \quad g_5 = b\bar{r}, \quad g_6 = b\bar{g}, \\
&g_7 = \frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g}), \\
&g_8 = \frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})
\end{aligned}
\tag{1}
$$

(a) **(10 points)** Show that the color factor due to 1-gluon exchange between 2 quarks of the same color, as shown in Fig. 1, is given by $+2/3$. Show this explicitly for a blue and a red quarks.

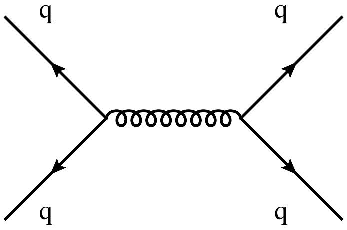

Figure 1: 1-gluon exchange potential between 2 quarks

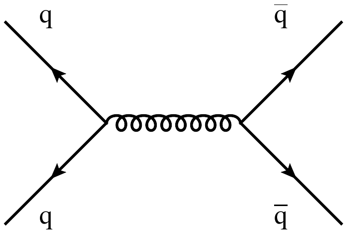

Figure 2: 1-gluon exchange potential between a quark and an antiquark

(b) **(10 points)** Calculate next the color factor arising from 1-gluon exchange between a quark and an anti-quark in a color singlet state:

$$|q\bar{q}\rangle_{\text{singlet}} = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b}) \tag{2}$$

as shown in Fig. 2. Show, that this color factor is given by $-8/3$, yielding an attractive interaction. Note that an antiquark has opposite color charge as a quark.

(c) **(10 points)** Likewise, show that the color factor arising from 1-gluon exchange between a quark and an antiquark in a color octet state is given by $+1/3$, yielding a repulsive interaction (choose one state, it is not necessary to show for all 8 of them).

(d) **(10 points)** Show that the color factor due to 1-gluon exchange in the color singlet 3-quark state

$$|qqq\rangle_{\text{singlet}} = \frac{1}{\sqrt{6}}(rgb + gbr + brg - rbg - grb - bgr) \tag{3}$$

is given by -4, also giving rise to strong attraction. **Hint:** consider one initial state, for instance rgb, and at the end multiply result by 6.

2 Quantization of the scalar field (40 points)

Consider the quantum field theory of a real scalar field governed by the Lagrangian,

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{4}$$

(a) **(30 points)** Evaluate the generating functional Z[J] perturbatively, keeping all terms up to and including terms of $\mathcal{O}(\lambda)$ as follows. First, show that $Z[J]$ can be written in the following form,

$$Z[J] = \mathcal{N}\left[1 - \frac{i\lambda}{4!}\int d^4y\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^4 + \mathcal{O}(\lambda^2)\right]\exp\left\{-\frac{i}{2}\int d^4x_1 d^4x_2 J(x_1)\Delta_F(x_1 - x_2)J(x_2)\right\}, \tag{5}$$

where $\mathcal{N}$ is the J-independent constant. Then, carry out the functional derivatives with respect to J, keeping all terms up to and including terms of $\mathcal{O}(\lambda)$. Using the result just obtained for $Z[J]$, obtain an expression for the generating functional for the connected Green functions, W[J], keeping all terms up to and including terms of $\mathcal{O}(\lambda)$.

(b) (10 points) Using the result of part **(a)** for W[J], compute the four-point connected Green function.

Appendix A Structure Constants and Color Matrices

The 3×3 SU(3) generators, t_a, satisfy

$$[t_a, t_b] = i f_{abc} t_c \tag{6}$$

where f_{abc} are the antisymmetric SU(3) structure constants with non-zero values given by

$$f_{123} = 1, \quad f_{147} = \frac{1}{2}, \quad f_{246} = \frac{1}{2}, \quad f_{257} = \frac{1}{2}, \quad f_{367} = -\frac{1}{2}, \quad f_{678} = \frac{\sqrt{3}}{2} \tag{7}$$

and $f_{156}, f_{345}, f_{458}$, can be found in Problem **1.2(b)**.

A convenient representation of the $t_a = \lambda_a/2$ matrices is the one introduced by Gell-Mann in which

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{8}$$

The structure constants d_{abc} are defined according to

$$\{t_a, t_b\} = \frac{1}{3}\delta_{ab} + d_{abc} t_c \, . \tag{9}$$

The t_a matrices satisfy

$$\begin{aligned}
t_a t_b &= \frac{1}{2}\left[\frac{1}{3}\delta_{ab} + (d_{abc} + i f_{abc}) t_c\right] , \\
t_a^{ij} t_a^{kl} &= \frac{1}{2}\left[\delta_{il}\delta_{jk} - \frac{1}{3}\delta_{ij}\delta_{kl}\right] , \\
\mathrm{Tr}(t_a) &= 0 \, , \\
\mathrm{Tr}(t_a t_b) &= \frac{1}{2}\delta_{ab} \, , \\
\mathrm{Tr}(t_a t_b t_c) &= \frac{1}{4}(d_{abc} + i f_{abc}) \, , \\
\mathrm{Tr}(t_a t_b t_a t_c) &= -\frac{1}{12}\delta_{bc} \, .
\end{aligned} \tag{10}$$

The structure constants satisfy the following Jacobi identities

$$\begin{aligned}
f_{abe} f_{ecd} + f_{cbe} f_{aed} + f_{dbe} f_{ace} &= 0 \, , \\
f_{abe} d_{ecd} + f_{cbe} d_{ced} + f_{dbe} d_{ace} &= 0 \, .
\end{aligned} \tag{11}$$

In addition,

$$f_{abc} f_{cde} = \frac{2}{3}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + (d_{ace} d_{bde} - d_{bce} d_{ade}) \, . \tag{12}$$